ON THE DYNAMICAL BEHAVIOR OF THE ABC MODEL

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Abstract. We consider the ABC dynamics, with equal density of the three species, on the discrete ring with N sites. In this case, the process is reversible with respect to a Gibbs measure with a mean field interaction that undergoes a second order phase transition. We analyze the relaxation time of the dynamics and show that at high temperature it grows at most as N^2 while it grows at least as N^3 at low temperature.

1. Introduction

The *ABC model* has been introduced by Evans et al. [12, 13], it exhibits the peculiar feature of being a one-dimensional stochastic conservative dynamics with local jump rates whose invariant measure undergoes a phase transition. The ABC model is a system consisting of three species of particles, traditionally labeled A, B, C, on a discrete ring with N sites. The system evolves by nearest neighbors particles exchanges with the following asymmetric rates: $AB \to BA$, $BC \to CB$, $CA \to AC$ with rate $q \in (0,1]$ and $BA \to AB$, $CB \to BC$, $AC \to CA$ with rate $1/q$. In particular, the total numbers of particles N_α, $\alpha \in \{A, B, C\}$, of each species are conserved and satisfy $N_A + N_B + N_C = N$. Observe that the case $q = 1$ corresponds to a three state version of the symmetric simple exclusion process. When $q \in (0,1)$, Evans et al. [12, 13] argued that in the thermodynamic limit $N \to \infty$ with $N_\alpha/N \to r_\alpha$ the system segregates into pure A, B, and C regions, with translationally invariant distribution of the phase boundaries. In the equal density case $N_A = N_B = N_C = N/3$ the dynamics is reversible and its invariant measure can be explicitly computed.

As discussed by Clincy et al. [7], the natural scaling to investigate the asymptotic behavior of the ABC model is the *weakly asymmetric* regime $q = \exp\{-\frac{\beta}{2N}\}$, where the parameter $\beta \in [0, +\infty)$ plays the role of an inverse temperature. With this choice the reversible measure of the equal densities case $r_A = r_B = r_C = 1/3$ becomes a canonical Gibbs measure, that we denote by ν_N^β, with a mean field Hamiltonian. The measure ν_N^β undergoes a second order phase transition at $\beta_c = 2\pi\sqrt{3} \approx 10.88$. This phase transition has been further analyzed in [1] and it is described in terms of the free energy functional $\mathcal{F}_\beta$ associated to ν_N^β. The functional $\mathcal{F}_\beta$ is (apart an additive constant) the large deviations rate function for ν_N^β in the scaling limit in which the discrete ring with N sites is embedded in the one-dimensional torus and the particles configuration is described in terms of the corresponding densities profiles(ρ_A, ρ_B, ρ_C). In this limit, $\mathcal{F}_\beta(\rho_A, \rho_B, \rho_C)$ thus gives the asymptotic probability of observing the density profile (ρ_A, ρ_B, ρ_C). In particular, the minimizer of $\mathcal{F}_\beta$ describes the typical behavior of the system as $N \to \infty$. The phase

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transition of ν_N^β corresponds to the following behavior of the free energy functional. For $\beta \in [0, \beta_c]$ the minimum of $\mathcal{F}_\beta$ is uniquely achieved at the homogeneous profile $(1/3, 1/3, 1/3)$. For $\beta > \beta_c$ the functional $\mathcal{F}_\beta$ has instead a one-parameter family of minimizers which describe the phase segregation. As shown in [2], this phase transition can also be detected via the two-point correlation functions of ν_N^β which become singular when the system approaches the transition.

For unequal densities the invariant measure of the ABC dynamics on a ring is not reversible, that is the stationary state is no longer an equilibrium state, and cannot be computed explicitly. As discussed in [2, 7], a stability analysis of the homogeneous density profile shows that for $\beta > 2\pi\big[1 - 2(r_A^2 + r_B^2 + r_C^2)\big]^{-1/2}$ it becomes unstable. As stated there, one however expects that the phase transition, at least for particular values of the parameters r_A, r_B, r_C, becomes of the first order. Again in [2, 7], the asymptotic of the two-point correlation functions is computed in the homogeneous phase and the large deviation rate function $\mathcal{F}_\beta$ has been calculated up to order β^2. When the ABC dynamics is considered on an open interval with reflecting endpoints, the corresponding invariant measure is reversible for all values of the densities [1]. In particular, it has the same Gibbs form as the one in the ring for the equal density case.

Main purpose of the present paper is the discussion of the phase transition of the ABC model on a ring with N sites from a dynamical viewpoint. More precisely, we focus on the asymptotic behavior, as N diverges, of the *relaxation time* τ_N^β which measures the time the dynamics needs to reach the stationary probability. Our analysis is restricted to the equal density case $r_A = r_B = r_C = 1/3$ in which the invariant measure ν_N^β is explicitly known and reversible. As usual, the relaxation time τ_N^β is defined as the inverse of the *spectral gap* of the generator L_N^β of the underlying Markov process. Observe that, in view of the reversibility, L_N^β is selfadjoint in $L^2(d\nu_N^\beta)$. Our main result implies that the asymptotic behavior of the relaxation time τ_N^β reflects the phase transition of the corresponding stationary measure ν_N^β. We indeed show that for β small enough τ_N^β is at most of order N^2 while for $\beta > \beta_c$ it is at least of order N^3.

The diffusive behavior $\tau_N^\beta \sim N^2$ is characteristic of conservative dynamics in the high temperature regime, the typical example being the Kawasaki dynamics for the Ising model. Indeed, this has been proven by different techniques in several contexts, see e.g. [3, 5, 17]. We here follow the approach introduced in [3] which is based upon a perturbative argument in β and can be directly applied to the case of mean field interactions. On the other hand, the behavior $\tau_N^\beta \sim N^3$ in the supercritical regime is characteristic of the system under consideration, we briefly discuss the heuristic picture. As stated in [1, 2], at time $O(N^2)$ the densities profiles of the three species (ρ_A, ρ_B, ρ_C) evolve according to the hydrodynamic equations

$$\begin{aligned} \partial_t \rho_A + \beta\, \nabla\big[\rho_A(\rho_C - \rho_B)\big] &= \Delta \rho_A \\ \partial_t \rho_B + \beta\, \nabla\big[\rho_B(\rho_A - \rho_C)\big] &= \Delta \rho_B \\ \partial_t \rho_C + \beta\, \nabla\big[\rho_C(\rho_B - \rho_A)\big] &= \Delta \rho_C \end{aligned} \tag{1.1}$$

where ∇ and Δ denote respectively the gradient and Laplacian on the macroscopic torus. As follows from microscopic reversibility, the evolution (1.1) can be obtained as a suitable gradient flow of the free energy $\mathcal{F}_\beta$. In particular, while the homogeneous profile $(1/3, 1/3, 1/3)$ is the unique, globally attractive, stationary solution

to (1.1) for $\beta < \beta_c$, the (one parameter family) minimizers of $\mathcal{F}_\beta$ are stationary solutions to (1.1) when $\beta > \beta_c$. According to the *fluctuating hydrodynamic theory*, we argue that, for large but finite N, the hydrodynamic equation (1.1) gives an accurate description of the system provided one adds in (1.1) a suitable noise term $O(1/\sqrt{N})$. At time $O(N^2)$ the ABC model then behaves as a Brownian motion on the set of minimizers of $\mathcal{F}_\beta$ with diffusion coefficient proportional to $1/N$. The time to thermalize is thus $O(N^3)$.

2. Notation and results

The ABC process. Given a positive integer N, we let $\mathbf{Z}_N = \{0, \cdots, N-1\}$ be the ring of the integers modulo N. The configuration space with N sites is $\widetilde{\Omega}_N := \{A, B, C\}^{\mathbf{Z}_N}$, elements of $\widetilde{\Omega}_N$ are denoted by ζ, for $x \in \mathbf{Z}_N$ the species of the particle at the site x is thus $\zeta(x) \in \{A, B, C\}$. We also let $\eta_\alpha \colon \widetilde{\Omega}_N \to \{0,1\}^{\mathbf{Z}_N}$, $\alpha \in \{A, B, C\}$, be the α occupation numbers namely, $[\eta_\alpha(\zeta)](x) := \mathbf{1}_{\{\alpha\}}(\zeta(x))$ in which $\mathbf{1}_E$ stands for the indicator function of the set E. Note that for each $x \in \mathbf{Z}_N$ we have $\eta_A(x) + \eta_B(x) + \eta_C(x) = 1$. Whereas $\eta = (\eta_A, \eta_B, \eta_C)$ is a function of the configuration ζ we shall omit to write explicitly the dependence on ζ.

Given $x \in \mathbf{Z}_N$ and $\zeta \in \widetilde{\Omega}_N$ we denote by $\zeta^{x,x+1}$ the configuration obtained from ζ by exchanging the particles at the sites x and $x+1$, i.e.

$$\left(\zeta^{x,x+1}\right)(y) := \begin{cases} \zeta(x+1) & \text{if } \ y = x, \\ \zeta(x) & \text{if } \ y = x+1, \\ \zeta(y) & \text{otherwise.} \end{cases} \tag{2.1}$$

The ABC process is the Markov chain on the state space $\widetilde{\Omega}_N$ whose generator $L_N = L_N^\beta$ acts on functions $f \colon \widetilde{\Omega}_N \to \mathbf{R}$ as

$$L_N^\beta f(\zeta) = \sum_{x \in \mathbf{Z}_N} c_x^\beta(\zeta) \big[f(\zeta^{x,x+1}) - f(\zeta) \big]. \tag{2.2}$$

For $\beta \geq 0$, the jump rates $c_x^\beta = c_{x,N}^\beta$ are given by

$$c_{x,N}^\beta(\zeta) := \begin{cases} \exp\{-\frac{\beta}{2N}\} & \text{if } \ (\zeta(x), \zeta(x+1)) \in \{(A,C), (C,B), (B,A)\} \\ \exp\{\frac{\beta}{2N}\} & \text{otherwise.} \end{cases} \tag{2.3}$$

As follows from (2.2), the ABC dynamics conserves the total number of particles of each species. Therefore, given three positive integers N_α, $\alpha \in \{A, B, C\}$ such that $N_A + N_B + N_C = N$, we have a well defined process on the linear manifold $\sum_{x \in \mathbf{Z}_N} \eta_\alpha(x) = N_\alpha$, $\alpha \in \{A, B, C\}$. As straightforward to check, the ABC dynamics is irreducible when restricted to such manifold; hence the process is ergodic and admits a unique invariant measure. In the case $\beta = 0$ this measure is the uniform probability. On the other hand, when $\beta > 0$ the explicit expression of the invariant measure is in general not known. However, as we next discuss, in the case $N_A = N_B = N_C$ the ABC process satisfies the detailed balance condition with respect to a mean field Gibbs measure [12, 13].

Invariant measure in the equal densities case. We assume that N is a multiple of 3 and we restrict to the case in which $N_A = N_B = N_C$. We shall then consider

the ABC process on

$$\Omega_N := \Big\{ \zeta \in \widetilde{\Omega}_N \,:\, \sum_{x\in\mathbb{Z}_N} \eta_A(x) = \sum_{x\in\mathbb{Z}_N} \eta_B(x) = \sum_{x\in\mathbb{Z}_N} \eta_C(x) = \frac{N}{3} \Big\}. \tag{2.4}$$

The Hamiltonian $H_N\colon \Omega_N \to \mathbf{R}$ is defined by

$$H_N(\zeta) := \frac{1}{N^2} \sum_{0\le x<y\le N-1} \big[\eta_A(x)\eta_C(y) + \eta_B(x)\eta_A(y) + \eta_C(x)\eta_B(y)\big]. \tag{2.5}$$

In view of the equal densities constraint, an elementary computation shows that the right hand side above does not depend on the choice of the origin. Equivalently, H_N is a translation invariant function on Ω_N. Given $\beta \ge 0$, we denote by ν_N^β the probability measure on Ω_N defined by

$$\nu_N^\beta(\zeta) := \frac{1}{Z_N^\beta} \exp\big\{ -\beta N H_N(\zeta) \big\} \tag{2.6}$$

where Z_N^β, the partition function, is the proper normalization constant. In the sequel, given a function f on Ω_N we denote respectively by $\nu_N^\beta(f)$ and $\nu_N^\beta(f,f)$ the expectation and variance of f with respect to ν_N^β.

As observed in [12, 13], the ABC process is reversible with respect to ν_N^β. In other worlds, the generator L_N^β in (2.2) is a self-adjoint operator on $L^2(\Omega_N, \nu_N^\beta)$ and in particular ν_N^β is the invariant measure.

Asymptotic of the spectral gap. The spectrum of L_N^β in (2.2), considered as a self-adjoint operator on $L^2(\Omega_N, \nu_N^\beta)$, is a finite subset of the negative real axes and, in view of the ergodicity of the process, zero is a simple eigenvalue of L_N^β. The *spectral gap* of L_N^β, denoted by $\mathrm{gap}(L_N^\beta)$, is the absolute value of the second largest eigenvalue. The spectral gap can be characterized in variational terms as follows: $\mathrm{gap}(L_N^\beta)$ is largest constant $\lambda \ge 0$ such that the Poincaré inequality

$$\lambda\, \nu_N^\beta(f,f) \le -\nu_N^\beta\big(f L_N^\beta f\big) \tag{2.7}$$

holds for any $f \in L^2(d\nu_N^\beta)$. The spectral gap controls the speed of convergence to equilibrium of the associated process in the following sense. For each $f \in L^2(d\nu_N^\beta)$,

$$\nu_N^\beta\big(e^{tL_N^\beta} f\,,\, e^{tL_N^\beta} f\big) \le e^{-2\,\mathrm{gap}(L_N^\beta)\,t}\, \nu_N^\beta(f,f).$$

Our main result concerns the asymptotic behavior of $\mathrm{gap}(L_N^\beta)$ as N diverges. In particular we show this behavior differs in the subcritical and supercritical regimes.

Theorem 2.1.

(i) *There exist constants $\beta_0, C_0 > 0$ such that for any $\beta \in [0, \beta_0]$ and any N*

$$\mathrm{gap}(L_N^\beta) \ge C_0\, \frac{1}{N^2}.$$

(ii) *Let $\beta_c := 2\pi\sqrt{3}$. For any $\beta > \beta_c$ there exists a constant $C(\beta) > 0$ such that for any N*

$$\mathrm{gap}(L_N^\beta) \le C(\beta)\, \frac{1}{N^3}.$$

The above statements raise two natural issues. As discussed in the Introduction, the $1/N^2$ asymptotic of the spectral gap is a common feature of conservative stochastic dynamics in the high temperature regime. One then expects that this behavior holds for any $\beta \in [0, \beta_c)$. The methods used in the present paper are based on a perturbation argument around $\beta = 0$ and their extension to the the whole subcritical regime does not appear feasible. In principle, the techniques developed in [5, 17], which require as an input a strong spatial mixing of the stationary probability, can be applied up to the critical temperature. Those techniques have been however developed for short range interactions and they do not seem, at least directly, applicable to mean field Hamiltonians.

The second, somehow more fundamental, issue is whether $1/N^3$ is the right scaling of the spectral gap in the supercritical regime. We mention that this behavior is also the one expected for the Kawasaki dynamics for the low temperature two dimensional Ising model with plus boundary condition (pure state) [4]. Indeed, in this case the heuristic picture presented in the Introduction corresponds to the diffusion of the Wulff bubble. While the statement (ii) in Theorem 2.1 is proven by exhibiting a suitable slowly varying test function, a proof of a matching lower bound appears considerably harder. The ABC model is however much simpler then short range models and it therefore might be a useful starting point toward the understanding of conservative dynamics in the phase transition region.

We next discuss the behavior of the spectral gap of the ABC process on an interval with zero flux condition at the endpoints. As shown in [1], in such a case the process is reversible with respect to a mean field Gibbs probability for all values of the densities. In the high temperature regime $\beta \ll 1$, the methods here developed can be directly applied to get the diffusive behavior $1/N^2$. As far the low temperature regime is concerned, the case of equal densities is the same as the one on the ring and we can therefore conclude that the lower bound $1/N^3$ holds also in this setting. On the other hand, as proven in [1], in the unequal densities case the free energy has always a unique minimizer, it seems reasonable to expect that on an interval with unequal densities the spectral gap of the ABC process behaves as $1/N^2$ for all values of β.

3. Asymptotic of the Gibbs measure

The upper bound on the spectral gap in the supercritical regime requires the law of large numbers for the empirical density with respect to the Gibbs measure ν_N^β. This result is proven by combining the large deviations principle for ν_N^β with the analysis of the minimizers of the free energy in [1]. As ν_N^β is a Gibbs measure with a mean field interaction, the associated large deviations principle can be proven by standard tools. As the specific application to the ABC model has not however been detailed in the literature, we present here the whole argument.

Empirical density. We let $\mathbf{T} := \mathbf{R}/\mathbf{Z}$ be the one-dimensional torus of side length one; the coordinate on $\mathbf{T}$ is denoted by $r \in [0, 1)$. The inner product in $L^2(\mathbf{T}, dr; \mathbf{R}^3)$ is denoted by $\langle \cdot, \cdot \rangle$. We set $\widetilde{\mathcal{M}} := L^\infty\big(\mathbf{T}, dr; [0, 1]^3\big)$ and denote by $\rho = (\rho_A, \rho_B, \rho_C)$ its elements. We consider $\widetilde{\mathcal{M}}$ endowed with the weak* topology. Namely, a sequence $\{\rho^n\}$ converges to ρ in $\mathcal{M}$ iff $\langle \rho^n, \phi \rangle \to \langle \rho, \phi \rangle$ for any function $\phi \in L^1(\mathbf{T}, dr; \mathbf{R}^3)$, equivalently for any smooth function $\phi \in C^\infty(\mathbf{T}; \mathbf{R}^3)$. Note that $\widetilde{\mathcal{M}}$ is a compact Polish space, i.e. separable, metrizable, and complete.

We introduce

$$\mathcal{M} := \Big\{ \rho \in \widetilde{\mathcal{M}} \,:\, \rho_A + \rho_B + \rho_C = 1\,,\ \int_0^1 dr\, \rho_\alpha(r) = \frac{1}{3}\,,\ \alpha \in \{A,B,C\} \Big\} \tag{3.1}$$

noticing it is a closed subset of $\widetilde{\mathcal{M}}$ that we consider equipped with the relative topology and the associated Borel σ-algebra. The set of Borel probability measures on $\mathcal{M}$, denoted by $\mathcal{P}(\mathcal{M})$, is endowed with the topology induced by the weak convergence of probability measures; namely, $\mathcal{P}_n \to \mathcal{P}$ iff for each continuous function $F : \mathcal{M} \to \mathbf{R}$ we have $\int d\mathcal{P}_n\, F \to \int d\mathcal{P}\, F$. Note that also $\mathcal{P}(\mathcal{M})$ is a compact Polish space.

We define the *empirical density* as the map $\pi_N \colon \Omega_N \to \mathcal{M}$ given by

$$\pi_N(\zeta)\,(r) \;:=\; \sum_{x \in \mathbb{Z}_N} \eta(x)\, \mathbf{1}_{[x/N,(x+1)/N)}(r)\,, \quad r \in \mathbf{T}, \tag{3.2}$$

recall $\eta = \eta(\zeta)$ is the map defined at the beginning of Section 2. We set $\mathcal{P}_N^\beta := \nu_N^\beta \circ \pi_N^{-1}$ namely, $\mathcal{P}_N^\beta$ is the law of π_N when ζ is distributed according to ν_N^β. Note that $\{\mathcal{P}_N^\beta\}$ is a sequence in $\mathcal{P}(\mathcal{M})$.

Large deviations principle. The *entropy* is the convex lower semicontinuous functional $\mathcal{S} \colon \mathcal{M} \to [0, +\infty)$ defined by

$$\mathcal{S}(\rho) := \int_0^1 dr\, \Big[\rho_A(r) \log \frac{\rho_A(r)}{1/3} + \rho_B(r) \log \frac{\rho_B(r)}{1/3} + \rho_C(r) \log \frac{\rho_C(r)}{1/3} \Big] \tag{3.3}$$

and the *energy* is the continuous functional $\mathcal{H} \colon \mathcal{M} \to \mathbf{R}$ defined by

$$\mathcal{H}(\rho) := \int_0^1 dr \int_r^1 dr'\, \Big[\rho_A(r)\rho_C(r') + \rho_B(r)\rho_A(r') + \rho_C(r)\rho_B(r') \Big]. \tag{3.4}$$

For $\beta \ge 0$ the *free energy* is finally the functional $\mathcal{F}_\beta \colon \mathcal{M} \to \mathbf{R}$ defined by

$$\mathcal{F}_\beta := \mathcal{S} + \beta\, \mathcal{H}. \tag{3.5}$$

Theorem 3.1. *The sequence $\{\mathcal{P}_N^\beta\}$ satisfies a large deviation principle with rate function $\mathcal{I}_\beta = \mathcal{F}_\beta - \inf \mathcal{F}_\beta$. Namely, for each closed set $\mathcal{C} \subset \mathcal{M}$ and each open set $\mathcal{O} \subset \mathcal{M}$*

$$\begin{aligned} \varlimsup_{N\to\infty} \frac{1}{N} \log \mathcal{P}_N^\beta(\mathcal{C}) \;&\le\; -\inf_{\rho \in \mathcal{C}} \mathcal{I}_\beta(\rho) \\ \varliminf_{N\to\infty} \frac{1}{N} \log \mathcal{P}_N^\beta(\mathcal{O}) \;&\ge\; -\inf_{\rho \in \mathcal{O}} \mathcal{I}_\beta(\rho). \end{aligned}$$

Since the beautiful Lanford's lectures [16], large deviations principles for Gibbs measures has become a basic topic in equilibrium statistical mechanics, see in particular [11] for the case of mean field interactions. On the other hand, the current setting is not completely standard as we are looking to large deviations of the empirical density for canonical Gibbs measures. We therefore give a detailed proof of the above result. The first step is the large deviations principle when $\beta = 0$; recall that $\mathcal{P}_N^0 = \nu_N^0 \circ \pi_N^{-1}$ is the law of π_N when ζ is distributed according to ν_N^0 which is the uniform probability on Ω_N.

Lemma 3.2. *The sequence $\{\mathcal{P}_N^0\}$ satisfies a large deviation principle with rate function $\mathcal{S}$. Namely, for each closed set $\mathcal{C} \subset \mathcal{M}$ and each open set $\mathcal{O} \subset \mathcal{M}$*

$$\varlimsup_{N\to\infty} \frac{1}{N} \log \mathcal{P}_N^0(\mathcal{C}) \;\le\; -\inf_{\rho\in\mathcal{C}} \mathcal{S}(\rho) \tag{3.6}$$

$$\varliminf_{N\to\infty} \frac{1}{N} \log \mathcal{P}_N^0(\mathcal{O}) \;\ge\; -\inf_{\rho\in\mathcal{O}} \mathcal{S}(\rho). \tag{3.7}$$

Proof. The proof is split is few steps.

Step 1. Set

$$\mathcal{A} := \Big\{ \lambda \in C^1(\mathbf{T};\mathbf{R}^3) : \int_0^1 dr\, \frac{e^{\lambda_\alpha(r)}}{e^{\lambda_A(r)} + e^{\lambda_B(r)} + e^{\lambda_C(r)}} = \frac{1}{3},\, \alpha \in \{A,B,C\} \Big\} \tag{3.8}$$

and let $\Lambda\colon \mathcal{A} \to \mathbf{R}$ be the functional

$$\Lambda(\lambda) := \int_0^1 dr\, \log\Big[\frac{1}{3}\Big(e^{\lambda_A(r)} + e^{\lambda_B(r)} + e^{\lambda_C(r)}\Big)\Big]. \tag{3.9}$$

We shall prove that for each $\lambda \in \mathcal{A}$

$$\lim_{N\to\infty} \frac{1}{N} \log \int d\mathcal{P}_N^0(\rho)\, \exp\big\{N\langle\lambda,\rho\rangle\big\} = \Lambda(\lambda). \tag{3.10}$$

We denote by $\lambda^N(x)$ the average of λ in the interval $\big[x/N, (x+1)/N\big)$,

$$\lambda_\alpha^N(x) := N \int_{\frac{x}{N}}^{\frac{x+1}{N}} dr\, \lambda_\alpha(r)\,, \qquad x \in \mathbf{Z}_N\,,\;\; \alpha \in \{A,B,C\}\,.$$

From the very definition of the measure $\mathcal{P}_N^0$,

$$\begin{aligned} \int d\mathcal{P}_N^0(\rho)\, \exp\big\{N\langle\lambda,\rho\rangle\big\} &= \sum_{\zeta\in\Omega_N} \nu_N^0(\zeta) \exp\big\{N\langle\lambda,\pi^N(\zeta)\rangle\big\} \\ &= \sum_{\zeta\in\Omega_N} \nu_N^0(\zeta) \prod_{x\in\mathbb{Z}_N} \exp\Big\{ \sum_{\alpha\in\{A,B,C\}} \lambda_\alpha^N(x)\eta_\alpha(x)\Big\} \end{aligned} \tag{3.11}$$

We denote by μ_N^λ the product measure on $\widetilde{\Omega}_N = \{A,B,C\}^{\mathbb{Z}_N}$ with marginals $\mu_{N,x}^\lambda$ given by

$$\mu_{N,x}^\lambda(\alpha) = \frac{e^{\lambda_\alpha^N(x)}}{e^{\lambda_A^N(x)} + e^{\lambda_B^N(x)} + e^{\lambda_C^N(x)}}\,, \qquad \alpha \in \{A,B,C\}.$$

When $\lambda = 0$ we drop the superscript λ from the notation so that μ_N is the uniform measure on $\widetilde{\Omega}_N$.

Set $\Xi_N(\lambda) := \prod_{x\in\mathbb{Z}_N} \Big(e^{\lambda_A^N(x)} + e^{\lambda_B^N(x)} + e^{\lambda_C^N(x)}\Big)$. As $\nu_N^0 = \mu_N(\cdot\,|\Omega_N)$, from (3.11) we get

$$\int d\mathcal{P}_N^0(\rho)\, \exp\big\{N\langle\lambda,\rho\rangle\big\} = \frac{\Xi_N(\lambda)}{3^N}\, \frac{\mu_N^\lambda(\Omega_N)}{\mu_N(\Omega_N)}.$$

Since μ_N is the uniform probability on $\widetilde{\Omega}_N$, $\lim_N \frac{1}{N}\log\mu_N(\Omega_N) = 0$. We claim that also $\lim_N \frac{1}{N}\log\mu_N^\lambda(\Omega_N) = 0$. The proof of this step is then completed by observing that $\frac{1}{N}\log\big[\Xi_N(\lambda)/3^N\big] \to \Lambda(\lambda)$.

To prove the claim, we write

$$\mu_N^\lambda\big(\Omega_N\big) = \mu_N^\lambda\Big(\frac{1}{N}\sum_{x\in\mathbb{Z}_N} \eta_\alpha(x) = \frac{1}{3},\, \alpha \in \{A,B,C\}\Big).$$

In view of the smoothness of λ and the constraints in (3.8), for each $\alpha \in \{A, B, C\}$

$$\frac{1}{N}\sum_{x\in\mathbb{Z}_N} \mu_N^\lambda\big(\eta_\alpha(x)\big) = \frac{1}{N}\sum_{x\in\mathbb{Z}_N} \frac{e^{\lambda_\alpha^N(x)}}{e^{\lambda_A^N(x)}+e^{\lambda_B^N(x)}+e^{\lambda_C^N(x)}}$$
$$= \int_0^1 dr\, \frac{e^{\lambda_\alpha(r)}}{e^{\lambda_A(r)}+e^{\lambda_B(r)}+e^{\lambda_C(r)}} + O\Big(\frac{1}{N}\Big) = \frac{1}{3} + O\Big(\frac{1}{N}\Big)$$

The claim now follows from an application of the local central limit theorem for triangular arrays, see e.g. [18, Ch. VII].

Step 2. We here prove the large deviations upper bound (3.6). Given $\lambda \in \mathcal{A}$ let $\mathcal{P}_N^{0,\lambda}$ be the probability on $\mathcal{M}$ defined by

$$d\mathcal{P}_N^{0,\lambda} := \exp\big\{N\big[\langle\lambda,\cdot\rangle - \Lambda_N(\lambda)\big]\big\}\, d\mathcal{P}_N^0 \tag{3.12}$$

where

$$\Lambda_N(\lambda) = \frac{1}{N}\log\int d\mathcal{P}_N^0(\rho)\, e^{N\langle\lambda,\rho\rangle}$$

Given a measurable subset $\mathcal{B}$ of $\mathcal{M}$, we then have

$$\mathcal{P}_N^0(\mathcal{B}) = \int_{\mathcal{B}} d\mathcal{P}_N^{0,\lambda}\, \frac{d\mathcal{P}_N^0}{d\mathcal{P}_N^{0,\lambda}} \;\le\; \sup_{\rho\in\mathcal{B}} \exp\big\{-N\big[\langle\lambda,\rho\rangle - \Lambda_N(\lambda)\big]\big\}$$

In view of Step 1, $\Lambda_N(\lambda) \to \Lambda(\lambda)$ as $N\to\infty$. We thus get

$$\varlimsup_{N\to\infty} \frac{1}{N}\log\mathcal{P}_N^0(\mathcal{B}) \le -\inf_{\rho\in\mathcal{B}}\big\{\langle\lambda,\rho\rangle - \Lambda(\lambda)\big\}\,.$$

By optimizing with respect to $\lambda \in \mathcal{A}$ and using a mini-max lemma, see e.g. Lemmata 3.2 and 3.3 in [15, App. 2], we deduce that for each compact $\mathcal{K}\subset\mathcal{M}$

$$\varlimsup_{N\to\infty} \frac{1}{N}\log\mathcal{P}_N^0(\mathcal{K}) \le -\inf_{\rho\in\mathcal{K}}\sup_{\lambda\in\mathcal{A}}\big\{\langle\lambda,\rho\rangle - \Lambda(\lambda)\big\} = -\inf_{\rho\in\mathcal{K}}\mathcal{S}(\rho).$$

By the compactness of $\mathcal{M}$ this concludes the proof of the upper bound.

Step 3. Given two probability measures P and Q, we denote by $\mathrm{Ent}(Q|P) = \int dQ\, \log[dQ/dP]$ the *relative entropy* of Q with respect to P. A simple computation based on Jensen inequality, see e.g. [14, Prop. 4.1], shows that the large deviations lower bound (3.7) can be deduced from the following statement. For each $\rho\in\mathcal{M}$ there exists a sequence of probability measures $\{\mathcal{Q}_N^\rho\}$ such that

$$\mathcal{Q}_N^\rho \to \delta_\rho \qquad \text{and} \qquad \varlimsup_{N\to\infty}\frac{1}{N}\mathrm{Ent}\big(\mathcal{Q}_N^\rho\big|\mathcal{P}_N^0\big) \le \mathcal{S}(\rho) \tag{3.13}$$

We here construct the sequence $\{\mathcal{Q}_N^\rho\}$ when ρ is continuously differentiable. For such a ρ let $\lambda = \lambda(\rho)$ be such that

$$\rho_\alpha = \frac{e^{\lambda_\alpha}}{e^{\lambda_A}+e^{\lambda_B}+e^{\lambda_C}}\,, \qquad \alpha\in\{A,B,C\}.$$

Observe that $\lambda\in\mathcal{A}$ since ρ is continuously differentiable. Recalling (3.12), we claim that $\{\mathcal{P}_N^{0,\lambda(\rho)}\}$ fulfils the condition (3.13). The law of large numbers $\mathcal{P}_N^{0,\lambda(\rho)} \to \delta_\rho$ can be indeed checked by the same computations of Step 1. Furthermore, in view of such law of large numbers and Step 1,

$$\lim_{N\to\infty}\frac{1}{N}\mathrm{Ent}\big(\mathcal{P}_N^{0,\lambda(\rho)}\big|\mathcal{P}_N^0\big) = \langle\lambda,\rho\rangle - \Lambda(\lambda) = \mathcal{S}(\rho)$$

where the last equality follows from the choice of λ.

Step 4. The proof of the lower bound can be concluded by an approximation argument. Let $\mathcal{M}_\circ$ be the subset of $\mathcal{M}$ given by the continuously differentiable profiles. The condition that a large deviation rate function is lower semicontinuous is not restrictive. More precisely, if a sequence of probabilities satisfies the large deviations lower bound for some rate function, then the lower bound still holds with the lower semicontinuous envelope of such rate function. If we let $\mathcal{S}^\circ$ be the functional equal to $\mathcal{S}$ on $\mathcal{M}_\circ$ and $\mathcal{S}^\circ(\rho) = +\infty$ otherwise, in view of Step 3, the proof of the lower bound (3.7) is concluded if we show that the lower semicontinuous envelope of $\mathcal{S}^\circ$ is $\mathcal{S}$. This amounts to prove that

$$\mathcal{S}(\rho) = \sup_{\mathcal{O}\ni\rho} \inf_{\hat\rho\in\mathcal{O}\cap\mathcal{M}_\circ} \mathcal{S}(\hat\rho)$$

where the first supremum is carried over all the open neighborhoods of ρ. The previous identity is easily proven by considering a sequence of continuously differentiable profiles $\{\rho^n\}$ which converges to ρ a.e. in $\mathbf{T}$. □

In view of the continuity of the functional $\mathcal{H}$ on $\mathcal{M}$, the large deviations principle for the sequence $\{\mathcal{P}_N^\beta\}$ is straightforward consequence of Lemma 3.2 and Laplace-Varadhan theorem.

Proof of Theorem 3.1. Recalling definitions (2.5), (3.2) and (3.4), we claim that for each $\zeta \in \Omega_N$

$$H_N(\zeta) = \mathcal{H}(\pi^N(\zeta)) \,. \tag{3.14}$$

It is indeed enough to notice that by writing explicitly the right hand side above the diagonal terms vanish since $\eta_A(x) + \eta_B(x) + \eta_C(x) = 1$, $x \in \mathbf{Z}_N$.

Recall that ν_N^0 is the uniform probability on Ω_N and let $\mathcal{B}$ be a measurable subset of $\mathcal{M}$. From (3.14) and the definitions of the measures $\mathcal{P}_N^\beta$ and ν_N^β, see (2.6)

$$\begin{aligned}\mathcal{P}_N^\beta(\mathcal{B}) &= \sum_{\substack{\zeta\in\Omega_N\\ \pi^N(\zeta)\in\mathcal{B}}} \nu_N^\beta(\zeta) = \frac{|\Omega_N|}{Z_N^\beta} \sum_{\substack{\zeta\in\Omega_N\\ \pi^N(\zeta)\in\mathcal{B}}} \nu_N^0(\zeta)\, e^{-\beta N H_N(\zeta)} \\ &= \frac{|\Omega_N|}{Z_N^\beta} \int_{\mathcal{B}} d\mathcal{P}_N^0\, e^{-\beta N\, \mathcal{H}(\pi_N)}.\end{aligned}$$

In particular, by taking $\mathcal{B} = \mathcal{M}$,

$$\frac{Z_N^\beta}{|\Omega_N|} = \int d\mathcal{P}_N^0\, e^{-\beta N\, \mathcal{H}(\pi_N)}.$$

Since $\mathcal{M}$ is compact and $\mathcal{H}\colon \mathcal{M} \to \mathbf{R}$ is continuous, by using Lemma 3.2 and Laplace-Varadhan theorem, see e.g. [8, Thm. 4.3.1], we deduce

$$\lim_{N\to\infty} \frac{1}{N} \log \frac{Z_N^\beta}{|\Omega_N|} = \sup_{\rho\in\mathcal{M}} \big\{ -\beta\mathcal{H}(\rho) - \mathcal{S}(\rho)\big\} = -\inf_{\rho\in\mathcal{M}} \mathcal{F}_\beta(\rho).$$

Let $\mathcal{C}$ and $\mathcal{O}$ be respectively a closed and an open subset of $\mathcal{M}$. Again from Lemma 3.2 and Laplace-Varadhan theorem, see e.g. [8, Ex. 4.3.11], we deduce

$$\begin{aligned}\varlimsup_{N\to\infty} \frac{1}{N} \log \int_{\mathcal{C}} d\mathcal{P}_N^0\, e^{-\beta N\, \mathcal{H}(\pi^N)} &\le -\inf_{\rho\in\mathcal{C}} \mathcal{F}_\beta(\rho) \\ \varliminf_{N\to\infty} \frac{1}{N} \log \int_{\mathcal{O}} d\mathcal{P}_N^0\, e^{-\beta N\, \mathcal{H}(\pi^N)} &\ge -\inf_{\rho\in\mathcal{O}} \mathcal{F}_\beta(\rho).\end{aligned}$$

The theorem follows readily. □

Minimizers of the free energy. We here recall the results in [1] concerning the minimizers of the free energy $\mathcal{F}_\beta$ in (3.5) that are needed in our analysis. As discussed in [1], the Euler-Lagrange equation $\delta\mathcal{F}_\beta = 0$ can be, equivalently, written as the system of ordinary differential equation

$$\begin{aligned} \rho'_A &= \beta\,\rho_A(\rho_C - \rho_B) \\ \rho'_B &= \beta\,\rho_B(\rho_A - \rho_C) \\ \rho'_C &= \beta\,\rho_C(\rho_B - \rho_A). \end{aligned} \tag{3.15}$$

Note that the above condition is equivalent to the statement that $\rho = (\rho_A, \rho_B, \rho_C)$ is a stationary solution to the hydrodynamic equation (1.1).

Let $\beta_{\rm c} := 2\pi\sqrt{3}$. In [1] it is proven that for $\beta \in [0, \beta_{\rm c}]$ the unique solution to (3.15) in $\mathcal{M}$ is the constant solution $\bar\rho := \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. On the other hand, when $\beta > \beta_{\rm c}$ there are non trivial solutions. In particular, there exists a unique $\rho \in C^\infty\left(\mathbf{R}; \mathbf{R}^3\right)$ satisfying the following conditions: (i) ρ solves (3.15), (ii) ρ is periodic with period 1, (iii) ρ satisfies the constraints in (3.1) and can therefore by thought as an element in $\mathcal{M}$, (iv) the center of mass of the B species is $1/2$, i.e. $3\int_0^1 dr\, r\rho_B(r) = 1/2$. We shall denote this solution by $\rho^* = \rho^{*,\beta}$. Note that any translation of $\rho^{*,\beta}$ satisfies conditions (i)–(iii) above but not (iv). We emphasize that the condition (ii) requires the minimal period to be one; indeed, as discussed in [1], when $\beta > n\beta_{\rm c}$ for some integer n, there are solutions of (3.15) with period $1/n$.

Given $s \in \mathbf{T}$ we denote by $\tau_s : \mathcal{M} \to \mathcal{M}$ the translation by s, namely $(\tau_s\rho)(r) = \rho(r - s)$. If $\mathcal{P}$ is a probability on $\mathcal{M}$, the corresponding translation is $\mathcal{P} \circ \tau_s^{-1}$. The following statement is a (partial) rewriting of Theorems 4.1 and 5.2 in [1].

Theorem 3.3.

(i) *If $\beta \in [0, \beta_{\rm c}]$ then*

$$\operatorname{arg\,inf} \mathcal{F}_\beta = \{\bar\rho\};$$

namely, the unique minimizer of $\mathcal{F}_\beta$ is $\bar\rho$.

(ii) *If $\beta \in (\beta_{\rm c}, +\infty)$ then*

$$\operatorname{arg\,inf} \mathcal{F}_\beta = \left\{\tau_s\rho^{*,\beta}\,,\ s \in \mathbf{T}\right\};$$

namely, $\mathcal{F}_\beta$ has a one-parameter family of minimizers which are obtained by translating $\rho^{*,\beta}$.

Law of large numbers for the empirical density. As a corollary of the previous statements, we here prove the law of large numbers for the sequence $\{\mathcal{P}_N^\beta\}$. The corresponding limit point charges the set of minimizers of the free energy only. In the supercritical case we show that each $\tau_s\rho^{*,\beta}$, $s \in \mathbf{T}$, is chosen with uniform probability.

Theorem 3.4.

(i) *If $\beta \in [0, \beta_{\rm c}]$ then the sequence $\{\mathcal{P}_N^\beta\}$ converges to $\delta_{\bar\rho}$.*

(ii) *If $\beta \in (\beta_{\rm c}, +\infty)$ then the sequence $\{\mathcal{P}_N^\beta\}$ converges to $\int_0^1 ds\,\delta_{\tau_s\rho^{*,\beta}}$.*

Proof. Item (i) follows immediately from the large deviations principle stated in Theorem 3.1 and the uniqueness of minimizers of $\mathcal{F}_\beta$ stated in item (i) of Theorem 3.3.

To prove item (ii), let $\vartheta : \Omega_N \to \Omega_N$ be the microscopic translation, i.e. $\vartheta\zeta$ is the configuration defined by $(\vartheta\zeta)(x) = \zeta(x - 1)$, $x \in \mathbf{Z}_N$. As follows from definition

(2.6), the probability ν_N^β is translation invariant, i.e. $\nu_N^\beta \circ \vartheta^{-1} = \nu_N^\beta$. This implies that the probability $\mathcal{P}_N^\beta$ is invariant by discrete translations: $\mathcal{P}_N^\beta \circ \tau_{x/N}^{-1} = \mathcal{P}_N^\beta$, $x \in \mathbf{Z}_N$. By the compactness of $\mathcal{M}$, there exists a probability $\mathcal{P} \in \mathcal{P}(\mathcal{M})$ and a subsequence $\{\mathcal{P}_N^\beta\}$ such that $\mathcal{P}_N^\beta \to \mathcal{P}$. We claim that $\mathcal{P}$ is translation invariant. Indeed, fix a continuous function F on $\mathcal{M}$ and $s \in \mathbf{T}$. Observe that, in view of the compactness of $\mathcal{M}$, F is uniformly continuous. Pick now a sequence $\{x_N \in \mathbf{Z}_N\}$ such that $x_N/N \to s$. The uniform continuity of F implies that $\tau_{x_N/N} F$ converges uniformly to $\tau_s F$. Since $\int d\mathcal{P}_N^\beta \, \tau_{x_N/N} F = \int d\mathcal{P}_N^\beta \, F$, by taking the limit $N \to \infty$ we deduce that $\int d\mathcal{P} \, \tau_s F = \int d\mathcal{P} \, F$. In view of the arbitrariness of F we conclude $\mathcal{P} \circ \tau_s^{-1} = \mathcal{P}$. Moreover, Theorem 3.1 and item (ii) in Theorem 3.3 imply that the support of $\mathcal{P}$ is a subset of $\{\tau_s \rho^{*,\beta}, \ s \in \mathbf{T}\} =: \mathcal{T}$. Let now $\phi \colon \mathcal{T} \to \mathbf{T}$ be the bijection defined by $\tau_s \rho^{*,\beta} \mapsto s$ and set $\lambda := \mathcal{P} \circ \phi^{-1}$. Since $\mathcal{P} = \mathcal{P} \circ \tau_s^{-1}$, $s \in \mathbf{T}$, we deduce that λ is a translation invariant probability measure on $\mathbf{T}$. As the Lebesgue measure dr is the unique translation invariant probability measure on $\mathbf{T}$ we deduce $\lambda(dr) = dr$. To proof is now completed by observing that for each continuous $F \colon \mathcal{M} \to \mathbf{R}$ the previous identity imply $\int d\mathcal{P}(\rho) \, F(\rho) = \int_0^1 ds \, F(\tau_s \rho^*)$. □

4. Lower bound on the spectral gap in the subcritical case

In this section we prove the first statement in Theorem 2.1. This result is derived from an analysis of a perturbed interchange process, that is detailed in Appendix A, and a comparison of the corresponding Dirichlet forms. This method has been introduced in [19] and applied in different contexts, see e.g. [3].

We start by defining the ABC process on the complete graph with N vertices. Given a (unoriented) bond $\{x, y\} \subset \mathbf{Z}_N$, $x \neq y$, and a function $f \colon \Omega_N \to \mathbf{R}$, we introduce the gradient

$$\left(\nabla_{x,y} f\right)(\zeta) := f(\zeta^{x,y}) - f(\zeta) \tag{4.1}$$

where, as in (2.1), $\zeta^{x,y}$ denotes the configuration obtained from ζ exchanging the particles in x and y. The ABC dynamics on the complete graph is then defined by the Markov generator

$$\mathcal{L}_N^\beta f := \sum_{\{x,y\} \subset \mathbf{Z}_N} c_{x,y}^\beta \nabla_{x,y} f \tag{4.2}$$

where, recalling (2.5), the jump rates $c_{x,y}^\beta = c_{x,y}^{\beta,N} \colon \Omega_N \to (0, +\infty)$ are given by

$$c_{x,y}^{\beta,N} := \frac{1}{N} \exp\Big\{ -\frac{\beta N}{2} \nabla_{x,y} H_N \Big\}. \tag{4.3}$$

In particular, the above rates satisfy the detailed balance with respect to the probability measure ν_N^β defined in (2.6). Recalling (2.3) and (2.5), we also observe that $N \, c_{x,x+1}^{\beta,N} = c_x^{\beta,N}$.

In Appendix A we prove that, provided β is small enough, the spectral gap of $\mathcal{L}_N^\beta$ is of order one uniformly in N.

Lemma 4.1. *There exist reals $\beta_0, C_1 \in (0, +\infty)$ such that for any $\beta \in [0, \beta_0]$ and any $N \geq 3$*

$$\mathrm{gap}\left(\mathcal{L}_N^\beta\right) \geq \frac{1}{C_1}.$$

We denote by D_N^β and $\mathcal{D}_N^\beta$ the Dirichlet forms associated to the generators L_N^β and $\mathcal{L}_N^\beta$, respectively. That is, given $f\colon \Omega_N \to \mathbf{R}$,

$$D_N^\beta(f) := \nu_N^\beta\big(f\,(-L_N^\beta)f\big) = \frac{1}{2}\sum_{x=1}^{N} \nu_N^\beta\big(c_x^\beta\,\big[\nabla_{x,x+1}f\big]^2\big), \tag{4.4}$$

$$\mathcal{D}_N^\beta(f) := \nu_N^\beta\big(f\,(-\mathcal{L}_N^\beta)f\big) = \frac{1}{2}\sum_{\{x,y\}\subset \mathbf{Z}_N} \nu_N^\beta\big(c_{x,y}^\beta\,\big[\nabla_{x,y}f\big]^2\big). \tag{4.5}$$

Lemma 4.2. *The inequality*

$$\mathcal{D}_N^\beta(f) \le 2\,e^{3\beta}\,N^2\,D_N^\beta(f)$$

holds for any $\beta \ge 0$ and any function $f\colon \Omega_N \to \mathbf{R}$.

Proof. Given $\{x,y\} \subset \mathbf{Z}_N$ we let $T_{x,y}\colon \Omega_N \to \Omega_N$ be the involution defined by $T_{x,y}\zeta := \zeta^{x,y}$. We use the same notation for the corresponding linear map on the set of functions $f\colon \Omega_N \to \mathbf{R}$, i.e. $\big(T_{x,y}f\big)(\zeta) := f(T_{x,y}\zeta) = f\big(\zeta^{x,y}\big)$. As it is simple to check, the long jump $T_{x,y}$ can be decomposed in terms of nearest neighbors jumps as follows

$$T_{x,y} = T_{x+1,x}\,T_{x+2,x+1}\,\cdots\,T_{y-1,y-2}\,T_{y-1,y}\,T_{y-2,y-1}\,\cdots\,T_{x+1,x+2}\,T_{x,x+1}$$

We then write the telescopic sum

$$\begin{aligned}
&T_{x,y}f - f\\
&\quad= \big[T_{x,x+1}\cdots T_{y,y-1}\cdots T_{x+2,x+1}T_{x+1,x}\,f \;-\; T_{x,x+1}\cdots T_{y,y-1}\cdots T_{x+2,x+1}\,f\big]\\
&\quad+ \big[T_{x,x+1}\cdots T_{y,y-1}\cdots T_{x+3,x+2}T_{x+2,x+1}\,f \;-\; T_{x,x+1}\cdots T_{y,y-1}\cdots T_{x+3,x+2}\,f\big]\\
&\quad+ \cdots + \big[T_{x,x+1}\cdots T_{y-2,y-1}T_{y,y-1}\,f \;-\; T_{x,x+1}\cdots T_{y-2,y-1}\,f\big]\\
&\quad+ \cdots + \big[T_{x,x+1}f - f\big]
\end{aligned}$$

Whence

$$\begin{aligned}
\nabla_{x,y}f(\zeta) =& \big(\nabla_{x,x+1}f\big)\big(T_{x+1,x}\cdots T_{y-1,y}\cdots T_{x+1,x+2}\zeta\big)\\
&+ \big(\nabla_{x+1,x+2}f\big)\big(T_{x+1,x}\cdots T_{y-1,y}\cdots T_{x+2,x+3}\zeta\big)\\
&+ \cdots + \big(\nabla_{y-1,y}f\big)\big(T_{x+1,x}\cdots T_{y-2,y-1}\zeta\big) + \cdots + \big(\nabla_{x,x+1}f\big)(\zeta)
\end{aligned} \tag{4.6}$$

In view of (2.5) and (2.6), for any $\beta \in \mathbf{R}_+$, any $z \in \mathbf{Z}_N$, and any positive function $g\colon \Omega_N \to \mathbf{R}_+$

$$\nu_N^\beta\big(T_{z,z+1}g\big) \le \exp\big\{N\beta\|H_N\|_\infty\big\}\,\nu_N^\beta(g) = e^{\beta/N}\,\nu_N^\beta(g) \tag{4.7}$$

By Schwarz inequality in (4.6) and using recursively the previous estimate we then get, for $1 \le x < y \le N$

$$\nu_N^\beta\big(\big[\nabla_{x,y}f\big]^2\big) \le 2\,[2(y-x)-1]\,e^{2\beta}\sum_{z=x}^{y-1}\nu_N^\beta\big(\big[\nabla_{z,z+1}f\big]^2\big) \tag{4.8}$$

Indeed, in the generic term on the right hand side of (4.6) there is the composition of nearest neighbors exchanges $T_{x+k,x+k+1}$ whose number is at most $2(y-x)-2 \le 2N$. In view of (4.7) this yields the factor $e^{2\beta}$. As the number of terms on the right hand side of (4.6) is $2(y-x)-1$ and each bond $\{z,z+1\}$, $z = x,\cdots,y-1$ is used at most two times, the bound (4.8) follows.

To conclude the proof of the lemma it is now enough to observe that the jump rates in the Dirichlet forms (4.4) and (4.5) respectively satisfy the bounds $c_x^\beta \ge$

$e^{-\beta/(2N)}$ and $Nc^{\beta}_{x,y} \le e^{\beta/2}$. In view of (4.8) elementary computations now yield the statement. □

Proof of Theorem 2.1, item (i). Recall the Rayleigh-Ritz variational characterization of the spectral gap (2.7). By Lemmata (4.1) and (4.2) we then deduce the statement with $C_0 = \frac{1}{2}e^{-3\beta}C_1$. □

5. Upper bound on the spectral gap in the supercritical case

We discuss here the upper bound on the spectral gap when $\beta > \beta_c$. In view of the Rayleigh-Ritz variational characterization (2.7), the proof will be achieved by exhibiting a suitable test function. The naive picture is the following. When $\beta > \beta_c$ and N is large, the ABC process essentially performs a random walk on the set of minimizers of the free energy $\mathcal{F}_\beta$, which in the supercritical case is homeomorphic to the one-dimensional torus. We thus choose as test function the one that corresponds to the slow mode of such random walk and conclude the argument.

Proof of Theorem 2.1, item (ii). Pick a Lipschitz function $\phi\colon \mathbf{T} \to \mathbf{R}$ such that $\int_0^1 dr\, \phi(r) = 0$ to be chosen later and let $f_N\colon \Omega_N \to \mathbf{R}$ be the function

$$f_N := \frac{1}{N}\sum_{x=1}^{N} \eta_B(x)\phi\big(\tfrac{x}{N}\big).$$

By the Rayleigh-Ritz principle,

$$\mathrm{gap}(L^{\beta}_N) \le \frac{D^{\beta}_N(f_N, f_N)}{\nu^{\beta}_N(f_N, f_N)} \tag{5.1}$$

where the Dirichlet form D^{β}_N has been defined in (4.4). We next estimate from below the denominator and from above the numerator.

To bound the variance of f_N, we first observe that, since ϕ has mean zero, we have $\lim_N \nu^{\beta}_N(f_N) = \frac{1}{3}\int_0^1 dr\, \phi(r) = 0$. Recall (3.1) and let $F\colon \mathcal{M} \to \mathbf{R}$ be defined by

$$F(\rho) = \int_0^1 dr\, \rho_B(r)\, \phi(r).$$

The continuity of ϕ implies

$$\lim_{N\to\infty} \sup_{\zeta\in\Omega_N} \big| f_N(\zeta) - F\big(\pi_N(\zeta)\big)\big| = 0,$$

where the empirical density $\pi_N\colon \Omega_N \to \mathcal{M}$ has been defined in (3.2). By assumption, $\beta > \beta_c$ and therefore Theorem 3.4, item (ii) implies

$$\lim_{N\to\infty} \nu^{\beta}_N(f_N^2) = \lim_{N\to\infty} \nu^{\beta}_N\big(F(\pi_N)^2\big) = \int_0^1 ds \Big[\int_0^1 dr\, \rho^{*,\beta}_B(r-s)\phi(r)\Big]^2.$$

We can choose ϕ such that the right hand side above is strictly positive. It is indeed enough to observe that, since $\rho^{*,\beta}$ is not constant, there exists a Lipschitz, mean zero, function ϕ such that $\int_0^1 dr\, \rho^{*,\beta}_B(r)\phi(r) \neq 0$. For such choice of ϕ we deduce there exists a constant $C(\beta) \in (0, +\infty)$ such that $\nu^{\beta}_N(f_N, f_N) \ge C(\beta)$ for any $N \ge 3$.

We next bound the Dirichlet form. A straightforward computation yields

$$\nabla_{x,x+1} f_N = \frac{1}{N}\Big[\phi\big(\tfrac{x+1}{N}\big) - \phi\big(\tfrac{x}{N}\big)\Big]\,\big[\eta_B(x) - \eta_B(x+1)\big].$$

Since $c_x \le \exp\{\frac{\beta}{2N}\}$, we then get

$$\mathcal{E}_N^\beta(f_N, f_N) \le \exp\{\tfrac{\beta}{2N}\}\, \frac{1}{2N^2} \sum_{x=1}^N \Big[\phi\big(\tfrac{x+1}{N}\big) - \phi\big(\tfrac{x}{N}\big)\Big]^2.$$

Therefore, letting C_ϕ be the Lipschitz constant of ϕ,

$$\varlimsup_{N\to\infty} N^3\, \mathcal{E}_N^\beta(f_N, f_N) \le \frac{1}{2}\, C_\phi^2$$

which concludes the proof. □

Appendix A. Gap for high temperature exchange process

We prove here a general result on the spectral gap on suitable Markov chains on the set of permutations of $\{1, \dots, N\}$. The jumps of this chain are obtained by randomly choosing a transposition. As reference process we consider the so-called interchange process on $\{1, \dots, N\}$, see [6, 9]. This process can be realized as the simple random walk on the graph with vertex set given by the symmetric group S_N and edges given by the collection of transpositions. Accordingly, the reference invariant measure is the uniform probability on the symmetric group. We then perturb this measure according to the standard Gibbs formalism and consider an associated reversible chain. Under general conditions on the energy, we show that – at high enough temperature – the relaxation time of perturbed chain behaves, for large N, as the one of the reference random walk. The ABC dynamics on the complete graph (4.2) can be realized by looking at the previous chain in a colorblind way, that is resolving only 3 out of of the N colors.

Let $V_N := \{1, \dots, N\}$ and $B_N := \{b \subset V_N : |b| = 2\}$. The complete graph on N vertices is $G_N := (V_N, B_N)$ and $S_N := \{\sigma : V_N \to V_N, \text{bijective}\}$ is the set of permutations on V_N. For any $\sigma \in S_N$, $\{x, y\} \in B_N$ define $\sigma^{\{x,y\}} \in S_N$ as the permutation obtained from σ applying the transposition which exchanges x and y

$$\sigma^{\{x,y\}}(z) := \begin{cases} \sigma(y) & \text{if } z = x \\ \sigma(x) & \text{if } z = y \\ \sigma(z) & \text{otherwise.} \end{cases}$$

Let $E_N : S_N \to \mathbf{R}$ be the energy function and $\beta \ge 0$ the inverse of temperature, we define canonical measure

$$\pi_N(\sigma) = \pi_N^\beta(\sigma) := \frac{1}{Z_N} \exp\{-\beta E_N(\sigma)\}.$$

where $Z_N = Z_N^\beta$ is the partition function. For any $f\colon S_N \to \mathbf{R}$, $a \in B_N$ define $f^a(\sigma) := f(\sigma^a)$, $\nabla_a f := f^a - f$ and the Markov generator

$$\mathcal{G}_N^\beta f = \mathcal{G}_N f := \sum_{a \in B_N} c_a\, \nabla_a f. \tag{A.1}$$

The transition rates are given by

$$c_b = c_{b,N}^\beta := \frac{1}{N} \exp\{-\tfrac{\beta}{2}\, \nabla_b E_N\} \tag{A.2}$$

and satisfy the detailed balance condition with respect to the probability π^N, i.e.,

$$\pi_N\big(c_a g\big) = \pi_N\big(c_a g^a\big). \tag{A.3}$$

The operator $\mathcal{G}_N$ is self-adjoint in $L^2(\pi_N)$ and the corresponding Dirichlet form is

$$\mathcal{E}_N(f,f) = \mathcal{E}_N^\beta(f,f) := -\pi_N(f\mathcal{G}_N f) = \frac{1}{2}\sum_{a\in B_N} \pi_N\big[c_a\,(\nabla_a f)^2\big].$$

We next show that for β small enough the spectral gap of $\mathcal{G}_N^\beta$ is strictly positive uniformly in N.

Theorem A.1. *Assume* $\sup_N \sup_b \|\nabla_b E_N\|_\infty < +\infty$. *Then there exist* $\beta_0, k_0 > 0$ *such that for any* $\beta \in [0,\beta_0]$ *and any* N

$$k_0 \le \operatorname{gap}\big(\mathcal{G}_N^\beta\big) \le \frac{1}{k_0}. \tag{A.4}$$

Proof of the upper bound. In view of the variational characterization (2.7) of the spectral gap $\mathcal{G}_N$, it is enough to exhibit a suitable test function. We next show that by choosing $f(\sigma) = \mathbf{1}_{\{1\}}(\sigma(1))$, the upper bound in (A.4) follows.

The variance of f is clearly $\pi_N(f,f) = \pi_N[\sigma(1)=1]\big(1-\pi_N[\sigma(1)=1]\big)$. To compute the Dirichlet form, we first observe that

$$(\nabla_b f_N)^2(\sigma) = \mathbf{1}_{\{1,y\}}(b)\big[\mathbf{1}_{\{1\}}(\sigma(1)) + \mathbf{1}_{\{1\}}(\sigma(y))\big].$$

Whence, in view of (A.2),

$$\begin{aligned}\mathcal{E}_N(f,f) &= \frac{1}{2}\sum_y \pi_N\Big(c_{\{1,y\}}\big[\mathbf{1}_{\{1\}}(\sigma(1)) + \mathbf{1}_{\{1\}}(\sigma(y))\big]\Big)\\ &\le \exp\Big\{\frac{\beta}{2}\sup_{N,b}\|\nabla_b E_N\|_\infty\Big\}\,\pi_N[\sigma(1)=1].\end{aligned}$$

Therefore

$$\operatorname{gap}(\mathcal{G}_N^\beta) \le \frac{\exp\Big\{\frac{\beta}{2}\sup_{N,b}\|\nabla_b E_N\|_\infty\Big\}}{1-\pi_N[\sigma(1)=1]}.$$

It remains to show that the denominator above is bounded away from 0. We claim that

$$\frac{1}{N}\exp\Big\{-\beta\sup_{N,b}\|\nabla_b E_N\|_\infty\Big\} \le \pi_N[\sigma(1)=1] \le \frac{1}{N}\exp\Big\{\beta\sup_{N,b}\|\nabla_b E_N\|_\infty\Big\}. \tag{A.5}$$

Indeed, fix $k \in V_N$ and observe

$$\begin{aligned}\pi_N[\sigma(1)=1] &= \frac{1}{Z_N}\sum_\sigma \pi_N(\sigma)\mathbf{1}_{\{1\}}(\sigma(1)) = \frac{1}{Z_N}\sum_\sigma \pi_N(\sigma^{\{1,k\}})\mathbf{1}_{\{1\}}(\sigma(k))\\ &= \frac{1}{Z_N}\sum_\sigma \pi_N(\sigma)\frac{\pi_N(\sigma^{\{1,k\}})}{\pi_N(\sigma)}\mathbf{1}_{\{1\}}(\sigma(k)) = \pi_N\Big[e^{-\beta\nabla_{\{1,k\}}E_N}\,\mathbf{1}_{\{1\}}(\sigma(k))\Big].\end{aligned}$$

This yields

$$\exp\Big\{-\beta\sup_{N,b}\|\nabla_b E_N\|_\infty\Big\} \le \frac{\pi_N[\sigma(k)=1]}{\pi_N[\sigma(1)=1]} \le \exp\big\{\beta\sup_{N,b}\|\nabla_b E_N\|_\infty\Big\}.$$

Summing over $k \in V_N$ and observing that $\sum_k \pi_N[\sigma(k)=1] = 1$ we get (A.5). □

Proof of the lower bound. An application of spectral theorem shows, see e.g., [3], that $\mathrm{gap}(\mathcal{G}_N)$ is the largest constant k such that for any $f\colon S_N \to \mathbf{R}$:

$$\pi_N\big[(\mathcal{G}_N f)^2\big] \ge k\,\mathcal{E}_N(f,f) = \frac{k}{2}\sum_{a\in B_N} \pi_N\left[c_a(\nabla_a f)^2\right]. \tag{A.6}$$

To prove the lower bound in (A.4) it is therefore enough there exist a constant k independent of N such that (A.6) holds. We proceed in two steps. We first show that

$$\begin{aligned}&\pi_N\big[(\mathcal{G}_N f)^2\big]\\ &\quad\ge \sum_{\substack{a,b\in B_N\\ a\cap b\neq\emptyset}} \pi_N\big[c_a c_b \nabla_a f \nabla_b f\big] + \frac{1}{2}\sum_{\substack{a,b\in B_N\\ a\cap b=\emptyset}} \pi_N\Big[c_a c_b\Big(1-\frac{c_b^a}{c_b}\Big)\nabla_a f\nabla_b f\Big].\end{aligned} \tag{A.7}$$

Then we prove there exists a constant k independent of N such that

$$\begin{aligned}&\frac{k}{2}\sum_{a\in B_N}\pi_N\big[c_a(\nabla_a f)^2\big]\\ &\quad\le \sum_{\substack{a,b\in B_N\\ a\cap b\neq\emptyset}} \pi_N\big[c_a c_b \nabla_a f \nabla_b f\big] + \frac{1}{2}\sum_{\substack{a,b\in B_N\\ a\cap b=\emptyset}} \pi_N\Big[c_a c_b\Big(1-\frac{c_b^a}{c_b}\Big)\nabla_a f\nabla_b f\Big].\end{aligned} \tag{A.8}$$

While the inequality (A.7) can be obtained as a consequence of Corollary 2.3 and Proposition 2.4 in [3], we next give a direct proof in the present setting. Observe that

$$\begin{aligned}\pi_N\big[(\mathcal{G}_N f)^2\big] &= \sum_{a,b\in B_N}\pi_N\big[c_a c_b\nabla_a f\nabla_b f\big]\\ &= \sum_{a\cap b\neq\emptyset}\pi_N\big[c_a c_b\nabla_a f\nabla_b f\big] + \sum_{a\cap b=\emptyset}\pi_N\big[c_a c_b\nabla_a f\nabla_b f\big].\end{aligned} \tag{A.9}$$

We rewrite last term as

$$\begin{aligned}&\sum_{a\cap b=\emptyset}\pi_N\big[c_a c_b\nabla_a f\nabla_b f\big]\\ &\quad= \frac{1}{2}\sum_{a\cap b=\emptyset}\pi_N\Big[c_a c_b\Big(1-\frac{c_b^a}{c_b}\Big)\nabla_a f\nabla_b f\Big] + \frac{1}{2}\sum_{a\cap b=\emptyset}\pi_N\Big[c_a c_b\Big(1+\frac{c_b^a}{c_b}\Big)\nabla_a f\nabla_b f\Big].\end{aligned}$$

We claim that last term on the right hand side above is positive. This statement together with (A.9) trivially implies (A.7). To prove the previous claim, fix $a,b\in B_N$, with $a\cap b=\emptyset$ and observe that in this case $(\nabla_a f)^a(\nabla_b f)^a = -\nabla_a f\nabla_b f^a$. The detailed balance condition (A.3) now implies

$$\begin{aligned}\pi_N\Big[c_a c_b\Big(1+\frac{c_b^a}{c_b}\Big)\nabla_a f\nabla_b f\Big] &= \pi_N\Big[c_a\Big(c_b+c_b^a\Big)\nabla_a f\nabla_b f\Big]\\ &= \pi_N\big[c_a\big(c_b^a+c_b\big)(\nabla_a f)^a(\nabla_b f)^a\big] = -\pi_N\big[c_a\big(c_b^a+c_b\big)\nabla_a f\nabla_b f^a\big].\end{aligned}$$

Then

$$\begin{aligned}\pi_N\Big[c_a c_b\Big(1+\frac{c_b^a}{c_b}\Big)\nabla_a f\nabla_b f\Big] &= \frac{1}{2}\,\pi_N\Big[c_a\Big(c_b+c_b^a\Big)\big(\nabla_a f\nabla_b f - \nabla_a f\nabla_b f^a\big)\Big]\\ &= -\frac{1}{2}\,\pi_N\big[c_a\big(c_b+c_b^a\big)\nabla_a f\nabla_a\nabla_b f\big],\end{aligned} \tag{A.10}$$

Furthermore, by direct computation,

$$c_a\big(c_b + c_b^a\big) = c_b\big(c_a + c_a^b\big). \tag{A.11}$$

By using (A.10), (A.11), detailed balance (A.3), and (A.11) again we obtain

$$\begin{aligned}
&\pi_N\Big[c_a c_b\Big(1+\frac{c_b^a}{c_b}\Big)\nabla_a f\nabla_b f\Big] \\
&\quad= -\frac{1}{2}\,\pi_N\big[c_a\big(c_b + c_b^a\big)\nabla_a f\nabla_a\nabla_b f\big] = -\frac{1}{2}\,\pi_N\big[c_b\big(c_a + c_a^b\big)\nabla_a f\nabla_a\nabla_b f\big] \\
&\quad= \frac{1}{2}\,\pi_N\big[c_b\big(c_a + c_a^b\big)\nabla_a f^b\nabla_a\nabla_b f\big] = \frac{1}{2}\,\pi_N\big[c_a\big(c_b + c_b^a\big)\nabla_a f^b\nabla_a\nabla_b f\big].
\end{aligned}$$

Averaging the previous equation and (A.10) we obtain

$$\begin{aligned}
\pi_N\Big[c_a c_b\Big(1+\frac{c_b^a}{c_b}\Big)\nabla_a f\nabla_b f\Big] &= \frac{1}{4}\,\pi_N\big[c_a\big(c_b + c_b^a\big)\big(\nabla_a f^b - \nabla_a f\big)\nabla_a\nabla_b f\big] \\
&= \frac{1}{4}\,\pi_N\big[c_a\big(c_b + c_b^a\big)(\nabla_a\nabla_b f)^2\big] \geq 0
\end{aligned}$$

which concludes the proof of the claim.

In order to prove (A.8) we observe that

$$\sum_{a\cap b\neq\emptyset}\pi_N\big[c_a c_b\nabla_a f\nabla_b f\big] = \sum_a \pi_N\big[c_a^2(\nabla_a f)^2\big] + \sum_{\substack{a\cap b\neq\emptyset\\ a\neq b}}\pi_N\big[c_a c_b\nabla_a f\nabla_b f\big]. \tag{A.12}$$

Furthermore, given $a, b \in B_N$ such that $a\cap b\neq\emptyset$ and $a\neq b$ there exists a unique *triangle* T such that $a, b\in T$. A triangle here is an element of

$$\mathcal{T}_N := \big\{\{a,b,c\}\subset B_N : |\{a,b,c\}| = 3, a\cap b\neq\emptyset, a\cap c\neq\emptyset, b\cap c\neq\emptyset\big\}.$$

Therefore

$$\sum_{\substack{a\cap b\neq\emptyset\\ a\neq b}}\pi_N\big[c_a c_b\nabla_a f\nabla_b f\big] = \sum_{T\in\mathcal{T}_N}\sum_{\substack{a,b\in T\\ a\neq b}}\pi_N\big[c_a c_b\nabla_a f\nabla_b f\big].$$

Note that

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_N}\sum_{\substack{a,b\in T\\ a\neq b}}\pi_N\big[c_a c_b\nabla_a f\nabla_b f\big] \\
&\quad= \sum_{T\in\mathcal{T}_N}\sum_{a,b\in T}\pi_N\big[c_a c_b\nabla_a f\nabla_b f\big] - \sum_{T\in\mathcal{T}_N}\sum_{a\in T}\pi_N\big[c_a^2(\nabla_a f)^2\big] \\
&\quad= \sum_{T\in\mathcal{T}_N}\sum_{a,b\in T}\pi_N\big[c_a c_b\nabla_a f\nabla_b f\big] - \sum_a\big|\{T\in\mathcal{T}_N : T\ni a\}\big|\,\pi_N\big[c_a^2(\nabla_a f)^2\big] \\
&\quad= \sum_{T\in\mathcal{T}_N}\sum_{a,b\in T}\pi_N\big[c_a c_b\nabla_a f\nabla_b f\big] - (N-2)\sum_a\pi_N\big[c_a^2(\nabla_a f)^2\big].
\end{aligned}$$

By plugging this result in (A.12) we get

$$\sum_{a\cap b\neq\emptyset}\pi_N\big[c_a c_b\nabla_a f\nabla_b f\big] = \sum_{T\in\mathcal{T}_N}\sum_{a,b\in T}\pi_N\big[c_a c_b\nabla_a f\nabla_b f\big] - (N-3)\sum_a\pi_N\big[c_a^2(\nabla_a f)^2\big].$$

For any $T\in\mathcal{T}_N$ define the set of vertexes of T as $\widetilde{T} := \bigcup_{a\in T} a$. Then

$$\sum_{a,b\in T}\pi_N\big[c_a c_b\nabla_a f\nabla_b f\big] = \pi_N\Big[\sum_{a,b\in T}\pi_N\big[c_a c_b\nabla_a f\nabla_b f\,\big|\,\sigma(z) : z\notin\widetilde{T}\big]\Big]. \tag{A.13}$$

We prove in Lemma A.2 below that there exists a constant $C_1(\beta) > 0$ satisfying $\lim_{\beta\downarrow 0} C_1(\beta) = 1$ such that for any $N \geq 3$, any $f\colon S_N \to \mathbf{R}$, and any $\sigma \in S_N$

$$\frac{N}{3} \sum_{a,b\in T} \pi_N\big[c_a c_b \nabla_a f \nabla_b f \,\big|\, \sigma(z) : z \notin \widetilde{T}\big] \geq \frac{C_1(\beta)}{2} \sum_{a\in T} \pi_N\big[c_a (\nabla_a f)^2 \,\big|\, \sigma(z) : z \notin \widetilde{T}\big].$$

By plugging this bound into (A.13), we deduce

$$\begin{aligned}
&\sum_{a\cap b\neq\emptyset} \pi_N\big[c_a c_b \nabla_a f \nabla_b f\big] \\
&\quad \geq \frac{3C_1(\beta)}{2N} \sum_{T\in\mathcal{T}_N} \sum_{a\in T} \pi_N\big[c_a (\nabla_a f)^2\big] - (N-3) \sum_a \pi_N\big[c_a^2 (\nabla_a f)^2\big] \\
&\quad = \frac{3C_1(\beta)(N-2)}{2N} \sum_a \pi_N\big[c_a (\nabla_a f)^2\big] - (N-3) \sum_a \pi_N\big[c_a^2 (\nabla_a f)^2\big] \\
&\quad \geq \Big(\frac{3C_1(\beta)(N-2)}{2N} - (N-3) \sup_a \|c_a\|_\infty\Big) \sum_a \pi_N\big[c_a (\nabla_a f)^2\big].
\end{aligned} \tag{A.14}$$

In view of (A.2),

$$\sup_a \|c_a\|_\infty \leq \frac{1}{N} \exp\Big\{\frac{\beta}{2} \sup_{N,a} \|\nabla_a E_N\|_\infty\Big\}. \tag{A.15}$$

Recalling the hypotheses $\sup_N \sup_b \|\nabla_b E_N\|_\infty < +\infty$, from (A.14) we then deduce there exists a constant $C_2(\beta) > 0$ satisfying $\lim_{\beta\downarrow 0} C_2(\beta) = 1$ such that

$$\sum_{a\cap b\neq\emptyset} \pi_N\big[c_a c_b \nabla_a f \nabla_b f\big] \geq \frac{C_2(\beta)}{2} \sum_a \pi_N\big[c_a (\nabla_a f)^2\big]. \tag{A.16}$$

To conclude the proof of (A.8) we show that the second term on its right hand side is, for β small enough, of order β. By Schwarz inequality and (A.11)

$$\begin{aligned}
&\Big|\frac{1}{2} \sum_{\substack{a,b\in B_N \\ a\cap b=\emptyset}} \pi_N\Big[c_a c_b \Big(1 - \frac{c_b^a}{c_b}\Big) \nabla_a f \nabla_b f\Big]\Big| \leq \frac{1}{2} \sum_{a\cap b=\emptyset} \pi_N\Big[c_a c_b \Big|1 - \frac{c_b^a}{c_b}\Big| \,|\nabla_a f|\,|\nabla_b f|\Big] \\
&\quad \leq \frac{1}{4}\Big\{ \sum_{a\cap b=\emptyset} \pi_N\Big[c_a c_b \Big|1 - \frac{c_b^a}{c_b}\Big| (\nabla_a f)^2\Big] + \sum_{a\cap b=\emptyset} \pi_N\Big[c_a c_b \Big|1 - \frac{c_b^a}{c_b}\Big| (\nabla_b f)^2\Big]\Big\} \\
&\quad = \frac{1}{2} \sum_{a\cap b=\emptyset} \pi_N\Big[c_a c_b \Big|1 - \frac{c_b^a}{c_b}\Big| (\nabla_a f)^2\Big] = \frac{1}{2} \sum_a \pi_N\Big[c_a (\nabla_a f)^2 \sum_{b\,:\, b\cap a=\emptyset} c_b \Big|1 - \frac{c_b^a}{c_b}\Big|\Big].
\end{aligned}$$

The hypotheses of the theorem implies $\sup_N \sup_{a,b} \|\nabla_a \nabla_b E_N\|_\infty < +\infty$. Recalling (A.15), for β small enough we then have

$$\begin{aligned}
\sum_{b\,:\, b\cap a=\emptyset} c_b \Big|1 - \frac{c_b^a}{c_b}\Big| &\leq \sup_a \|c_a\|_\infty \sum_{b\,:\, b\cap a=\emptyset} \big|1 - e^{-\frac{\beta}{2} \nabla_a \nabla_b E_N}\big| \\
&\leq \beta \sup_a \|c_a\|_\infty \sum_{b\,:\, b\cap a=\emptyset} \big|\nabla_a \nabla_b E_N\big| \leq C_3 \beta
\end{aligned}$$

for some constant C_3 independent of N. Therefore

$$\frac{1}{2} \sum_{\substack{a,b\in B_N \\ a\cap b=\emptyset}} \pi_N\Big[c_a c_b \Big(1 - \frac{c_b^a}{c_b}\Big) \nabla_a f \nabla_b f\Big] \geq -\frac{C_3\beta}{2} \sum_a \pi_N\big[c_a (\nabla_a f)^2\big],$$

which together with (A.16) completes the proof of (A.8). □

Lemma A.2. *Assume* $\sup_N \sup_b \|\nabla_b E_N\|_\infty < +\infty$. *Then there exists a constant* $C_1(\beta)$ *satisfying* $\lim_{\beta\downarrow 0} C_1(\beta) = 1$ *such that for any* $N \geq 3$, *any* $T \in \mathcal{T}_N$, *any* $f\colon S_N \to \mathbf{R}$, *and any* $\sigma \in S_N$

$$\frac{N}{3} \sum_{a,b\in T} \pi_N \big[c_a c_b \nabla_a f \nabla_b f \,\big|\, \sigma(z) : z \notin \widetilde{T}\big] \geq \frac{C_1(\beta)}{2} \sum_{a\in T} \pi_N \big[c_a (\nabla_a f)^2 \,\big|\, \sigma(z) : z \notin \widetilde{T}\big]$$

where we recall $\widetilde{T} := \bigcup_{a\in T} a$.

Proof. The argument relies on two ingredients. The first is that, given a triangle $T \in \mathcal{T}_N$, the conditional probability $\pi_N^\beta[\cdot \,|\, \sigma(z) :\notin \widetilde{T}]$ is, for β small enough, close to the uniform measure. Namely, there exist $C_4(\beta)$ satisfying $\lim_{\beta\downarrow 0} C_4(\beta) = 1$ independent of N, $T \in \mathcal{T}_N$, and $\sigma \in S_N$, such that

$$\frac{1}{C_4(\beta)} \leq \frac{\pi_N^\beta\big[\sigma \,\big|\, \sigma(z) : z \notin \widetilde{T}\big]}{\pi_N^0\big[\sigma \,\big|\, \sigma(z) : z \notin \widetilde{T}\big]} \leq C_4(\beta). \tag{A.17}$$

The second ingredient is that the spectral gap of the interchange process on a graph with 3 vertices is equal to 1. This process can be realized as the random walk on S_3 in which the allowed transition occur, with transition probability $1/3$, along transpositions. This statement readily implies

$$\pi_N^0\big[f, f \,\big|\, \sigma(z) : z \notin \widetilde{T}\big]\big] \leq \frac{1}{6} \sum_{a\in T} \pi_N^0\big[(\nabla_a f)^2 \,\big|\, \sigma(z) : z \notin \widetilde{T}\big]. \tag{A.18}$$

We first show how the lemma follows from these ingredients. Since $\pi_N^0[\cdot \,|\, \sigma(z) : z \notin \widetilde{T}]$ is the uniform measure on a set of cardinality 6, in view of (A.18)

$$\begin{aligned}
\pi_N^\beta\big[f, f \,\big|\, \sigma(z) : z \notin \widetilde{T}\big] &\leq C_4(\beta)^2 \pi_N^0\big[f, f \,\big|\, \sigma(z) : z \notin \widetilde{T}\big] \\
&\leq \frac{C_4(\beta)^2}{6} \sum_{a\in T} \pi_N^0\big[(\nabla_a f)^2 \,\big|\, \sigma(z) : z \notin \widetilde{T}\big] \\
&\leq \frac{1}{3} C_4(\beta)^3 N e^{\frac{\beta}{2} \sup_{N,a} \|\nabla_a E_N\|_\infty} \frac{1}{2} \sum_{a\in T} \pi_N^\beta\big[c_a (\nabla_a f)^2 \,\big|\, \sigma(z) : z \notin \widetilde{T}\big].
\end{aligned}$$

Whence

$$\begin{aligned}
&\sum_{a,b\in T} \pi_N^\beta\big[c_a c_b \nabla_a f \nabla_b f \,\big|\, \sigma(z) : z \notin \widetilde{T}\big] \\
&\qquad \geq \frac{3}{C_4(\beta)^3\, N} \exp\big\{ -\tfrac{\beta}{2} \sup_{N,a} \|\nabla_a E_N\|_\infty \big\} \frac{1}{2} \sum_{a\in T} \pi_N^\beta\big[c_a (\nabla_a f)^2 \,\big|\, \sigma(z) : z \notin \widetilde{T}\big]
\end{aligned}$$

which, for a suitable $C_1(\beta)$, is the thesis of the lemma.

The estimate (A.17) follows from assumption and standard arguments. Firstly note that for any $a \in T$

$$\frac{\pi_N^\beta\big[\sigma^a \,\big|\, \sigma(z) : z \notin \widetilde{T}\big]}{\pi_N^\beta\big[\sigma \,\big|\, \sigma(z) : z \notin \widetilde{T}\big]} = \exp\big\{ -\beta\, \nabla_a E_N(\sigma) \big\},$$

Therefore, by observing that any two given permutations in S_3 can be connected at most by two transpositions, a telescopic argument yields that for any $\sigma, \sigma' \in S_N$

such that $\sigma(z) = \sigma'(z)$ for $z \notin \widetilde{T}$

$$\exp\big\{ -2\beta \sup_{N,a} \|\nabla_a E_N\|_\infty \big\} \le \frac{\pi_N^\beta\big[\sigma' \,\big|\, \sigma'(z) : z \notin \widetilde{T}\big]}{\pi_N^\beta\big[\sigma \,\big|\, \sigma(z) : z \notin \widetilde{T}\big]} \le \exp\big\{2\beta \sup_{N,a} \|\nabla_a E_N\|_\infty\big\}.$$

By averaging the above inequality over σ' the bound (A.17) follows.

The spectral gap of the interchange process on $\{1,2,3\}$ can be obtained from the general results in [6, 9]. An elementary proof can however also be obtained by writing out the 6×6 matrix corresponding to the generator and computing its eigenvalues, as in the example at page 50 of [10]. We order the 6 permutations of S_3 as $\binom{123}{123}$, $\binom{123}{132}$, $\binom{123}{231}$, $\binom{123}{213}$, $\binom{123}{312}$, and $\binom{123}{321}$. With this choice, the generator of the interchange process is written as the matrix

$$\begin{pmatrix} -1 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & -1 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & -1 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & -1 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & -1 & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & -1 \end{pmatrix}$$

whose eigenvalues are 0 (simple), -1 (with multiplicity four), and -2 (simple). □

We finally show, as a corollary of the previous result, that the spectral gap of the ABC dynamics on the complete graph is of order one.

Proof of Lemma 4.1. Fix N multiple of three and let $\chi_N \colon S_N \to \Omega_N$ be the projection defined by

$$\chi_N \sigma\,(x) := \begin{cases} A & \text{if } \sigma(x) = 1 \mod 3, \\ B & \text{if } \sigma(x) = 2 \mod 3, \\ C & \text{if } \sigma(x) = 3 \mod 3. \end{cases}$$

Namely, χ_N resolves only three out of the original N colors. Recalling (2.5), let $E_N \colon S_N \to \mathbf{R}$ be defined by $E_N := H_N \circ \chi_N$. For this choice the ABC dynamics with long jumps , i.e. the process with the generator (4.2), can be realized as the χ_N-projection of the process with generator (A.1). In particular, $\nu_N^\beta = \pi_N^\beta \circ \chi_N^{-1}$ and $\mathrm{gap}\,(\mathcal{L}_N^\beta) \ge \mathrm{gap}\,(\mathcal{G}_N^\beta)$. Since $H_N \circ \chi_N$ satisfies the hypotheses in Theorem A.1, the statement follows. □

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