Stochastic Mechanics 6 CFU
Part II 9.7.2010

Exercise 1 Let W_t be a Brownian motion. Illustrate Ito's formula and use it to calculate
a $d(X_t^{2n})$ for $n \geq 1$, where $dX_t = dt + (t+1)dW_t$
b $dW_t^{2n} + dW_t^{2n+1}$ for $n \geq 1$
c $\int_0^T [W_t^2 + W_t]\, dW_t$

Exercise 2 Given the non linear SDE

$$\begin{cases} dX_t & = [-\frac{1}{2}\, e^{2X_t} + 2]dt + dW_t \\ X_0 = 1 \end{cases}$$

a find the value of $c \in \mathbb{R}$ such that using the substitution $y = h(x) = e^{-cx}$ it becomes a linear SDE;
b find the solution.

Exercise 3 Given the SDE

$$dX_t = (1 + \beta X_t)dt + \gamma dW_t$$

with $X_{t=0} = X_0 \sim \mathcal{N}(0, \sigma^2)$ and $\sigma, \beta, \gamma \in \mathbb{R}$, $\sigma, \beta, \gamma \neq 0$.
a find the solution;
b calculate $E(X_t)$ and $\text{Var}(X_t)$ and study their behavior when $t \to \infty$.
c find a condition on the parameters which assures that the $\text{Var}(X_t)$ is independent of time.

Exercise 4 Use Feynman-Kac formula to solve the following PDE with final condition, $x \in \mathbb{R}$ and $t \in [0, T]$,

$$\begin{cases} \frac{\partial f}{\partial t} - x\frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial x^2} = t^2 f \\ f(x, T) = e^x \end{cases}$$

Exercise 5 Illustrate and discuss the Chapman-Kolmogorov equation.