Stochastic Mechanics 6 CFU
Part I 27.4.2009

Exercise 1
a Let $\Omega = \{1, 2, 3\}$. Complete $\{\{2\}, \{3\}\}$ to obtain a σ-algebra. Add as few sets as possible.
b What are the atoms in the following σ-algebra:

$$F_1 = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{3,4\}, \{2,3,4\}, \{1,3,4\}, \{1,2,3,4\}\}$$

Exercise 2
a Give the definition of random variable.
b Find the smallest σ-algebras on $\Omega = \{-2, -1, 0, 1, 2\}$ such that the following function is a random variable:

$$X(\omega) = |\omega|$$

c What is the smallest number of elements of a σ-algebra if a function $X : \Omega \to \mathbb{R}$ taking exactly n different values is to be a random variable with respect to this σ-field?

Exercise 3
a Let $\Omega = \{1, 2, 3, 4\}$ with uniform probability and let $A = \{1, 2\}$. List all $B \subset \Omega$ such that A and B are independent.
b A die is rolled twice; if the sum of outcomes is even, you win 1 euro, and if it is odd, you lose 1 euro; X is the amount won or lost and Y is the outcome of the first roll. Find an explicit formula for $E(X|Y)$. Find and compare the σ-algebras generated by the random variables Y and $E(X|Y)$.

Exercise 4
Let X be a Bernoulli variable of parameter $p = \frac{1}{3}$, ($P(X = 1) = p$ and $P(X = 0) = 1 - p$) and Y a normal random variable $Y \sim N(0, 1)$. X and Y are independent. Calculate the characteristic function of $Z = 2X + Y$.

Exercise 5
a Give the definition of Brownian motion.
b Let W_t and $\hat{W}_t$ be two independent Brownian motions, define $X_t = aW_t + b\hat{W}_t$. Find $a, b \in \mathbb{R}$ such that X_t is a Brownian motion.

Exercise 6
Give the definition of Ito integral for a step function.