Stochastic Mechanics 6 CFU
Part I 9.7.2010

Exercise 1
a If $\mathcal{F}$ is a σ-algebra, show that if A, $B \in \mathcal{F}$ then their symmetric difference $A \triangle B$ also belongs to $\mathcal{F}$.
b Show that $P(A \triangle B) = P(A) + P(B) - 2P(A \cap B)$

Exercise 2
a Two dice are rolled. Let X be the larger of he two numbers shown. Compute $P(\{X \in [2, 4]\})$.
b Let $\Omega = \{-4, -2, 0, 1, 2, 3\}$ find the smallest σ-algebra such that $X = \omega + 1$ is a random variable.

Exercise 3
a Show that if $A \cap B = \emptyset$ and $P(C) \neq 0$, then $P(A \cup B|C) = P(A|C) + P(B|C)$;
b A die is rolled twice; if the sum of outcomes is even you win 1 euro, if it is odd you lose 1 euro. X is the amount won or lost and Y is the outcome of the first roll. Find $E(X|Y)$, find and compare the σ-algebra generated by Y and by $E(X|Y)$. How many elements do these σ-algebras have?

Exercise 4
Find the characteristic function $\phi_X(t)$ of a random variable X having distribution with density

$$f_X(x, \theta) = \frac{1}{2} e^{-|x-\theta|} \ \ x \in \mathbb{R}, \ \theta \in \mathbb{R}$$

and find the value of θ such that the caracterisc function is a real function.

Exercise 5
a Give the definition of Brownian motion.
b Let W_t, $\hat{W}_t$ and $\tilde{W}_t$ be three independent Brownian motions, define $X_t = \alpha W_{a^2 t} + \beta \hat{W}_{b^2 t} + \gamma \tilde{W}_{c^2 t}$. Find a condition on α, β, γ, a, b, $c \in \mathbb{R}$ such that X_t is a Brownian motion.

Exercise 6
Let W_t be a Wiener process. Apply the property of Ito itegral that $E(\int_0^T G dW_s)^2 = \int_0^T E(G^2) ds$ for any function $G \in L^2(0, T)$ to calculate

$$E(\int_0^T W_s \, dW_s)^2$$