Stochastic Mechanics 6 CFU

Part I 28.6.2011

Exercise 1

a If $\mathcal{F}$ is a σ-algebra, show that if A, $B \in \mathcal{F}$ then their difference $A \setminus B$ also belongs to $\mathcal{F}$.

b Show that $P(A \setminus B) = P(A) - P(A \cap B)$

Exercise 2

a Two dice are rolled. Let X be the smaller of the two numbers shown. Compute $P(\{X \in [3,5]\})$.

b Let $\Omega = \{-2, -1, 0, 2, 3\}$ find the smallest σ-algebra such that $X = \omega^2 + 1$ is a random variable.

Exercise 3

a Show that if A, $B \subset \Omega$, then $P(B) = P(B|A)P(A) + P(B|A^c)P(A^c)$;

b A die is rolled twice; if the sum of outcomes is > 5 you win 1 euro, if it is ≤ 5 you lose 1 euro. X is the amount won or lost and Y is the outcome of the first roll. Find $E(X|Y)$, find and compare the σ-algebra generated by Y and by $E(X|Y)$. How many elements do these σ-algebras have?

Exercise 4

Find the characteristic function $\phi_X(t)$ of a random variable X having distribution with density

$$f_X(x) = \begin{cases} \frac{1}{2}x & x \in [0,2] \\ 0 & \text{otherwise} \end{cases}$$

Exercise 5

Let W_t be a Brownian motion and $T \in \mathbb{R}$, define

$$X_t = \begin{cases} W_t & t \leq T \\ 2W_T - W_t & t > T \end{cases}$$

a Prove that X_t is a Brownian motion.

b Calculate $\int_0^t X_s^2 dX_s$.

Exercise 6

Let W_t be a Wiener process. Apply the property of Ito itegral that $E(\int_0^T G dW_s)^2 = \int_0^T E(G^2) ds$ for any function $G \in L^2(0,T)$ to calculate

$$E(\int_0^T e^{2s - W_s} dW_s)^2$$