Stochastic Mechanics 6 CFU
Part II 14.6.2011

Exercise 1 Let W_t be a Brownian motion.
a Illustrate Ito's formula and use it to calculate the differential of the process $X_t = W_t e^{2W_t}$.
b Calculate $\int_0^T e^{W_s} dW_s$.

Exercise 2 Let W_t be a Brownian motion and define $W_{\tau(t)} = W_{e^{2t}} - W_1$,
a verify that $W_{\tau(t)}$ is a Brownian motion.
Given the linear SDE

$$\begin{cases} dX_t = -X_t dt + e^{-t} dW_{\tau(t)} \\ X_{t=0} = X_0 \end{cases}$$

where $X_0 \in \mathbb{R}$.
b find the solution;
c calculate $E(X_t)$ and $\mathrm{Var}(X_t)$, and their behavior when $t \to \infty$;
d calculate the transition probability density of X_t.

Exercise 3 Given the SDE

$$\begin{cases} dX_t = (1 - X_t)dt + \sqrt{2} dW_t \\ X_{t=0} = X_0 \end{cases}$$

a find the solution;
b calculate $E(X_t)$ and $\mathrm{Var}(X_t)$ and study their behavior when $t \to \infty$;
c calculate the probability density function of the process;
d write the Fokker-Planck equation and find the stationary solution.

Exercise 4 Use Feynman-Kac formula to solve the following PDE with final condition, $x \in \mathbb{R}$ and $t \in [0, T]$,

$$\begin{cases} \frac{\partial f}{\partial t} - x\frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial x^2} = (2t^2 + 1) f \\ f(x, T) = e^{2x} \end{cases}$$

Exercise 5 Give the definition of *diffusion process* and discuss an example.