Stochastic Mechanics 6 CFU
Part II 12.6.2012

Exercise 1 Let W_t be a Brownian motion.
a llustrate Ito's formula and use it to calculate $d(e^{\sigma W_t^2})$
b Let $H \in L^2(0,T)$ be a step process with partition $P = \{0 = t_0 < t_1 < \cdots < t_n = T\}$. Define the (Ito) Riemann sums of H with respect to the Brownian motion W_t for this partition P, by

$$S(H,P) = \sum_{i=0}^{n-1} H_{t_i}(W_{t_{i+1}} - W_{t_i})$$

Calculate the mean and the variance of $S(H,P)$.

Exercise 2
a Let $g = g(y)$ be a given function of y, and suppose that $y = f(w)$ is a solution of the ODE

$$dy = g(y)dw,$$

that is, $f'(w) = g(f(w))$. Show that $X_t = f(W_t)$ then is a solution of the SDE:

$$dX_t = \frac{1}{2}g(X_t)g'(X_t)dt + g(X_t)dW_t$$

b Use point **a** above to solve the SDE

$$dX_t = \frac{\sigma^2}{4}dt + \sigma\sqrt{X_t}dW_t$$

with $X_0 \geq 0$. (*Hint*: Take $g(x) = \sigma\sqrt{x}$).

Exercise 3 Consider the boundary value problem for the backwards heat equation with a drift term (a being a constant):

$$\begin{cases} \frac{\partial u}{\partial s} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x} = 0, \quad x \in \mathbb{R},\ s < T \\ u(x,T) = e^{2x} \end{cases}$$

a Write the solution of the associated SDE and calculate its transition probability density.
b Use Feynman-Kac formula to solve it.

Exercise 4 Given the SDE

$$\begin{cases} dX_t = (2 - X_t)dt + \sqrt{2}dW_t \\ X_0 = 1 \end{cases}$$

a Solve the equation and calculate $E(X_t)$ and $Var(X_t)$.
b Calculate the transition probability density of X_t.
c Write the Kolmogorov forward equation for the process and find its stationary solution.