Stochastic Mechanics 6 CFU
Part I 13.4.2011

Exercise 1
Let Ω and $\tilde{\Omega}$ be arbitrary sets and let $X\colon \tilde{\Omega} \to \Omega$ be any function. Show that if $\mathcal{F}$ is a σ-algebra on Ω then $\tilde{\mathcal{F}} = \{X^{-1}(A)\colon A \in \mathcal{F}\}$ is a σ-algebra on $\tilde{\Omega}$.

Exercise 2
a Show that if X is constant function, then it is a random variable with respect to any σ-algebra.
b Let $\Omega = \{1, 2, 3, 4\}$ and $\mathcal{F} = \{\emptyset, \Omega, \{1\}, \{2, 3, 4\}\}$. Is $X(\omega) = 1 + \omega$ a random variable with respect to the σ-algebra $\mathcal{F}$? If not give an example of a non-costant function which is.

Exercise 3
a Show that if A, B, and C are pairwise independent events and $P(B \cap C) \neq 0$, then the following conditions are equivalent (i) $P(A) = P(A|B \cap C)$; (ii) $P(A \cap B \cap C) = P(A)P(B)P(C)$.
b A fair die is rolled twice. What is the conditional expectation of the sum of outcomes given the first roll shows 1?

Exercise 4
Find the characteristic function $\phi_X(t)$ of a random variable X with uniform distribution over the interval $[-1, 1]$. Shetch the graph of this characteristic function. Calculate $E(X^3)$ (facultative).

Exercise 5
a Give the definition of a 3 dimensional Brownian motion.
b Let W_t and $\hat{W}_t$ be two independent Brownian motions and define $X_t = aW_t + \frac{b}{c}\hat{W}_{c^2 t}$. Find $a, b, c \in \mathbb{R}$ and $c \neq 0$, such that X_t is a Brownian motion.

Exercise 6
Let W_t be a Wiener process. Calculate

$$E\left(\int_0^T [W_s^{\frac{5}{2}} + W_s^{\frac{1}{2}}]dW_s\right)^2$$