Stochastic Mechanics 6 CFU
Part I 15.7.2009

Exercise 1
a Which of the following families of sets is a field:

$$\begin{aligned}\mathcal{F}_1 &= \{\emptyset,\ \{a\},\ \{b\},\ \{a,c\},\ \{a,b,c\}\}\\ \mathcal{F}_2 &= \{\emptyset,\ [0,1],\ (\frac{1}{4},1],\ [0,\frac{1}{4}],\ [0,\frac{1}{4}),\ \{\frac{1}{4}\},\ [\frac{1}{4},1],\ [0,1]\setminus\{\frac{1}{4}\}\}\\ \mathcal{F}_3 &= \{\emptyset,\{1\},\ \{2\},\ \{3\},\{1,2\},\{2,3\},\ \{2,3,4\}\ \{1,2,3\}\}\end{aligned}$$

b What are the atoms in the following σ-algebra:

$$\mathcal{F} = \{\emptyset,\{3\},\ \{4\},\ \{1,2\},\ \{3,4\},\ \{2,3,4\},\{1,3,4\},\{1,2,3,4\}\}$$

Exercise 2
Let $\Omega = [0,1]$ with the σ-algebra $\mathcal{F}$ of Borel sets B contained in $[0,1]$.
a Is $X(x) = x$ a random variable on Ω with respect to $\mathcal{F}$?
b Is $Y(x) = |x - \frac{1}{3}|$ a random variable on Ω with respect to $\mathcal{F}$?

Exercise 3
a Let $\Omega = H_1 \cup H_2$ and $H_1 \cap H_2 = \emptyset$. Assuming that $P(A|H_1)$, $P(A|H_2)$, $P(H_1), P(H_2) \neq 0$ are known, find an expression for $P(A)$
b From a bag containing four balls numbered $1, 2, 3, 4$ you draw two. If at least one of the numbers drawn is greater than 2, you win 1 euro, otherwise you lose 1 euro. Let X be the amount won or lost and let Y the first number drawn. Find an explicit formula for $E(X|Y)$.

Exercise 4
Let X and Y be random variables such that $Y = aX+b$ where $a, b \in \mathbb{R}$. Show that

$$\phi_Y(t) = e^{itb}\phi_X(at)$$

Exercise 5
a Give the definition of Ito integral and discuss an example.
b Let W_t^1 and W_t^2 be two independent Brownian motions, define $X_t = aW_t^1 + bW_{2t}^2$. Find $a, b \in \mathbb{R}$ such that X_t is a Brownian motion.

Exercise 6
Let W_t be a Wiener process. Apply the property of Ito itegral that $E(\int_0^T G dW_s)^2 = \int_0^T E(G^2)ds$ for any function $G \in L^2(0,T)$ to calculate

$$E(\int_0^T e^{W_s - s} dW_s)^2$$