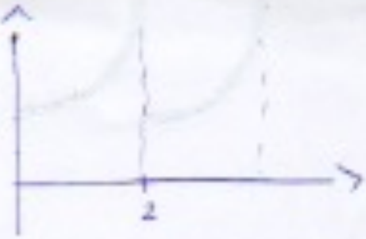
④

Es. 6

$f(x) = e^{-2x} \qquad T = 2 \qquad x \in [0,2]$



$\omega = \frac{2\pi}{T} = \pi$

$$a_0 = \frac{2}{T}\int_0^T f(x)\,dx = \int_0^2 e^{-2x}\,dx = -\frac{1}{2}e^{-2x}\Big|_0^2 = \frac{1}{2}(1-e^{-4})$$

$$a_k = \frac{2}{T}\int_0^T f(x)\cos(k\omega x)\,dx = \int_0^2 e^{-2x}\cos(k\omega x)\,dx$$

$$= \int_0^2 e^{-2x}\,\frac{e^{ik\omega x}+e^{-ik\omega x}}{2}\,dx = \int_0^2 \frac{e^{(-2+ik\omega)x}+e^{(-2-ik\omega)x}}{2}\,dx =$$

$$= \frac{1}{2}\left[\frac{1}{-2+ik\omega}e^{(-2+ik\omega)x} + \frac{1}{-2-ik\omega}e^{(-2-ik\omega)x}\Big|_0^2\right] =$$

$$= \frac{1}{2}\left[\frac{1}{-2+ik\omega}\left(e^{-4}-1\right) + \frac{1}{-2-ik\omega}\left(e^{-4}-1\right)\right] =$$

$$= \frac{(e^{-4}-1)}{2}\left(\frac{1}{-2+ik\omega}+\frac{1}{-2-ik\omega}\right) = \frac{(e^{-4}-1)}{2}\left(\frac{-2-ik\omega-2+ik\omega}{4+k^2\omega^2}\right) =$$

$$= \frac{4\,(1-e^{-4})}{2\,(4+k^2\omega^2)} = \frac{2\,(1-e^{-4})}{4+k^2\omega^2} \qquad \omega=\pi \qquad e^{\pm i n\pi 2} = 1$$

$$b_k = \frac{2}{T}\int_0^T f(x)\,\mathrm{sen}(k\omega x)\,dx = \int_0^2 e^{-2x}\,\mathrm{sen}(k\omega x)\,dx =$$

$$= \int_0^2 e^{-2x}\,\frac{e^{ik\omega x}-e^{-ik\omega x}}{2i} = \frac{1}{2i}\int_0^2 \left(e^{(-2+ik\omega)x}-e^{(-2-ik\omega)x}\right)dx$$

$$= \frac{1}{2i}\left(e^{-4}-1\right)\left(\frac{1}{-2+ik\omega}-\frac{1}{-2-ik\omega}\right) = \frac{e^{-4}-1}{2i}\;\frac{-2-ik\omega+2-ik\omega}{4+k^2\omega^2}$$

$$= (1-e^{-4})\,\frac{k\omega}{4+k^2\omega^2}$$

$$f(x) \sim (1-e^{-4})\left[\frac{1}{4}+\sum_{k=1}^{\infty}\left[\frac{2}{4+k^2\omega^2}\cos(\omega k x) + \frac{k\omega}{4+k^2\omega^2}\,\mathrm{sen}(\omega k x)\right]\right]$$

La funzione è regolare a tratti e continua in $(0,2)$
$\Rightarrow$ la serie di Fourier converge ad $f(x)$ $\forall x \in (0,2)$
in $x=0$ e $x=2$ converge a $\frac{f(0)+f(2)}{2} = \frac{1+e^{-4}}{2}$
Dato che $f(0) \neq f(\pi)$ la serie non è derivabile termine a termine.