$$f(x) = e^{-x+1} \qquad x \in [0,1] \qquad T = 1 \qquad \omega = \frac{2\pi}{T} = 2\pi$$

$$a_k = \frac{2}{T}\int_0^T f(x)\cos(2k\pi x)\,dx \qquad k = 0, 1, 2, \dots$$

$$a_0 = 2\int_0^1 e^{-x+1}\,dx = 2e\left(-e^{-x}\right)_0^1 = 2e(1-e^{-1})$$

$$a_k = 2e\int_0^1 e^{-x}\cos(2\pi k x)\,dx = \cancel{2}e\int_0^1 \frac{e^{(-1+i2\pi k)x} + e^{(-1-2\pi i k)x}}{\cancel{2}}\,dx$$

$$= e\left[\frac{1}{-1+2\pi i k}(e^{-1}-1) + \frac{1}{-1-2\pi i k}(e^{-1}-1)\right] = e(e^{-1}-1)\left[\frac{-1\cancel{-2\pi i k}-1\cancel{+2\pi i k}}{1+4\pi^2k^2}\right]$$

$$= \frac{2e(1-e^{-1})}{1+4\pi^2k^2}$$

$$b_k = \frac{e}{i}(e^{-1}-1)\,\frac{\cancel{-1}-2\pi i k\cancel{+1}-2\pi i k}{4\pi^2k^2+1} = \frac{4\pi k\, e(1-e^{-1})}{4\pi^2k^2+1}$$

$$f(x) \sim e(1-e^{-1}) + e(1-e^{-1})\sum_{k=1}^{\infty}\left[\frac{2}{1+4\pi^2k^2}\cos(2k\pi x) + \frac{4\pi k}{1+4\pi^2k^2}\,\mathrm{sen}(2k\pi x)\right]$$

$$\int_0^1 f^2(x)\,dx = e^2\int_0^1 e^{-2x}\,dx = \frac{e^2}{2}(1-e^{-2})$$

$$\frac{\cancel{e^2}}{2}(1-e^{-2}) = \frac{1}{2}\left[2\cancel{e^2}(1-e^{-1})^2 + \cancel{e^2}(1-e^{-1})^2\sum_{k=1}^{\infty}\left(\frac{4+16\pi^2k^2}{(1+4\pi^2k^2)^2}\right)\right]$$

$$\Rightarrow \cancel{2}\,\frac{1-e^{-2}-(1-e^{-1})^2}{(1-e^{-1})^2} = \overset{2}{\cancel{4}}\sum_{k=1}^{\infty}\frac{1}{1+4\pi^2k^2}$$

$$\sum_{k=1}^{\infty}\frac{1}{1+4\pi^2k^2} = \frac{1}{2}\,\frac{1-e^{-2}-(1-e^{-1})^2}{(1-e^{-1})^2}$$