Stochastic Mechanics 6 CFU
Part II 28.6.2011

Exercise 1 Let W_t be a Brownian motion. Illustrate Ito's formula and use it to calculate
a $d(W_t e^{W_t})$
b $\int_0^T [e^{-W_t} + W_t]\, dW_t$

Exercise 2 Given the non linear SDE

$$\begin{cases} dX_t &= -\alpha\, e^{-4X_t} dt + 2dW_t \\ X_0 &= 1 \end{cases}$$

a find the value of α, $c \in \mathbb{R}$ such that using the substitution $y = h(x) = e^{-cx}$ it becomes a linear SDE;
b find the solution.

Exercise 3 Given the SDE

$$dX_t = X_t(\beta dt + \gamma dW_t)$$

with $X_0 \sim \mathcal{N}(0, \sigma^2)$ independent of and $\sigma, \beta, \gamma \in \mathbb{R}$, $\sigma, \beta, \gamma \neq 0$.
a find the solution;
b calculate $E(X_t)$ and $\text{Var}(X_t)$ and study their behavior when $t \to \infty$.

Exercise 4 Given the following PDE with final condition, $x \in \mathbb{R}$ and $t \in [0, T]$,

$$\begin{cases} \frac{\partial f}{\partial t} - x\frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial x^2} = t^3 f \\ f(x, T) = e^{x+1} \end{cases}$$

a Write the solution of the associated SDE and calculate its transition density probability.
b Use Feynman-Kac formula to solve it.

Exercise 5 Consider a Brownian motion W_t.
a Write its transition probability density.
b Verify it satisfies the Kolmogorov forward equation.