Stochastic Mechanics 6 CFU
Part I 12.6.2012

Exercise 1

a Is $\mathcal{F} = \{A \subset \Omega \colon A \,\text{is a finite set}\}$ always a σ-algebra?
b Verify the inequality $P(A \triangle C) \leq P(A \triangle B) + P(B \triangle C)$. Remember that $A \triangle B := A \cup B \setminus A \cap B$.

Exercise 2
a What is the smallest number of elements of a σ-algebra if a function $X \colon \Omega \to \mathbb{R}$ taking exactly n different values is to be a random variable with respect to this σ-algebra?
b Let $\Omega = [0, 1]$ with Borel sets and Lebesgue measure. Find $P(X \in [0, \frac{1}{2}))$ if $X(x) = x^2$.

Exercise 3
a Let A be an event. Prove that the following conditions are equivalent:
$i)$ A, B are independent for any event B,
$ii)$ $P(A) = 0$ or 1.
b Show that if X and Y are independent random variables and Y is discrete, then $E(X \,|\, Y) = E(X)$.

Exercise 4
Calculate the characteristic function of a normal random variable X with mean 0 and variance 1.

Exercise 5
a Give the definition of a 1 dimensional Brownian motion.
b Let $\sigma > 0$ and $s < t$. Show that

$$E(e^{-\frac{1}{2}\sigma^2 t + \sigma W_t} \,|\, W_s) = e^{-\frac{1}{2}\sigma^2 s + \sigma W_s}$$

Exercise 6
Let W_t be a Wiener process. Calculate

$$E(\int_0^T s\, dW_s \int_0^T W_s^2 dW_s)$$