Stochastic Mechanics 6 CFU
Part I 14.6.2011

Exercise 1
Let $\mathcal{F}$ be a σ-algebra.
a Give the definition of the *atoms* of $\mathcal{F}$.
b Show that different atoms must be disjoint.

Exercise 2
a Let X be the number of tosses of a fair coin up to and including the first toss showing heads. Find $P(X \in 2\mathbb{N})$, where $2\mathbb{N} = \{2n \colon n = 1, 2, \cdots\}$ is the set of even integers. (*Hint*: $\sum_{n=0}^{\infty} a^n = (1-a)^{-1}$ for $0 < a < 1$).
b Find the smallest σ-algebra on $\Omega = \{-3, -2, -1, 0, 1, 2, 3\}$ such that the following function are random variables: $X(\omega) = |\omega|$, $Y(\omega) = 2\omega$.

Exercise 3
a Show that if $A = B$ and A, B are independent (that is A is independent of itself), then $P(A) = 0 \,\text{or}\, 1$.
b A coin is tossed three times behind a curtain and you win an amount X equal to the number of heads shown. If, peeping behind the curtain, you could see the result Y of the first toss, how would you asses your expected winning?

Exercise 4
Let $X_j \sim \mathcal{N}(m_j, \sigma_j^2)$ $j = 1, \cdots, n$ be independent gaussian random variables. Show that $Y_n = \sum_{j=1}^{n} X_j$ is a gaussian random variable and calculate its expected value and its variance.

Exercise 5
a Give the definition of a 1 dimensional Brownian motion.
b Let W_t a standard Brownian motion and define $X_t = W_{t+s} - W_s$ for $s > 0$. Prove that X_t is a Brownian motion.

Exercise 6
Let W_t be a Wiener process.
a Give the definition of Ito stochastic integral.
b Calculate

$$E(\int_0^T 2\, dW_s \int_0^T s\, dW_s)$$

(*Hint*: use the definition of the stochastic integral and the properties of Brownian motion)