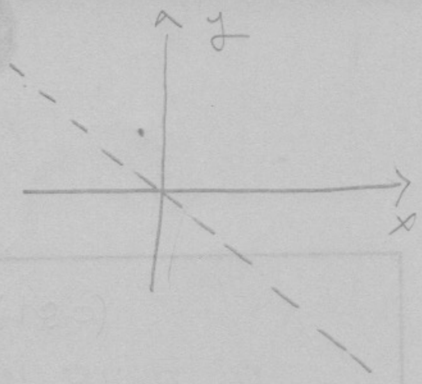
Es. 1

$$f(x,y) = y \lg\left[\frac{(x+y)^2}{x^2+y^2}\right]$$



$$\begin{cases} (x+y)^2 \neq 0 \\ x^2+y^2 \neq 0 \end{cases} \qquad \begin{matrix} x+y \neq 0 \\ (x,y) \neq (0,0) \end{matrix}$$

$D_f = \{(x,y) \in \mathbb{R}^2 : y \neq -x\}$ aperto, illimitato, non connesso

$$\lim_{(x,y)\to(0,0)} y \lg\left[\frac{(x+y)^2}{x^2+y^2}\right] = 0 \quad \text{infatti}$$

$$|f(x,y)| = |y| \left| \lg\left[\frac{(x+y)^2}{x^2+y^2}\right] \right| \leq |y| \left| \lg \frac{2x^2+2y^2}{x^2+y^2} \right| = |y| \lg 2 \xrightarrow[(x,y)\to(0,0)]{} 0$$

Possiamo definire $\tilde{f}(x,y) = \begin{cases} f(x,y) & \text{se } (x,y) \neq (0,0) \\ 0 & \text{se } (x,y) = (0,0) \end{cases}$

$D_{\tilde{f}} = D_f \cup \{(0,0)\}$

$$\frac{\partial f}{\partial x}(0,0) = \frac{d}{dx} f(x,0) \Big|_{x=0} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \frac{d}{dy} f(0,y) \Big|_{y=0} = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{\tilde{f}(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{y \lg \frac{(x+y)^2}{x^2+y^2}}{\sqrt{x^2+y^2}}$$

$y = mx \quad m \neq -1 \qquad \frac{m\,x}{|x|\sqrt{1+m^2}} \lg \frac{(1+m)^2}{1+m^2} \xrightarrow[x\to 0]{} sg(x) \frac{m}{\sqrt{1+m^2}} \lg \frac{(1+m)^2}{1+m^2}$

il limite dipende da m $\Rightarrow$ il limite non esiste

$\Rightarrow$ la funzione non è differenziabile in $(0,0)$.

Es. 2 $\quad f(x,y) = x^2y + x^2 + y^2 - 6y \quad D = \mathbb{R}^2$

$$\frac{\partial f}{\partial x} = 2xy + 2x \qquad \begin{cases} 2x\,(y+1) = 0 \\ x^2 + 2y - 6 = 0 \end{cases}$$

$$\frac{\partial f}{\partial y} = x^2 + 2y - 6$$

$x = 0 \quad y = 3$ $\qquad P_1(0,3) \quad P_2(-2\sqrt{2}, -1)$

$y = -1 \quad x^2 - 2 - 6 = 0 \quad x = \pm 2\sqrt{2}$ $\qquad P_3(2\sqrt{2}, -1)$

$$\frac{\partial^2 f}{\partial x^2} = 2y + 2 \qquad \frac{\partial^2 f}{\partial y^2} = 2 \qquad \frac{\partial^2 f}{\partial x \partial y} = 2x$$

$$Hf(0,3) = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}$$ entrambi gli autovalori sono positivi $\Rightarrow P_1$ pto di minimo

$$Hf(-2\sqrt{2}, -1) = \begin{pmatrix} 0 & -4\sqrt{2} \\ -4\sqrt{2} & 2 \end{pmatrix}$$ $\det Hf(-2\sqrt{2}, -1) = -(4\sqrt{2})^2 < 0$

$\Rightarrow P_2$ pto di sella

$$Hf(2\sqrt{2}, -1) = \begin{pmatrix} 0 & 4\sqrt{2} \\ 4\sqrt{2} & 2 \end{pmatrix}$$ $\det Hf(+2\sqrt{2}, -1) = -(4\sqrt{2})^2 < 0$

$\Rightarrow P_3$ pto di sella

$\frac{\partial f}{\partial x}(1,2) = 6 \qquad \frac{\partial f}{\partial y}(1,2) = -1 \qquad z = -5 + 6(x-1) - 1\,(y-2)$

$f(1,2) = 2 + 1 + 4 - 12 = -5 \qquad z = 6x - y + 8$

Dato che $f \in C^1(\mathbb{R}^2)$ f è differenziabile in $\mathbb{R}^2$ $\Rightarrow$ vale la formula del gradiente

$D_v f(1,2) = \nabla f(1,2) \cdot \hat{v} = (6, -1) \cdot (v_1, v_2) = 6v_1 - v_2$

Se $\hat{v} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \qquad D_{\hat{v}} f(1,2) = -\frac{6}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{7}{\sqrt{2}}$

Es. 3 $f(x,y) = y^2 + 2\sqrt{2}\,xy$ $\quad x^2+y^2=4$

$g(x,y) = x^2+y^2-4$ $\quad \nabla g = (2x, 2y)$ $\quad \nabla g = (0,0) \Leftrightarrow (x,y) = (0,0)$

$(0,0)$ non appartiene al vincolo $\Rightarrow$ il vincolo non ha pti critici

$$\mathcal{L} = f(x,y) - \lambda g(x,y) = y^2 + 2\sqrt{2}\,xy - \lambda(x^2+y^2-4)$$

$$\frac{\partial \mathcal{L}}{\partial x} = 2\sqrt{2}\,y - 2\lambda x \qquad \begin{cases} \sqrt{2}\,y - \lambda x = 0 \\ y + \sqrt{2}\,x - \lambda y = 0 \\ x^2+y^2-4=0 \end{cases}$$

$$\frac{\partial \mathcal{L}}{\partial y} = 2y + 2\sqrt{2}\,x - 2\lambda y$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(x^2+y^2-4)$$

$y = \frac{\lambda}{\sqrt{2}}x$ $\qquad \frac{\lambda}{\sqrt{2}}x + \sqrt{2}\,x - \frac{\lambda^2}{\sqrt{2}}x = 0$

$\frac{x}{\sqrt{2}}(\lambda + 2 - \lambda^2) = 0$ $\qquad x = 0 \quad y = 0$ non soddisfa l'eq. del vincolo

$\lambda^2 - \lambda - 2 = 0$ $\qquad \lambda = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = \begin{cases} 2 \\ -1 \end{cases}$

$\lambda = -1$ $\quad y = -\frac{1}{\sqrt{2}}x$ $\quad x^2 + \frac{1}{2}x^2 = 4$ $\quad x^2 = \frac{8}{3}$ $\quad x = \pm 2\sqrt{2/3}$

$\lambda = 2$ $\quad y = \sqrt{2}\,x$ $\quad x^2 + 2x^2 = 4$ $\quad x^2 = \frac{4}{3}$ $\quad x = \pm\frac{2}{\sqrt{3}}$

$P_1\left(-2\sqrt{\frac{2}{3}}, \frac{2}{\sqrt{3}}\right)$ $P_2\left(+2\sqrt{\frac{2}{3}}, -\frac{2}{\sqrt{3}}\right)$

$P_3\left(\frac{2}{\sqrt{3}}, \frac{2\sqrt{2}}{\sqrt{3}}\right)$ $P_4\left(-\frac{2}{\sqrt{3}}, -\frac{2\sqrt{2}}{3}\right)$

$$f\left(\pm\frac{2\sqrt{2}}{\sqrt{3}}, \mp\frac{2}{\sqrt{3}}\right) = -4$$

$$f\left(\pm\frac{2}{\sqrt{3}}, \pm\frac{2\sqrt{2}}{\sqrt{3}}\right) = \frac{8}{3} + 2\sqrt{2}\,\frac{4\sqrt{2}}{3} = \frac{24}{3} = 8$$

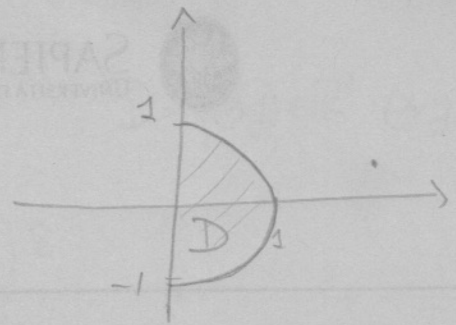
Il vincolo è un insieme chiuso e limitato, f è continua sul vincolo $\Rightarrow$ posso applicare il teorema di Weierstrass

$\Rightarrow$ P_1 e P_2 sono pti di minimo vincolato

P_3 e P_4 sono pti di max. vincolato

Es. 4

$$\iint_D \frac{x e^y}{\sqrt{x^2+y^2}}\,dx\,dy$$



Passiamo a coordinate polari

$$x = \rho\cos\vartheta \qquad 0 < \rho \le 1$$
$$y = \rho\,\text{sen}\,\vartheta \qquad -\tfrac{\pi}{2} < \vartheta < \tfrac{\pi}{2}$$

$$\int_0^1 \int_{-\pi/2}^{\pi/2} \frac{\cancel{\rho}\cos\vartheta\, e^{\rho\,\text{sen}\,\vartheta}}{\cancel{\rho}}\,\rho\,d\rho\,d\vartheta =$$

$$= \int_0^1 \left(\int_{-\pi/2}^{\pi/2} \rho\cos\vartheta\, e^{\rho\,\text{sen}\,\vartheta}\,d\vartheta \right) d\rho =$$

$$= \int_0^1 \left(e^{\rho\,\text{sen}\,\vartheta} \Big|_{-\pi/2}^{\pi/2} \right) d\rho = \int_0^1 \left(e^{\rho} - e^{-\rho}\right) d\rho =$$

$$= e^{\rho} + e^{-\rho} \Big|_0^1 = e - 1 + e^{-1} - 1 = e + \frac{1}{e} - 2$$

Es. 5)

$$\vec{F} = \frac{2xy}{(x^2+y^2)^2}\,\hat{\imath} + \frac{y^2-x^2}{(x^2+y^2)^2}\,\hat{\jmath} \qquad D = \{(x,y)\in\mathbb{R}^2 : (x,y)\neq(0,0)\}$$

$$\nabla\times\vec{F} = -\left(\partial_x \frac{y^2-x^2}{(x^2+y^2)^2} - \frac{\partial}{\partial y}\frac{2xy}{(x^2+y^2)^2}\right)\hat{k}$$

$$= \left(\frac{-2x}{(x^2+y^2)^2} - \frac{(y^2-x^2)4x}{(x^2+y^2)^3} - \left(\frac{2x}{(x^2+y^2)^2} - \frac{2xy\,4y}{(x^2+y^2)^3}\right)\right)$$

$$= -\frac{2x}{(x^2+y^2)^2}\left[1 + \frac{2y^2-2x^2}{x^2+y^2} + 1 - \frac{4y^2}{x^2+y^2}\right] =$$

$$= -\frac{2x}{(x^2+y^2)^3}\left[\cancel{2x^2}+\cancel{2y^2}+\cancel{2y^2}-\cancel{2x^2}-\cancel{4y^2}\right] = 0 \Rightarrow \vec{F} \text{ irrotazionale}$$

$\Rightarrow \vec{F}$ è localmente conservativo.

Calcolo il potenziale

$$\frac{\partial U}{\partial x} = \frac{2xy}{(x^2+y^2)^2} \Rightarrow U(x,y) = -\frac{y}{x^2+y^2} + g(y)$$

$$\frac{\partial U}{\partial y} = -\frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2} + g'(y) = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\cancel{y^2-x^2}}{\cancel{(x^2+y^2)^2}} + g'(y) = \frac{\cancel{y^2-x^2}}{\cancel{(x^2+y^2)^2}} \Rightarrow g'(y)=0 \Rightarrow g(y)=c$$

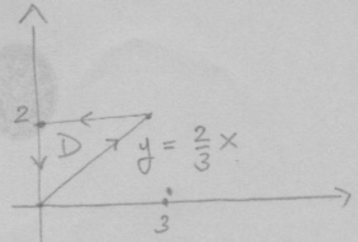
$\Rightarrow U(x,y) = -\dfrac{y}{x^2+y^2} + c$ definito in D $\Rightarrow$ il campo è conservativo in D

$P_1(1,0)$ $P_2=(-1,0)$

$$\int_\gamma \vec{F}\cdot d\vec{r} = U(P_2)-U(P_1)$$
$$= U(-1,0) - U(1,0) = c - c = 0$$

Es. 6 $\quad \vec{F} = \left(x^2y + 3x^2y^2,\ y^3x + x^2y^2\right)$

$\vec{F} = (P,\ Q)$



$$\int_{\gamma^+ D} \vec{F}\cdot d\vec{r} = \iint_D (Q_x - P_y)\,dx\,dy$$

$$= \iint_D \left(y^3 + 2xy^2 - x^2 - 6x^2y\right) dx\,dy = \int_0^3 \left(\int_{\frac{2}{3}x}^{2} \left(y^3 + 2xy^2 - x^2 - 6x^2y\right) dy \right) dx$$

$$= \int_0^3 \left(\frac{1}{4}y^4 + \frac{2}{3}xy^3 - x^2y - 3x^2y^2 \right)\Bigg|_{\frac{2}{3}x}^{2} dx =$$

$$= \int_0^3 \left[\frac{1}{4}\left(16 - \left(\frac{2}{3}\right)^4 x^4\right) + \frac{2}{3}x\left(8 - \left(\frac{2}{3}\right)^3 x^3\right) - 3x^2\left(4 - \left(\frac{2}{3}\right)^2 x^2\right) \right] dx$$

$$= 4x - \left(\frac{2}{3}\right)^4 \cdot \frac{1}{4}\cdot\frac{1}{5}x^5 + \frac{8}{3}x^2 - \left(\frac{2}{3}\right)^4\cdot\frac{1}{5}x^5 - 4x^3 + 3\left(\frac{2}{3}\right)^2\cdot\frac{1}{5}x^5 \Bigg|_0^3$$

$$= 12 - \frac{\cancel{5}}{\cancel{4}}\left(\frac{2}{\cancel{3}}\right)^{\cancel{4}\,2}\cdot\frac{1}{\cancel{5}}\,3^{\cancel{5}} + \frac{8}{\cancel{3}}\cdot 3^{\cancel{2}} - 4\cdot 3^3 + \cancel{3}\cdot\left(\frac{2}{3}\right)^2\cdot\frac{1}{5}\,3^5$$

$12 \cdot 9 \qquad \frac{4\cdot 3^4}{5}$

$$= \cancel{12} - \cancel{12} + 24 - 108 + \frac{4\cdot 3^4}{5} = \frac{-84\cdot 5 + 81\cdot 4}{5} = -\frac{96}{5}$$