Stochastic Mechanics 6 CFU
Part I 2.4.2012

Exercise 1
Let $\mathcal{F}$ be a σ algebra on Ω and let $A \subset \Omega$. Show that

a If $\mathcal{F}$ is a σ-algebra on Ω then $\tilde{\mathcal{F}} = \{A \cap B \colon B \in \mathcal{F}\}$ is a σ-algebra on A.
b $B \to P(B|A)$ is a probability measure on $\tilde{\mathcal{F}}$

Exercise 2
a Show that if X is a random variable with respect to a σ-algebra $\mathcal{F}$ and $\mathcal{F} \subset \mathcal{G}$ for some σ-algebra $\mathcal{G}$, then X is a random variable with respect to $\mathcal{G}$.
b Let $\Omega = \{1, 2, 3, 4, 5\}$ and $\mathcal{F} = \{\emptyset, \Omega, \{1\}, \{2, 3, 4\}\}$. Is $X(\omega) = 2\omega$ a random variable with respect to the σ-algebra $\mathcal{F}$? If not give an example of a non-costant function which is a random variable w.r.t. $\mathcal{F}$.

Exercise 3
a A fair coin is tossed three times and the following events are considered:
A= toss 1 and toss 2 produce different outcomes
B= toss 2 and toss 3 produce different outcomes
C= toss 3 and toss 1 produce different outcomes

Show that $P(A) = P(A|B) = P(A|C)$ but $P(A) \neq P(A|B \cap C)$. Comment the result.
b Let Y be a discrete random variable. Show that $E(E(X|Y)) = E(X)$.

Exercise 4
Find the characteristic function $\phi_X(t)$ of a random variable X with Poisson distribution ($P(X = k) = \frac{\lambda^k}{k!}e^{-\lambda}$, $k = 0, 1, \cdots$). Calculate $E(X^2)$ (facultative).

Exercise 5
a Give the definition of a 1 dimensional Brownian motion.
b Let W_t and $\hat{W}_t$ be two independent Brownian motions and define $X_t = a(W_{t+s} - W_s) + b(\hat{W}_{t+s} - \hat{W}s)$ for $s > 0$. Find $a, b \in \mathbb{R}$ such that X_t is a Brownian motion.

Exercise 6
Let W_t be a Wiener process. Calculate

$$E(\int_0^T s\,dW_s \int_0^T W_s dW_s)$$