Stochastic Mechanics 6 CFU
Part I 22.6.2010

Exercise 1
Find two σ-algebra such that their union is not a σ algebra.

Exercise 2
a Give the definition of a simple function.
b Give an example of a simple function X and construct the σ-algebra generated by X.
c Let $\Omega = \{-4, -2, 0, 1, 2, 3\}$ find the smallest σ-algebra such that $X = 16 - \omega^2$ is a random variable.

Exercise 3
a Let A, B, C be independent events. Show that B and $A \cap C$ are independent;
b Three coins, 5, 10 and 20 cents, are tossed and you can take those showing heads; X is the total amount you take and Y is the number of heads. Find $E(X|Y)$, find and compare the σ-algebra generated by Y and by $E(X|Y)$. How many elements do these σ-algebras have?

Exercise 4
Find the characteristic function $\phi_X(t)$ of a random variable X having distribution with density

$$f_X(x) = \begin{cases} 1 - |x| & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Exercise 5
a Give the definition of a 2 dimensional Brownian motion.
b Let W_t and $\hat{W}_t$ be two independent Brownian motions, define $X_t = \frac{a}{b} W_{b^2 t} + \frac{1}{c} \hat{W}_{c^2 t}$. Find $a, b, c \in \mathbb{R}$ and $b, c \neq 0$, such that X_t is a Brownian motion.

Exercise 6
Let W_t be a Wiener process. Apply the property of Ito itegral that $E(\int_0^T G dW_s)^2 = \int_0^T E(G^2) ds$ for any function $G \in L^2(o, T)$ to calculate

$$E(\int_0^T [W_s^{\frac{3}{2}} + W_s^{\frac{1}{2}}] dW_s)^2$$