Stochastic Mechanics 6 CFU
Part I 26.6.2009

Exercise 1
a Let $\Omega = [0,1]$. Tell if any of the following families is a σ-algebra (explain your answer):

$$\begin{aligned}
\mathcal{F}_1 &= \{\emptyset,\, (0,1),\, (0,\frac{1}{2}),\, (\frac{1}{2},1)\} \\
\mathcal{F}_2 &= \{\emptyset,\, (0,1),\, (0,\frac{1}{2}),\, [\frac{1}{2},1),\, (0,\frac{2}{3}],\, (\frac{2}{3},1)\} \\
\mathcal{F}_3 &= \{\emptyset,\, (0,1),\, (0,\frac{2}{3}),\, [\frac{2}{3},1)\}
\end{aligned}$$

b What are the atoms in the following σ-algebra: $\mathcal{F} = \{\emptyset,\, (0,1)\, (0,\frac{1}{3}),\, [\frac{1}{3},1)\}$

Exercise 2
a Let $\Omega = \{-1,-2,-3,-4\}$ and $\mathcal{F} = \{\emptyset, \Omega, \{-2\}, \{-1,-3,-4\}\}$. Is $X(\omega) = 3 - 2\omega$ a random variable with respect to the σ-field $\mathcal{F}$? If not give an example of a non-costant function which is a random variable with respect to $\mathcal{F}$.
b Find the smallest σ-algebras on $\Omega = \{-3,-1,0,1,3\}$ such that the following functions are a random variables: $X_1(\omega) = \omega^3, \quad X_2(\omega) = 2\omega^2$

Exercise 3
a Show that for any events A and B, the following conditions are equivalent: (i) A and B are independent, (ii) $\Omega \setminus A$ and $\Omega \setminus B$ are independent,
b A bag contains 3 coins, but one has heads on both sides. Two coins are drawn and tossed. Find $E(X|Y)$ if X is the number of heads and and Y is the number of genuine coins among tha drawn ones.

Exercise 4
Consider n independent random variables $X_1, \cdots, X_n$, all having the same normal distribution with mean μ and variance σ^2, then calculate the characteristic function of the random variable $X = X_1 + \cdots + X_n$.

Exercise 5
a Give the definition of Ito integral and discuss an example.
b Let W_t be a Brownian motion, define $X_t = \frac{a}{\sqrt{b}} W_{b^2 t}$. Find $a, b \in \mathbb{R}$, $b > 0$ such that X_t is a Brownian motion.

Exercise 6
Let W_t be a Wiener process. Apply the property of Ito itegral that $E(\int_0^T G dW_s)^2 = \int_0^T E(G^2) ds$ for any function $G \in L^2(o,T)$ to calculate

$$E(\int_0^T W_s dW_s)^2$$