Attitude Control of Spacecraft

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Introduction

- In the near future large flexible structures will be used in the space for various uses (ISS, Earth observation, communication, etc.)
- Pointing precision, shape control and integrity of the structures are prior mission requirements
- Large platforms, antennas, solar arrays, etc., of dimensions ranging from some meters possibly up to various kilometers
- To put in orbit these space structures at reasonable costs, their wieght has to be minimized
- It must be possible to compactly store these structures to diminish the costs to put them in orbit (weight and payload space are crucial issues)
- Their dynamic behavior can be difficult to predict analytically (main problems arise from the unreliability or impracticality of structural tests on Earth) and the performances of controls designed on the basis of perfect model knowledge can be deteriorated leading to on–orbit behaviors which can be substantially different from preflight ground test measures or analytic predictions

Introduction

- When the structure dimensions increase, the frequencies of the first natural modes decrease
- The elastic modes of these light structures may be poorly damped, and problem arise if their spectrum overlaps the controller bandwidth
- For these reasons the controller design may be critic and important is to ensure stability and performance

- Important control problems
  1. Fine alignment with prescribed attitude and precision (payload – sensors, antennas, etc. – precise pointing)
  2. Shape control and integrity of the structures (vibration damping)
  3. Great angular displacements for re–orienting the structure (minimum time with minimum fuel consumption)

  The control requirements for both the problems are highly demanding
- In attitude control the structure translation is not considered
  1. Center of mass translation influence only the spacecraft orbit
  2. The spacecraft must stay in a “box”
  3. Orbit corrections are performed periodically and when they occur the normal pointing operations are interrupted

- The required high control performance can be obtained only considering the rigid–elastic dynamic coupling (nonlinear model)
- Quick review of application of nonlinear control techniques to attitude control

RIGID: [Dwyer, IEEE TAC 1984], [Monaco, Stornelli, 1985], [Monaco, Normand-Cyrot, Stornelli, CDC 1986], [Dwyer, CDC 1987], [Wen, Kreutz-Delgado IEEE TAC 1991], [Crouch, IEEE TAC 1984], [Aeyels, S&CL 1985], [Lizarralde, IEEE TAC 1996], [Di Gennaro, Monaco, Normand-Cyrot, Pignatelli, 1997]

FLEXIBLE: [Balas, AIAA JG&C 1979], [Joshi, 1989], [Vadali, 1990], [Di Gennaro, CDC 1996], [Di Gennaro, AIAA JGCD 1998], [Di Gennaro, JOTA 1998], etc.

- Important aspect when applying nonlinear controls: state measurement
  - Attitude position and velocity (rigid main body)
  - Modal position and velocity variables (elastic deflection of the flexible appendages)

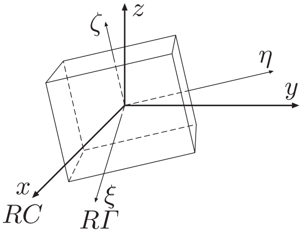
- Modal position and velocity variables are important for fine pointing and vibration damping – when they are not measured (no appropriate sensors can be used) the control ensuring the performance can not be implemented
- Dynamic nonlinear controllers can reconstruct the modal variables
- Dynamic controllers can also reconstruct main body angular velocity (in case of sensor failure)

Mathematical Model of a Flexible Spacecraft

- In the derivation of the model we consider first the kinematic equations and then the dynamic ones

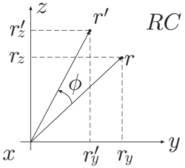
Kinematics

- Inertial frame $RC = \{O, x, y, z\}$ and non–inertial frame $R\Gamma = \{\Omega, \xi, \eta, \zeta\}$ (body–fixed frame)



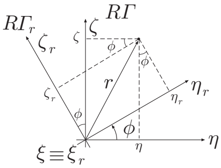
- It is usual to consider $\Omega \equiv O \equiv$ main body center of mass, and $RC \equiv R\Gamma$ at $t = 0$
- $R\Gamma$ determines the spacecraft attitude (i.e. its position in the space)

- When discussing a rotation, there are two possible conventions:
  - Rotation of vectors (the frame remains fixed!)



$$r' = \bar{\mathcal{R}}(\phi) r, \quad \bar{\mathcal{R}}(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$

  - Rotation of a frame $R\Gamma$ up to coincide with a frame $R\Gamma_r$ (the vector remains fixed!)



$$r' = \mathcal{R}(\phi) r, \quad \mathcal{R}(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}$$

- Note that $\bar{\mathcal{R}}^T = \mathcal{R}$

- Considering rotations of $R\Gamma$ of a positive (conterclockwise) angle ϕ
  - About the x–axis: The rotation matrix is

$$\mathcal{R}_1(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}$$

  - About the y–axis:

$$\mathcal{R}_2(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}$$

  - About the z–axis:

$$\mathcal{R}_3(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Observation: a matrix $\mathcal{R}(\phi)$ is a linear transformation of vectors (i.e. of the point pointed by r), and can be interpreted as

  1. operator which transforms a vector of $R\Gamma$ into a vector of another frame rotated w.r.t. $R\Gamma$.

     This vector is *rotated* and we still want to *express it in* $R\Gamma$.

     The angles are positive if *clockwise* (in fact, it is equivalent to rotate the reference frame counterclockwise or rotate the vector clockwise)

  2. operator which transforms vectors of $R\Gamma$ into vectors of another frame rotated w.r.t. $R\Gamma$. The vectors does not rotate, $R\Gamma$ rotates (positively if counterclockwise)

- In the following we are interested in the first interpretation. Hence the rotation will be positive if *clockwise*

- Note that $\mathcal{R}_i(\phi)$, $i = 1, 2, 3$ are
  - Orthogonal ($\mathcal{R}^{-1} = \mathcal{R}^T$)
  - With eigenvalues $\lambda_1 = 1$, $\lambda_{2,3} = e^{\pm j\phi} = \cos\phi \pm j \sin\phi$
  - With determinant equal to 1 (proper orthogonal matrix)
- Transformations represented by such matrices (with unitary eigenvalue) are rotation transformations of an angle ϕ about an axis determined by the corresponding eigenvalue

$$\mathcal{R}v = v, \quad \lambda = 1, \quad v \text{ eigenvector}$$

i.e. the axis is the subspace $\mathcal{V}$ generated by v, which is transformed into itself during the rotation (namely $\mathcal{V}$ is invariant under rotations)

- Euler Theorem *Every rotation sequence leaving fixed a point in the space is equivalent to a (certain) rotation of an angle Φ about a (certain) axis passing through this point. The axis is called Euler axis, and is determined by the unit vector ϵ, and Φ is called Euler angle*

Euler angles

- There exist various parameterizations of a rigid body kinematics
- A minimal set of of parameters allowing the determination of the attitude of a rigid body (spacecraft main body) in the space is that of the Euler angles
- Advantages of Euler angles:
  - have simple geometrical interpretation (rotations about the coordinate axes)
  - can be used to determine the closed form solution of the kinematic equations
  - can be used for spacecraft stabilized about the coordinate axes (small angle maneuvers)
- Disadvantages:
  - kinematics present singularities
  - transcendent functions appear in the kinematics and numeric disadvantages

- To define the attitude matrix describing the rotation from (inertial) frame RC to the spacecraft attitude, i.e. to the (non–inertial) frame $R\Gamma$, one considers a sequence of 3 rotation (each about a coordinate axis) bringing RC to superpose on $R\Gamma$
  - $r_1 = \mathcal{R}_i(\varphi)r$
  - $r_2 = \mathcal{R}_j(\vartheta)r_1 = \mathcal{R}_j(\vartheta)\mathcal{R}_i(\varphi)r$
  - $r' = r_3 = \mathcal{R}_k(\psi)r_2 = \mathcal{R}_k(\psi)\mathcal{R}_j(\vartheta)\mathcal{R}_i(\varphi)r$
  - Hence $\mathcal{R}_{ijk}(\varphi, \vartheta, \psi) = \mathcal{R}_k(\psi)\mathcal{R}_j(\vartheta)\mathcal{R}_i(\varphi)$
- i, j, k is the sequence of coordinate axes, about which the rotations are performed
- In total, there are 24 possible sequences

- A very used sequence is the 3–1–3 sequence

$$\mathcal{R}_{313}(\varphi,\vartheta,\psi) = \mathcal{R}_3(\psi)\mathcal{R}_1(\vartheta)\mathcal{R}_3(\varphi)$$
$$= \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & \sin\vartheta \\ 0 & -\sin\vartheta & \cos\vartheta \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \cos\psi\cos\varphi - \cos\vartheta\sin\psi\sin\varphi & \cos\psi\cos\varphi + \cos\vartheta\sin\psi\sin\varphi & \sin\vartheta\sin\psi \\ -\sin\psi\cos\varphi - \cos\vartheta\cos\psi\sin\varphi & -\sin\psi\sin\varphi + \cos\vartheta\cos\psi\cos\varphi & \sin\vartheta\cos\psi \\ \sin\vartheta\sin\varphi & -\sin\vartheta\cos\varphi & \cos\vartheta \end{pmatrix}$$

- If $\mathcal{R}_{313}(\varphi,\vartheta,\psi)$ is known (from measurements, etc.) it is possible to calculate the Euler angles (r_{ij} are the elements of $\mathcal{R}_{313}$)

$$\vartheta = \arccos r_{33}, \quad \varphi = -\arctan\frac{r_{31}}{r_{32}}, \quad \psi = \arctan\frac{r_{13}}{r_{23}}$$

- Indetermination in ϑ: $\vartheta \in [-\pi, 0)$ or $\vartheta \in [0, \pi)$

  Once solved, φ, ψ are uniquely determined

  If ϑ multiple of π: only $\varphi + \psi$ (ϑ even multiple), or $\varphi - \psi$ (ϑ odd multiple) are determined (usually $\sin\vartheta \geq 0$ or $0 \leq \vartheta < \pi$ is considered)

- Another popular sequence is the 3–1–2 sequence (“yaw, roll, pitch”)

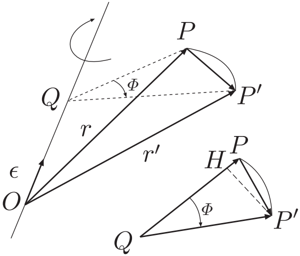
$$\mathcal{R}_{312}(\varphi,\vartheta,\psi) = \begin{pmatrix} \cos\psi\cos\varphi - \sin\vartheta\sin\psi\sin\varphi & \cos\psi\sin\varphi + \sin\vartheta\sin\psi\cos\varphi & -\cos\vartheta\sin\psi \\ -\cos\vartheta\sin\varphi & \cos\vartheta\cos\varphi & \sin\vartheta \\ \sin\psi\cos\varphi + \sin\vartheta\cos\psi\sin\varphi & \sin\psi\sin\varphi - \sin\vartheta\cos\psi\cos\varphi & \cos\vartheta\cos\psi \end{pmatrix}$$

- In this case $\vartheta = \arcsin r_{23}, \quad \varphi = -\arctan\frac{r_{21}}{r_{22}}, \quad \psi = -\arctan\frac{r_{13}}{r_{33}}$
- Indetermination in ϑ: $\vartheta \in [-\pi, -\frac{\pi}{2}) \cup [\frac{\pi}{2}, \pi)$ or $\vartheta \in [-\frac{\pi}{2}, \frac{\pi}{2})$ unless ϑ is an odd multiple of $\frac{\pi}{2}$ – in this case $\vartheta \in [-\frac{\pi}{2}, \frac{\pi}{2})$ ($\cos\vartheta \geq 0$)
- For small rotations

$$\mathcal{R}_{312}(\varphi,\vartheta,\psi) \simeq \begin{pmatrix} 1 & \varphi & -\psi \\ -\varphi & 1 & \vartheta \\ \psi & -\vartheta & 1 \end{pmatrix}$$

Euler parameters

- Using the Euler axis ϵ and angle Φ (positive if clockwise) to determine the attitude of the rigid main body



- Vector r is transformed into $r' = r + \overrightarrow{PP'} = r + \overrightarrow{PH} + \overrightarrow{HP'}$ with

$$\overrightarrow{QP} = -\vec{\epsilon} \times (\vec{\epsilon} \times \overrightarrow{OP}) = -\vec{\epsilon} \times (\vec{\epsilon} \times \vec{v})$$

$$\overrightarrow{PH} = -(1 - \cos\Phi)\overrightarrow{OP} = (1 - \cos\Phi)\vec{\epsilon} \times (\vec{\epsilon} \times r)$$

$$\overrightarrow{HP'} = -|\overrightarrow{QP'}| \sin\Phi \frac{\vec{\epsilon} \times \overrightarrow{QP}}{|\overrightarrow{QP}|} = -\sin\Phi\, \vec{\epsilon} \times \overrightarrow{OP} = -\sin\Phi\, \vec{\epsilon} \times r$$

- Hence: $r' = r + (1 - \cos\Phi)\, \epsilon \times (\epsilon \times r) - \sin\Phi\, \epsilon \times r$
- Setting

$$\epsilon\times = \begin{pmatrix} 0 & -\epsilon_3 & \epsilon_2 \\ \epsilon_3 & 0 & -\epsilon_1 \\ -\epsilon_2 & \epsilon_1 & 0 \end{pmatrix} \quad \longleftarrow \text{dyadic representation of } \epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix}$$

$$\epsilon \times v = \begin{pmatrix} 0 & -\epsilon_3 & \epsilon_2 \\ \epsilon_3 & 0 & -\epsilon_1 \\ -\epsilon_2 & \epsilon_1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\epsilon \times (\epsilon \times v) = \begin{pmatrix} 0 & -\epsilon_3 & \epsilon_2 \\ \epsilon_3 & 0 & -\epsilon_1 \\ -\epsilon_2 & \epsilon_1 & 0 \end{pmatrix}^2 \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -(\epsilon_2^2 + \epsilon_3^2) & \epsilon_1\epsilon_2 & \epsilon_1\epsilon_3 \\ \epsilon_1\epsilon_2 & -(\epsilon_1^2 + \epsilon_3^2) & \epsilon_2\epsilon_3 \\ \epsilon_1\epsilon_3 & \epsilon_2\epsilon_3 & -(\epsilon_1^2 + \epsilon_2^2) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

one gets ($\tilde{\epsilon}^2 = \epsilon\epsilon^T - I$)

$$r' = \mathcal{R}(\Phi) r = \Big[I + (1 - \cos\Phi)\, \tilde{\epsilon}^2 - \sin\Phi\, \tilde{\epsilon} \Big] r = \Big[\cos\Phi\, I + (1 - \cos\Phi)\, \epsilon\epsilon^T - \sin\Phi\, \tilde{\epsilon} \Big] r$$

$$\mathcal{R}(\Phi) = \begin{pmatrix} \epsilon_1^2 + (\epsilon_2^2 + \epsilon_3^2)\cos\Phi & \epsilon_1\epsilon_2(1 - \cos\Phi) + \epsilon_3 \sin\Phi & \epsilon_1\epsilon_3(1 - \cos\Phi) - \epsilon_2 \sin\Phi \\ \epsilon_1\epsilon_2(1 - \cos\Phi) - \epsilon_3 \sin\Phi & \epsilon_2^2 + (\epsilon_1^2 + \epsilon_3^2)\cos\Phi & \epsilon_1\epsilon_3(1 - \cos\Phi) + \epsilon_1 \sin\Phi \\ \epsilon_1\epsilon_3(1 - \cos\Phi) + \epsilon_2 \sin\Phi & \epsilon_1\epsilon_3(1 - \cos\Phi) - \epsilon_1 \sin\Phi & \epsilon_3^2 + (\epsilon_1^2 + \epsilon_2^2)\cos\Phi \end{pmatrix}$$

- This parameterization with the Euler axis uses 3 parametrers: Φ and the 3 components of ϵ (with the constraint $\|\epsilon\| = 1$)
- Since $\tilde{\epsilon}\epsilon = 0$, one deduces that ϵ is the eigenvector (determines the rotation axis)

$$\mathcal{R}(\Phi)\epsilon = \left[I + (1 - \cos\Phi)\ \tilde{\epsilon}^2 - \sin\Phi\ \tilde{\epsilon} \right]\epsilon = \epsilon$$

- ϵ has the same components in RC and in $R\Gamma$

The Unitary Quaternion or Euler symmetric parameters

- Quaternions, introduced by Hamilton in 1843, and used by Whittaker in 1937 to describe the rigid body motion, are defined by

$$q_0 = \cos\frac{\Phi}{2}, \quad q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} \sin\frac{\Phi}{2}$$

- Constraint: $q_0^2 + q_1^2 + q_2^2 + q_3^2 = q_0^2 + \|q\|^2 = 1$
- Since
$\sin\Phi = 2\cos\frac{\Phi}{2}\sin\frac{\Phi}{2}, \quad \cos\Phi = 1 - 2\sin^2\frac{\Phi}{2} \Rightarrow 1 - \cos\Phi = 2\sin^2\frac{\Phi}{2}$

$$\mathcal{R}(\Phi) = \mathcal{R}_{bi} = \begin{pmatrix} \epsilon_1^2 + (\epsilon_2^2 + \epsilon_3^2)\cos\Phi & \epsilon_1\epsilon_2(1-\cos\Phi) + \epsilon_3\sin\Phi & \epsilon_1\epsilon_3(1-\cos\Phi) - \epsilon_2\sin\Phi \\ \epsilon_1\epsilon_2(1-\cos\Phi) - \epsilon_3\sin\Phi & \epsilon_2^2 + (\epsilon_1^2 + \epsilon_3^2)\cos\Phi & \epsilon_1\epsilon_3(1-\cos\Phi) + \epsilon_1\sin\Phi \\ \epsilon_1\epsilon_3(1-\cos\Phi) + \epsilon_2\sin\Phi & \epsilon_1\epsilon_3(1-\cos\Phi) - \epsilon_1\sin\Phi & \epsilon_3^2 + (\epsilon_1^2 + \epsilon_2^2)\cos\Phi \end{pmatrix}$$

$$= \begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 + q_0q_3) & 2(q_1q_3 - q_0q_2) \\ 2(q_1q_2 - q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 + q_0q_1) \\ 2(q_1q_3 + q_0q_2) & 2(q_2q_3 - q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix} = I - 2(q_0 I - \tilde{q})\tilde{q}$$

- Hence $\begin{pmatrix} q_0 \\ q \end{pmatrix}$ describes the spacecraft attitude w.r.t. RC, and the transformation matrix describing the rotation that brings RC onto $R\Gamma$ is $\mathcal{R}_{bi}$
- The quaternion is the generalization of an imaginary number (Hamilton)

$$\mathrm{q} = q_0 + q_1 i + q_2 j + q_3 k = \begin{pmatrix} q_0 \\ q \end{pmatrix}, \qquad q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix},$$

where i, j, k are imaginary numbers such that

$$\begin{aligned} i^2 &= j^2 = k^2 = -1 & \qquad ij &= -ji = k \\ jk &= -kj = i & \qquad ki &= -ik = j \end{aligned}$$

and q_0 is the real or scalar part and $q = q_1 i + q_2 j + q_3 k$ is the imaginary or vectorial part

- Addition and subtraction are obvious
- Multiplication of 2 quaternions q, e is defined as for complex numbers (but the products of i, j, k are not commutative)

$$
\begin{aligned}
\mathrm{q}_r = \mathrm{qe} = & (q_0 + q_1 i + q_2 j + q_3 k)(e_0 + e_1 i + e_2 j + e_3 k) = \\
= & \quad (q_0 e_0 - q_1 e_1 - q_2 e_2 - q_3 e_3) + \\
& + (q_0 e_1 + q_1 e_0 + q_2 e_3 - q_3 e_2) i + \\
& + (q_0 e_2 - q_1 e_3 + q_2 e_0 + q_3 e_1) j + \\
& + (q_0 e_3 + q_1 e_2 - q_2 e_1 + q_3 e_0) k = (e_0 q_0 - e^T q, e_0 q + q_0 e - \tilde{e} q)
\end{aligned}
$$

i.e.

$$
\begin{aligned}
\begin{pmatrix} q_{r0} \\ q_r \end{pmatrix} = \begin{pmatrix} q_0 \\ q \end{pmatrix} \begin{pmatrix} e_0 \\ e \end{pmatrix} = & \begin{pmatrix} q_0 e_0 - q_1 e_1 - q_2 e_2 - q_3 e_3 \\ q_0 e_1 + q_1 e_0 + q_2 e_3 - q_3 e_2 \\ q_0 e_2 - q_1 e_3 + q_2 e_0 + q_3 e_1 \\ q_0 e_3 + q_1 e_2 - q_2 e_1 + q_3 e_0 \end{pmatrix} \\
= & \begin{pmatrix} q_0 & -q^T \\ q & q_0 I + \tilde{q} \end{pmatrix} \begin{pmatrix} e_0 \\ e \end{pmatrix} = \begin{pmatrix} e_0 & -e^T \\ e & e_0 I - \tilde{e} \end{pmatrix} \begin{pmatrix} q_0 \\ q \end{pmatrix}
\end{aligned}
$$

- The matrices multiplying $\begin{pmatrix} q_0 \\ q \end{pmatrix}$ e $\begin{pmatrix} e_0 \\ e \end{pmatrix}$ are orthogonal, so that

$$\begin{pmatrix} e_0 \\ e \end{pmatrix} = \begin{pmatrix} q_0 & q^T \\ -q & q_0 I - \tilde{q} \end{pmatrix} \begin{pmatrix} q_{r0} \\ q_r \end{pmatrix} = \begin{pmatrix} q_{r0} & q_r^T \\ q_r & -q_{r0} I + \tilde{q}_r \end{pmatrix} \begin{pmatrix} q_0 \\ q \end{pmatrix}$$

- If q, e represent 2 rotations (defined by the matrices $\mathcal{R}_1$, $\mathcal{R}_2$), the rotation defined by $\mathcal{R}_2 \mathcal{R}_1$ is represented by the quaternion $\mathrm{q}_r = \mathrm{qe}$
- Note the order of the matrices and that of the quaternions

$$\mathcal{R}(q_r) = \mathcal{R}_2(e)\mathcal{R}_1(q) \quad \leftrightarrow \quad \mathrm{q}_r = \mathrm{qe}$$

Quaternion dynamics

- To derive the quaternion dynamics, let

$$RC \rightsquigarrow R\Gamma_t \rightarrow \boxed{\begin{pmatrix} q_0(t) \\ q(t) \end{pmatrix}} \qquad RC \rightsquigarrow R\Gamma_{t+\Delta t} \rightarrow \boxed{\begin{pmatrix} q_0(t+\Delta t) \\ q(t+\Delta t) \end{pmatrix}}$$

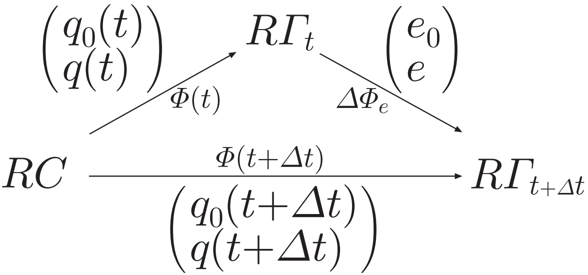
  the quaternions representing the spacecraft attitude w.r.t. RC at time t (reference $R\Gamma_t$) and $t+\Delta t$ ($R\Gamma_{t+\Delta t}$)

- Let

$$R\Gamma_t \rightsquigarrow R\Gamma_{t+\Delta t} \rightarrow \boxed{\begin{pmatrix} e_0 \\ e \end{pmatrix} = \begin{pmatrix} \cos\frac{\Delta\Phi_e}{2} \\ \epsilon_e \sin\frac{\Delta\Phi_e}{2} \end{pmatrix}}$$

  the error quaternion representing the spacecraft attitude at time $t+\Delta t$ (i.e. of ($R\Gamma_{t+\Delta t}$)) w.r.t. $R\Gamma_t$

  $\Delta\Phi_e = \Phi(t+\Delta t) - \Phi(t)$ is the rotation performed in Δt about ϵ_e



- For the quaternion multiplication law

$$\begin{pmatrix} q_0(t+\Delta t) \\ q(t+\Delta t) \end{pmatrix} = \begin{pmatrix} e_0 & -e^T \\ e & e_0 I - \tilde{e} \end{pmatrix} \begin{pmatrix} q_0(t) \\ q(t) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{\Delta\Phi_e}{2} & -\epsilon_e^T \sin\frac{\Delta\Phi_e}{2} \\ \epsilon_e \sin\frac{\Delta\Phi_e}{2} & \cos\frac{\Delta\Phi_e}{2} I - \tilde{\epsilon}_e \sin\frac{\Delta\Phi_e}{2} \end{pmatrix} \begin{pmatrix} q_0(t) \\ q(t) \end{pmatrix}$$

- As $\Delta t \to 0$ and for the small angle approximations

$$\cos\frac{d\Phi}{2} \approx 1, \qquad \sin\frac{d\Phi}{2} \approx \frac{d\Phi}{2} = \frac{1}{2}|\omega|\, dt$$

where

$$|\omega(t)| = \lim_{\Delta t \to 0} \frac{\Delta\Phi_e}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Phi(t+\Delta t) - \Phi(t)}{\Delta t} = \frac{d\Phi}{dt} \qquad \omega(t) = \epsilon_e |\omega(t)|$$

- Hence

$$\begin{pmatrix} q_0(t+dt) \\ q(t+dt) \end{pmatrix} = \begin{pmatrix} 1 & -\epsilon_e^T \frac{1}{2}|\omega(t)|\, dt \\ \epsilon_e \frac{1}{2}|\omega(t)|\, dt & I - \tilde{\epsilon}_e \frac{1}{2}|\omega(t)|\, dt \end{pmatrix} \begin{pmatrix} q_0(t) \\ q(t) \end{pmatrix}$$
$$= \begin{pmatrix} q_0(t) \\ q(t) \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & -\omega^T(t)\, dt \\ \omega(t)\, dt & -\tilde{\omega}(t)\, dt \end{pmatrix} \begin{pmatrix} q_0(t) \\ q(t) \end{pmatrix}.$$

$$\Rightarrow \begin{pmatrix} \dot{q}_0(t) \\ \dot{q}(t) \end{pmatrix} = \begin{pmatrix} \dfrac{q_0(t+dt) - q_0(t)}{dt} \\ \dfrac{q(t+dt) - q(t)}{dt} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & -\omega^T(t) \\ \omega(t) & -\tilde{\omega}(t) \end{pmatrix} \begin{pmatrix} q_0(t) \\ q(t) \end{pmatrix}$$

- Rearranging

$$\begin{pmatrix} \dot{q}_0 \\ \dot{q} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -q^T \\ R(q) \end{pmatrix} \omega = \frac{1}{2} \mathcal{Q}^T(q_0, q)\omega$$

$$\mathcal{Q}^T = \begin{pmatrix} -q^T \\ R(q) \end{pmatrix}$$

$$R(q_0, q) = q_0 T + \tilde{q} = \begin{pmatrix} q_0 & -q_3 & q_2 \\ q_3 & q_0 & -q_1 \\ -q_2 & q_1 & q_0 \end{pmatrix}$$

Note that ω is expressed in $R\Gamma = R\Gamma_t$

Note also that $\omega = 2\mathcal{Q}(q_0, q) \begin{pmatrix} \dot{q}_0 \\ \dot{q} \end{pmatrix}$

- This description is more appropriate for *rest–to–rest maneuvers* ((q_0, q) expresses the attitude error)
- Advantages of (nonminimal) parametrization, with respect to a minimal one (Euler angles)
  1. Absence of geometrical singularities
  2. Attitude matrix is algebraic and does not depend on transcendental functions
  3. Easy quaternion multiplication rule for successive rotations
  4. An attitude change is obtained by a single rotation about an appropriate axis (fuel and time optimum)

- In the case of attitude *tracking* kinematics are expressed with error quaternion

$$\boxed{\begin{pmatrix} \dot{e}_0 \\ \dot{e} \end{pmatrix} = \frac{1}{2}\mathcal{Q}^T(e_0, e)\omega_e} \qquad \omega_e = \omega - \omega_r$$

- ω_r has to be expressed in $R\Gamma$ (as ω)

$$\omega_r = \mathcal{R}(q_0, q)\mathcal{R}^T(q_{r0}, q_r)\mu(q_r, \dot{q}_r) \qquad \begin{aligned} \mathcal{R}(q_0, q) &= I - 2(q_0 I - \tilde{q})\tilde{q} \\ \mathcal{R}(q_{r0}, q_r) &= I - 2(q_{r0} - \tilde{q}_r)\tilde{q}_r \end{aligned}$$

with

$$\begin{pmatrix} \dot{q}_{r0} \\ \dot{q}_r \end{pmatrix} = \frac{1}{2}\mathcal{Q}^T(q_{r0}, q_r)\mu(q_r, \dot{q}_r) \quad \Rightarrow \quad \mu(q_r, \dot{q}_r) = 2\mathcal{Q}(q_{r0}, q_r)\begin{pmatrix} \dot{q}_{r0} \\ \dot{q}_r \end{pmatrix}.$$

Dynamic equations

- The dynamic equation for a satellite with flexible elements, such as solar panels, antennas, etc., can be obtained following the hybrid coordinates approach proposed by Likins [Likins, 1970] and used in [Monaco, Stornelli, CDC 1985], [Monaco, Normand-Cyrot, Stornelli, 1986], etc
- Angular velocity dynamics (Euler theorem)

$$\dot{L} = -\omega \times L + u_g + d_r = -(\tilde{\omega}_e + \tilde{\omega}_r)L + u_g + d_r$$

  - $L =$ total angular momentum (depends on the rigid and flexible dynamics)
  - $u_g =$ control torque acting (gas jets)
  - $d_r =$ disturbances on the main body
  - $\omega_e = \omega - \omega_r$ ($\omega_r =$ reference angular velocity)
  - $\omega \times = \tilde{\omega}$ (dyadic representation)

- Reaction wheels dynamics

$$\dot{\Omega} = -\dot{\omega} + J_r^{-1} u_r = -\dot{\omega}_e + J_r^{-1} u_r - \dot{\omega}_r$$

  - Ω = angular velocity of the reaction wheels with respect to the main body
  - u_r = reaction wheel driving torques
  - J_r = reaction wheel inertia matrix

- Flexible appendage dynamics

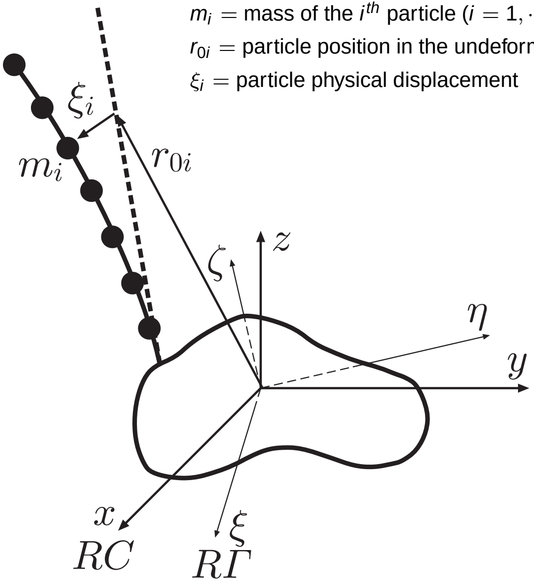
$$M_f \ddot{\xi} + C_f \dot{\xi} + K_f \xi = H_f + d_f$$

  - $\xi = 3N \times 1$ vector of physical displacements flexible appendages discretized with N particles
  - M_f, C_f K_f = $3N \times 3N$ mass, damping and stiffness matrices
  - $H_f = 3N \times 1$ vector of noninertial forces due to the main body rotation (centrifugal, Coriolis, due to the non uniform angular velocity variation)
  - $d_f = 3N \times 1$ vector of external disturbance on the flexible structure

m_i = mass of the i^{th} particle ($i = 1, \cdots, N$)

r_{0i} = particle position in the undeformed structure w.r.t. $R\Gamma$

ξ_i = particle physical displacement



- Total angular momentum sum of “rigid” and “flexible contributions”

$$L_r = J_t\omega + J_r\Omega$$

$$L_f = \sum_{i=1}^{N}(r_{0i}+\xi_i)\times m_i\dot{\xi}_i + L_f^c = \sum_{i=1}^{N} m_i(\tilde{r}_{0i}+\tilde{\xi}_i)\dot{\xi}_i + \sum_{i=1}^{N} m_i(\tilde{r}_{0i}^T\tilde{\xi}_i + \tilde{\xi}_i^T\tilde{r}_{0i} + \tilde{\xi}_i^T\tilde{\xi}_i)\omega$$

  - m_i = mass of the i^{th} particle ($i = 1, \cdots, N$)
  - r_{0i} = particle position in the undeformed structure w.r.t. $R\Gamma$
  - ξ_i = particle physical displacement
  - $J_t = J_{mb} + J_r + \sum_{i=1}^{N}\left(J_{0i} + m_i\tilde{r}_{0i}^T\tilde{r}_{0i}\right)$ = inertia matrix of the whole undeformed structure
  - J_{mb} = main body inertia matrix
  - J_{0i} = particle inertia matrix (expressed in the frame attached to the mass and with the axes parallel to the $R\Gamma$ frame); here $J_{0i} = 0$ since the particles are considered punctiform
  - Torsional motion neglected (it can be considered in a similar way)

- Noninertial forces $H_f = \begin{pmatrix} H_{f1} \\ \vdots \\ H_{fN} \end{pmatrix}$ on the particles

$$H_{fi} = m_i \left[(\tilde{r}_{0i} + \tilde{\xi}_i)\dot{\omega} - 2\tilde{\omega}\dot{\xi}_i + \tilde{\omega}^T \tilde{\omega}(r_{0i} + \xi_i) \right] \quad i = 1, \cdots, N$$

- Model for the control law design obtained assuming
  - Small deformations
  - Coriolis and centrifugal effects L_f^c neglected
  - First $N_e < N$ vibration modes (significant modes)
- Hence M_{f0}, C_{f0}, K_{f0} $N_e \times N_e$ matrices and

$$M_{f0}\ddot{\xi}_0 + C_{f0}\dot{\xi}_0 + K_{f0}\xi_0 = H_{f0} + d_{f0} \qquad \xi_0 = \begin{pmatrix} \xi_{01} \\ \vdots \\ \xi_{0N_e} \end{pmatrix}$$

$$H_f \simeq H_{f0} = \begin{pmatrix} m_1 \tilde{r}_{01} \\ \vdots \\ m_{N_e} \tilde{r}_{0N_e} \end{pmatrix} \dot{\omega} = -M_{f0}\delta_0\dot{\omega} \qquad \delta_0 = \begin{pmatrix} \delta_{01} \\ \vdots \\ \delta_{0N_e} \end{pmatrix} = \begin{pmatrix} \tilde{r}_{01}^T \\ \vdots \\ \tilde{r}_{0N_e}^T \end{pmatrix}$$

$$d_f \simeq d_{f0} = \begin{pmatrix} d_{f01} \\ \vdots \\ d_{f0N_e} \end{pmatrix} \qquad M_{f0} = \mathrm{diag}\{m_1 I, \cdots, m_{N_e} I\}$$

- Decoupling of the flexible dynamics (modal analysis)
  - $\eta = T\xi$ $N_e \times 1$ vector of the appendage modal displacements
  - T decouples the dynamics and is orthogonal ($T^{-1} = T^T$)

$$TM_{f0}T^T = I \qquad TK_{f0}T^T = K = \text{diag}\,\{\omega_1^2, \cdots, \omega_{N_e}^2\}$$

  - Note that

$$TC_{f0}T^T = C \simeq \text{diag}\,\{2\zeta_1\omega_1, \cdots, 2\zeta_{N_e}\omega_{N_e}\}$$

    the damping matrix C can be considered diagonal only in first approximation
  - $\lambda_i = \omega_i^2$ are the first N_e modal frequencies and ζ_i are the damping of the N_e modes

$$M_{f0}\ddot{\xi}_0 + C_{f0}\dot{\xi}_0 + K_{f0}\xi_0 = H_{f0} + d_{f0} = -M_{f0}\delta_0\dot{\omega} + d_{f0}$$

$$\Downarrow$$

$$TM_{f0}T^{-1}\,T\ddot{\xi} + TC_{f0}T^{-1}\,T\dot{\xi} + TK_{f0}T^{-1}\,T\xi = -TM_{f0}T^T\,T\delta_0\dot{\omega} + Td_{f0}$$

$$\Downarrow$$

$$\ddot{\eta} + C\dot{\eta} + K\eta = -T\delta_0\dot{\omega} + Td_{f0} = -\delta(\dot{\omega}_e + \dot{\omega}_r) + D_f \qquad T\delta_0 = \delta \qquad Td_{f0} = D_f$$

- The flexible dynamics are hence

$$\ddot{\eta} + C\dot{\eta} + K\eta = -T\delta_0\dot{\omega} + Td_{f0} = -\delta(\dot{\omega}_e + \dot{\omega}_r) + D_f$$

K, C stiffness and damping matrices, δ the $N_e \times 3$ coupling matrix

- Also the angular momentum due to the flexible appendages is approximated

$$L_f \simeq \sum_{i=1}^{N_e} m_i \tilde{r}_{0i}\dot{\xi}_i = \delta_0^T M_{f0}\dot{\xi} = \delta_0^T T^T \; TM_{f0}T^T \; T\dot{\xi} = \delta^T\dot{\eta}$$

- Hence

$$L = L_r + L_f = J_t(\omega_e + \omega_r) + J_r\Omega + \delta^T\dot{\eta}$$

- The angular velocity dynamics rewrite $\rightarrow \dot{L} = -(\tilde{\omega}_e + \tilde{\omega}_r)L + u_g + d_r$

$$J_t(\dot{\omega}_e + \dot{\omega}_r) + J_r\dot{\Omega} + \delta^T\ddot{\eta} = -(\tilde{\omega}_e + \tilde{\omega}_r)\left[J_t(\omega_e + \omega_r) + J_r\Omega + \delta^T\dot{\eta}\right] + u_g + d_r$$

- The mathematical model of the flexible spacecraft is

$$\dot{e}_0 = -\frac{1}{2} e^T \omega_e \quad \longleftarrow \text{redundant}$$

$$\dot{e} = \frac{1}{2} R(e) \omega_e$$

$$\dot{\omega}_e = J_{mb}^{-1} \Big[-(\tilde{\omega}_e + \tilde{\omega}_r)(J_t \omega_e + J_r \Omega + \delta^T z + J_t \omega_r) + \delta^T (Cz + K\eta) + u_g - u_r + D_r \Big] - \dot{\omega}_r$$

$$\dot{\Omega} = -\dot{\omega}_e + J_r^{-1} u_r - \dot{\omega}_r$$

$$\dot{\eta} = z$$

$$\dot{z} = -\delta \dot{\omega}_e - (Cz + K\eta) - \delta \dot{\omega}_r + D_f$$

Green = kinematics

Blue = rigid dynamics

Red = flexible dynamics

with $D_r = d_r - \delta^T D_f = d_r - \delta_0^T d_{f0}, \quad J_t = J_{mb} + J_r + \delta^T \delta$

- Sometime it is practical to substitute $\dot{\Omega}$ with

$$\dot{L} = -(\tilde{\omega}_e + \tilde{\omega}_r) L + u_g + d_r$$

- The disturbances acting on the structure are supposed negligible for simplicity

- In order to simplify the design of the control law, we use the variables

$$\xi_e = L - J_t\omega_r = J_t\omega_e + J_r\Omega + \delta^T\dot{\eta} = (J_{mb} + J_r)\omega_e + J_r\Omega + \delta^T\psi$$
$$\psi = \delta\omega_e + \dot{\eta}$$

  - $\xi_e =$ error between total angular momentum and reference angular momentum of the undeformed structure with idle reaction wheels
  - $\psi =$ difference between the total modal velocity $\delta\omega + \dot{\eta}$ and the reference velocity $\delta\omega_r$, in modal coordinates

- Therefore, the *mathematical model of a flexible spacecraft* can be rewritten

$$\dot{e}_0 = -\frac{1}{2}e^T\omega_e \quad \longleftarrow \text{redundant}$$
$$\dot{e} = \frac{1}{2}R(e)\omega_e$$
$$N(\omega_e, \xi_e, \omega_r) = (\tilde{\omega}_e + \tilde{\omega}_r)(\xi_e + J\omega_r)$$
$$\dot{\omega}_e = J_{mb}^{-1}\Big[-N(\omega_e, \xi_e, \omega_r) + \delta^T\Big(C\psi + K\eta - C\delta\omega_e\Big) + u_g - u_r\Big] - \dot{\omega}_r$$
$$\dot{\xi}_e = -N(\omega_e, \xi_e, \omega_r) + u_g - J\dot{\omega}_r$$
$$\begin{pmatrix}\dot{\eta}\\ \dot{\psi}\end{pmatrix} = A\begin{pmatrix}\eta\\ \psi\end{pmatrix} - AB\delta\omega_e - B\delta\dot{\omega}_r \quad \leftarrow A = \begin{pmatrix}0 & I\\ -K & -C\end{pmatrix},\; B = \begin{pmatrix}0\\ I\end{pmatrix}$$

- When dealing with *rest-to-rest maneuver* (i.e. $q_r = 0$, $\omega_r = 0$, $\dot{\omega}_r = 0$)

$$\dot{q}_0 = -\frac{1}{2} q^T \omega$$
$$\dot{q} = \frac{1}{2} R(e)\omega$$
$$\dot{\omega} = J_{mb}^{-1} \left[-N(\omega,\xi) + \delta^T \left(C\psi + K\eta - C\delta\omega \right) + u_g - u_r \right]$$
$$\dot{\xi} = -N(\omega,\xi) + u_g$$
$$\begin{pmatrix} \dot{\eta} \\ \dot{\psi} \end{pmatrix} = A \begin{pmatrix} \eta \\ \psi \end{pmatrix} - AB\delta\omega$$

with

$$\xi = L = J_t\omega + J_r\Omega + \delta^T \dot{\eta} = (J_{mb} + J_r)\omega + J_r\Omega + \delta^T \psi \qquad N(\omega,\xi) = \tilde{\omega}\xi$$

Problem Formulation

- *Rest–to–rest maneuvers*: Drive $R\Gamma$ to $R\Gamma_r$ (constant), damping out the induced flexible oscillations

$$\lim_{t\to\infty} q = 0 \qquad \lim_{t\to\infty} \eta = 0$$

- *Tracking maneuvers*: $R\Gamma$ tracks $R\Gamma_r$ (variable), damping out the induced flexible oscillations

$$\lim_{t\to\infty} e = 0, \qquad \lim_{t\to\infty} \eta = 0$$

- Problems solved for $\sigma(A) \in \mathbb{C}^-$ (non–negligible internal damping)

Rest–to–Rest Maneuvers

State/Output–Feedback Controllers

$$\dot{q}_0 = -\frac{1}{2} q^T \omega \qquad \text{No reaction wheels}$$
$$\dot{q} = \frac{1}{2} R(e)\omega$$
$$\dot{\omega} = J_{mb}^{-1}\left[-\omega \times J_{mb}\omega + u_g \right]$$

- Simple case: rigid spacecraft ⟶
- *State Feedback.* A simple proportional and derivative control $u_g = -k_p q - k_d \omega$, with $k_p > 0$, $k_d > 0$ scalars, globally asymptotically stabilizes a *rigid* spacecraft [Wie, AIAA JG 1985]
- In fact, take the following Lyapunov function candidate

$$V = k_p\left[(q_0 - 1)^2 + q^T q\right] + \frac{1}{2}\omega^T J_{mb}\omega$$
$$\Downarrow$$
$$\dot{V} = k_p q^T \omega + \omega^T \left[-\omega \times J_{mb}\omega - k_p q - k_d \omega \right] = -k_d \omega^T \omega \leq 0$$

- La Salle theorem:

$$x(t) \rightarrow \{q = 0, \omega = 0\} = \mathcal{E} \subseteq E = \left\{ x \in \mathbf{R}^n \mid \dot{V} = 0 \right\}$$

- Flexible spacecraft: $\longrightarrow$

$$\dot{q}_0 = -\frac{1}{2}q^T\omega \qquad \text{No reaction wheels}$$
$$\dot{q} = \frac{1}{2}R(e)\omega$$
$$\dot{\omega} = J_{mb}^{-1}\left[-\omega\times(J_{mb}\omega+\delta^T\psi)+\delta^T\left(C\psi+K\eta-C\delta\omega\right)+u_g\right]$$
$$\dot{\xi} = -\omega\times(J_{mb}\omega+\delta^T\psi)+u_g$$
$$\begin{pmatrix}\dot{\eta}\\ \dot{\psi}\end{pmatrix} = A\begin{pmatrix}\eta\\ \psi\end{pmatrix} - AB\delta\omega$$

- *State Feedback.* A similar proportional control can be designed (PD + terms accounting for the flexible dynamics)

$$u_g = -k_pq - k_d\omega - \delta^T\left[\begin{pmatrix}K\\ C\end{pmatrix} - P_1AB\right]\begin{pmatrix}\eta\\ \psi\end{pmatrix} \qquad P_1 = P_1^T > 0$$

- In fact

$$V = k_p\left[(q_0-1)^2+q^Tq\right] + \frac{1}{2}\omega^TJ_{mb}\omega + \frac{1}{2}\begin{pmatrix}\eta^T & \psi^T\end{pmatrix}P_1\begin{pmatrix}\eta\\ \psi\end{pmatrix}$$
$$\Downarrow$$
$$\dot{V} = \omega^T\left[k_pq - \omega\times\left(J_{mb}\omega+\delta^T\psi\right)+\delta^T\left(C\psi+K\eta-C\delta\omega\right)+u_g\right]$$
$$+\begin{pmatrix}\eta^T & \psi^T\end{pmatrix}P_1\left[A\begin{pmatrix}\eta\\ \psi\end{pmatrix}-AB\delta\omega\right] = -\omega^T(k_dI+\delta^TC\delta)\omega - \begin{pmatrix}\eta^T & \psi^T\end{pmatrix}Q_1\begin{pmatrix}\eta\\ \psi\end{pmatrix} \le 0$$

$\Rightarrow$ La Salle: $\mathcal{E} = \{x\in \mathbf{R}^n \mid q=0,\ \omega=0,\ \eta=0,\ \psi=0\}$

- Output Feedback. Modal position and velocity are often not measurable (sensors placed along the flexible structure necessary)
- Extend the previous controller: estimates $\hat{\eta}$, $\hat{\psi}$ of η, ψ
- Lyapunov function candidate $P_2 = P_2^T > 0$

$$\mathcal{V} = V + \frac{1}{2}\begin{pmatrix} e_\eta^T & e_\psi^T \end{pmatrix} P_2 \begin{pmatrix} e_\eta \\ e_\psi \end{pmatrix}, \qquad e_\eta = \eta - \hat{\eta}, \quad e_\psi = \psi - \hat{\psi}$$

$$\Downarrow$$

$$\dot{\mathcal{V}} = \dot{V} + \begin{pmatrix} e_\eta^T & e_\psi^T \end{pmatrix} P_2 \left[A \begin{pmatrix} \eta \\ \psi \end{pmatrix} - AB\delta\omega - \begin{pmatrix} \dot{\hat{\eta}} \\ \dot{\hat{\psi}} \end{pmatrix} \right]$$

- The control law is now slightly changed

$$u = -k_p q - k_d \omega - \delta^T \left[\begin{pmatrix} K \\ C \end{pmatrix} - P_1 AB \right] \begin{pmatrix} \hat{\eta} \\ \hat{\psi} \end{pmatrix}$$

and the update law is chosen

$$\begin{pmatrix} \dot{\hat{\eta}} \\ \dot{\hat{\psi}} \end{pmatrix} = A \begin{pmatrix} \hat{\eta} \\ \hat{\psi} \end{pmatrix} - AB\delta\omega + P_2^{-1} \left[\begin{pmatrix} K \\ C \end{pmatrix} - P_1 AB \right] \delta\omega$$

$$\Rightarrow \dot{V} = -\omega^T (k_d I + \delta^T C \delta)\omega - \begin{pmatrix} \eta^T & \psi^T \end{pmatrix} Q_1 \begin{pmatrix} \eta \\ \psi \end{pmatrix} - \begin{pmatrix} e_\eta^T & e_\psi^T \end{pmatrix} Q_2 \begin{pmatrix} e_\eta \\ e_\psi \end{pmatrix} \leq 0$$

$\Rightarrow$ La Salle $\Rightarrow$ global stability

Tracking Maneuvers

State/Output–Feedback Controllers

- La Salle theorem can not be applied – Barbalat theorem instead
- State Feedback. Take the PD–like controller

$$u_g = -k_p e - k_d \omega_e - \frac{1}{2} J_{mb} R(e)\omega_e + N - \delta^T (C\psi + K\eta - C\delta\omega_e) + J_{mb}\dot{\omega}_r$$

designed using

$$V(t,x) = (k_p + k_d)\left[(e_0 - 1)^2 + e^T e\right] + \frac{1}{2}(e+\omega_e)^T J_{mb}(e+\omega_e) + \frac{1}{2}\begin{pmatrix}\eta^T & \psi^T\end{pmatrix} P \begin{pmatrix}\eta \\ \psi\end{pmatrix}$$

- Deriving

$$\begin{aligned}\dot{V}(t,x) &= (k_p + k_d) e^T \omega_e + (e+\omega_e)^T \Bigg[\frac{1}{2} J_{mb} R(e)\omega_e - \bar{N}(\omega_e, \psi, \omega_r) \\ &\quad + \delta^T (C\psi + K\eta - C\delta\omega_e) + u_g - J_{mb}\dot{\omega}_r\Bigg] + \begin{pmatrix}\eta^T & \psi^T\end{pmatrix} P \left[A\begin{pmatrix}\eta \\ \psi\end{pmatrix} - AB\delta\omega_e - B\delta\dot{\omega}_r\right] \\ &\overset{u_g}{=} -k_p\|e\|^2 - k_d\|\omega_e\|^2 - \left\|\begin{matrix}\eta \\ \psi\end{matrix}\right\|_Q^2 - \begin{pmatrix}\eta^T & \psi^T\end{pmatrix} PAB\delta\omega_e - \begin{pmatrix}\eta^T & \psi^T\end{pmatrix} PB\delta\dot{\omega}_r \\ &\le -\lambda_m \|x\|^2 + \alpha\|\dot{\omega}_r\|\|x\|\end{aligned}$$

- Hence $\|x\| \to 0$ (Barbalat)

▸ Show the details of the proof ▸ Skip the details

- *Proof.* We suppose that $\omega_r \in L_\infty[0,\infty)$ and $\dot{\omega}_r \in L_2[0,\infty) \cap L_\infty[0,\infty)$
- Integrating both sides of $\dot{V}(t,x) - \lambda_m \|x\|^2 + \alpha \|\dot{\omega}_r\| \|x\|$ we have

$$V(t,x) - V(0,x_0) \leq -\lambda_m \int_0^t \|x(\tau)\|^2 d\tau + \alpha \int_0^t \|\dot{\omega}_r(\tau)\| \|x(\tau)\| d\tau \leq$$

Schwarz inequality $\longrightarrow$ $$\leq -\lambda_m \int_0^t \|x(\tau)\|^2 d\tau + \alpha \left[\int_0^t \|\dot{\omega}_r(\tau)\|^2 d\tau \right]^{1/2} \left[\int_0^t \|x(\tau)\|^2 d\tau \right]^{1/2}$$

and considering the limit as t tends to infinity ($\|\cdot\|_2$ is the L_2-norm) one has

$$V(\infty,x) - V(0,x_0) \leq -\lambda_m \|x\|_2^2 + \alpha \|\dot{\omega}_r\|_2 \|x\|_2$$

- Since $V(\infty,x) \geq 0$,

$$\lambda_m \|x\|_2^2 - \alpha \|\dot{\omega}_r\|_2 \|x\|_2 \leq V(0,x_0) - V(\infty,x) \leq V(0,x_0)$$

and because $\dot{\omega}_r \in L_2[0,\infty) \cap L_\infty[0,\infty)$ we obtain the bound

$$\|x\|_2 \leq \frac{1}{\sqrt{\lambda_m}} \left[V(0,x_0) + \frac{\alpha^2}{4\lambda_m} \|\dot{\omega}_r\|_2^2 \right]^{1/2} + \frac{\alpha}{2\lambda_m} \|\dot{\omega}_r\|_2 \quad \Rightarrow \quad x \in L_2[0,\infty)$$

- It follows that $V(\infty,x) < \infty$, i.e. $V(t,x)$ is uniformly bounded in t along the solution trajectories and, therefore, x is uniformly bounded
- Also $\dot{x}$ is uniformly bounded and, hence, x is uniformly continuous. Since x is a uniformly continuous function in $L_2[0,\infty)$, we have that (Barbalat theorem)

$$\lim_{t\to\infty} x = 0 \quad \text{and in particular} \quad \lim_{t\to\infty} e = 0, \quad \lim_{t\to\infty} \eta = 0$$

- Remark. The control u_{PD} is sufficient to solve the attitude tracking when $\omega_r, \dot{\omega}_r \in L_2[0,\infty) \cap L_\infty[0,\infty)$. In fact, taking

$$V(t,x) = (k_p + k_d)\left[(e_0 - 1)^2 + e^T e\right] + \frac{1}{2}(e^T + \omega_e^T)J_{mb}(e + \omega_e) + \frac{1}{2}\begin{pmatrix}\eta^T & \psi^T\end{pmatrix}\left[P + \begin{pmatrix}K & 0\\ 0 & I\end{pmatrix}\right]\begin{pmatrix}\eta\\ \psi\end{pmatrix}$$

finally one works out (α_1, α_2 are appropriate constants)

$$\dot{V}(t,x) \le -\lambda_m \|x\|^2 + \left(\alpha_1 \|\omega_r\| + \alpha_2 \|\dot{\omega}_r\|\right)\|x\| \quad \Rightarrow \quad \lim_{t\to\infty} x = 0$$

- The PD controller is robust in the sense that it does not need the parameter knowledge
- Moreover, can be used when measurements of the modal variables are not available
- Nevertheless, the gains k_p, k_d must be higher (no good in space)

- Output Feedback. If the modal variables η, ψ are not measured, use the same control but with estimates $\hat{\eta}$, $\hat{\psi}$

$$u_g = -k_p e - k_d \omega_e - \frac{1}{2} J_{mb} R(e) \omega_e + \hat{N} - \delta^T (C\hat{\psi} + K\hat{\eta} - C\delta\omega_e) + J_{mb}\dot{\omega}_r$$

- The design of the update controls $\dot{\hat{\eta}}$, $\dot{\hat{\psi}}$ is based on

$$\mathcal{V}(t, x, e_\eta, e_\psi) = V(t, x) + \frac{1}{2} \begin{pmatrix} e_\eta^T & e_\psi^T \end{pmatrix} \Gamma^{-1} \begin{pmatrix} e_\eta \\ e_\psi \end{pmatrix}$$

- Analogously to the previous case one eventually gets

$$\begin{pmatrix} \dot{\hat{\eta}} \\ \dot{\hat{\psi}} \end{pmatrix} = A \begin{pmatrix} \hat{\eta} \\ \hat{\psi} \end{pmatrix} - AB\delta\omega_e - B\delta\dot{\omega}_r + \Gamma \begin{pmatrix} K\delta \\ \delta(\tilde{\omega}_e + \tilde{\omega}_r) + C\delta \end{pmatrix} (e + \omega_e)$$

with

$$\hat{N} = (\tilde{\omega}_e + \tilde{\omega}_r)(J_{mb}\omega_e + \delta^T \hat{\psi} + J_s \omega_r)$$

Reaction wheels

When using reaction wheels the methods substantially is the same

1. Lyapunov $V(t,x) =$
$$(k_p+k_d)\left[(e_0-1)^2+e^Te\right]+\frac{1}{2}(e+\omega_e)^T J_{mb}(e+\omega_e)+\frac{1}{2}\begin{pmatrix}\eta^T & \psi^T\end{pmatrix}P\begin{pmatrix}\eta\\ \psi\end{pmatrix}+\frac{1}{2}\xi_e^T\xi_e$$
2. Choose appropriately
$$u_r = -\left[-k_p e-k_d\omega_e-\frac{1}{2}J_{mb}R(e)\omega_e+N-\delta^T(C\psi+K\eta-C\delta\omega_e)+J_{mb}\dot{\omega}_r\right]+\tilde{\xi}_e J\omega_r$$
$$N = (\tilde{\omega}_e+\tilde{\omega}_r)(\xi_e+J\omega_r)$$
3. Calculate $\dot{V}(t,x) \leq -\lambda_m\|x\|^2+\alpha_0\varphi(\omega_r,\dot{\omega}_r)\|x\|$
4. Under stronger hypothesis $\omega_r,\dot{\omega}_r \in L_2[0,\infty)\cap L_\infty[0,\infty)$
5. Barbalat $\lim_{t\to\infty} x = 0$, while $\lim_{t\to\infty}\Omega = J_r^{-1}J\omega_r$ (obvious)
6. When η, ψ are not measured use estimates
$$u_r = -\left[-k_p e-k_d\omega_e-\frac{1}{2}J_{mb}R(e)\omega_e+\hat{N}-\delta^T(C\hat{\psi}+K\hat{\eta}-C\delta\omega_e)+J_{mb}\dot{\omega}_r\right]+\tilde{\hat{\xi}}_e J\omega_r$$
$$\hat{N} = N(\omega_e,\hat{\xi}_e,\omega_r) = (\tilde{\omega}_e+\tilde{\omega}_r)(\hat{\xi}_e+J\omega_r)\quad \hat{\xi}_e = (J_{mb}+J_r)\omega_e+J_r\Omega+\delta^T\hat{\psi}$$
7. Prove stability using $\mathcal{V}(t,x,e_\eta,e_\psi) = V(t,x)+\frac{1}{2}\xi_e^T\xi_e+\frac{1}{2}\begin{pmatrix}e_\eta^T & e_\psi^T\end{pmatrix}\Gamma^{-1}\begin{pmatrix}e_\eta\\ e_\psi\end{pmatrix}$
8. Get $\begin{pmatrix}\dot{\hat{\eta}}\\ \dot{\hat{\psi}}\end{pmatrix} = A\begin{pmatrix}\hat{\eta}\\ \hat{\psi}\end{pmatrix}-AB\delta\omega_e-B\delta\dot{\omega}_r+\Gamma\begin{pmatrix}K\delta\\ \delta(\tilde{\omega}_e+\tilde{\omega}_r)+C\delta\end{pmatrix}(e+\omega_e)$
9. Barbalat again

Simulation Results (Tracking Maneuvers)

- The mathematical model of the spacecraft has been implemented on a digital computer
- Spillover has been studied by considering a model with more elastic modes than in the model used to derive the control law

Material	Aluminum	Shear modulus	$2.5\ 10^{10}$ N/m^2
Length	20 m	r_x	1.5 m
Density	$2.76\ 10^3$ Kg/m^3	r_y	2.3 m
Young modulus	$6.8\ 10^{10}$ N/m^2	r_z	-0.8 m

Characteristics of the flexible appendage

- Three elastic modes result from the modal analysis of the structure, with natural frequencies $\omega_{n1} = 19.38$, $\omega_{n2} = 77.98$, $\omega_{n3} = 157.22$ rad/s and dampings $\zeta_1 = 0.0001$, $\zeta_2 = 0.00005$, $\zeta_3 = 0.00001$
- Only the first two modes have been considered in the controller design
- Coupling matrix δ

$$\delta = \begin{pmatrix} 14.3961 & 8.37634 & -5.29354 \\ -20.4871 & 7.59188 & -6.08014 \\ 4.50401 & 11.5222 & -12.6033 \end{pmatrix} \ \mathrm{Kg}^{1/2}\mathrm{m}$$

- A payload of 30 Kg is present at the tip of the appendage
- The spacecraft is characterized by an inertia matrix

$$J_{mb} = \begin{pmatrix} 400 & 3 & 10 \\ 3 & 300 & 12 \\ 10 & 12 & 200 \end{pmatrix} \ \mathrm{Kg\,m}^2$$

- Reference trajectory described as

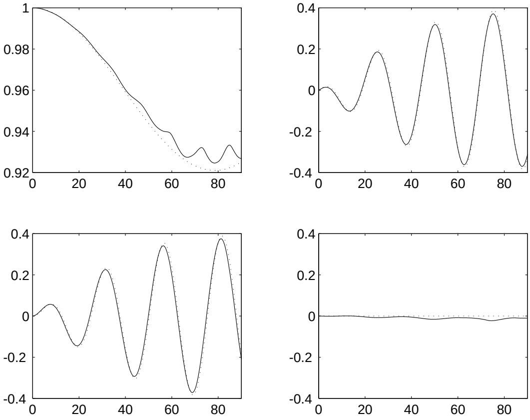
$$q_{r0} = \cos\frac{\phi_r}{2}, \quad q_r = \begin{pmatrix}\cos\ 0.5\,t \\ \sin\ 0.5\,t \\ 0\end{pmatrix}\sin\frac{\phi_r}{2}, \quad \phi_r = \sin\gamma t, \quad \gamma = 0.035\,\mathrm{rad/s}$$

- Trajectory corresponding to a spiral maneuver which, starting from the initial spacecraft attitude, diverges when ϕ_r increases and converges when ϕ_r decreases
- $R\Gamma \equiv R\Gamma_r$ for $t = 0$, i.e. $e_0(0) = 1$, $e_i(0) = 0$ $i = 1, 2, 3$; the initial error angular velocity is $\omega_e(0) = \omega_r(0)$
- $\eta(0) = 0$, $\psi(0) = \delta\omega_e(0) + \dot{\eta}(0) = \delta\omega_r(0)$ (undeformed flexible appendages)

- A comparison between the controllers with gas gets (state feedback and output feedback) has been conducted
- For both the controllers $k_p = 10^5$, $k_d = 3 \times 10^5$
- For the dynamic controller the gain matrix Γ has been set equal to the identity matrix, while the initial conditions for the estimated modal variables are $\hat{\eta}(0) = 0$, $\hat{\psi}(0) = 0$
- The simulations are rendered more realistic by respecting the fact that the gas jets work in a “bang–bang” manner, with saturation values at 60 Nm. This renders harder the control task

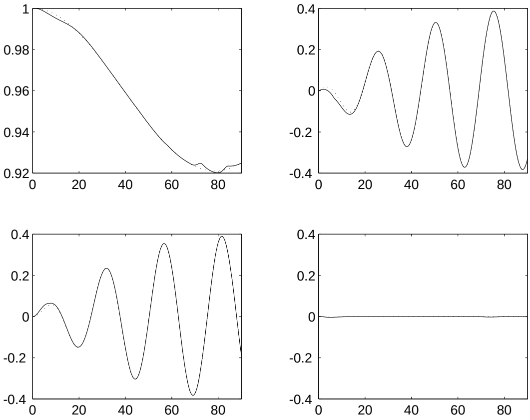
Simulation Results

- A PD controller is capable to track the desired trajectory when the angular velocity is low, but when it increases and the influence of the flexibility becomes too high and unstable, the input saturation



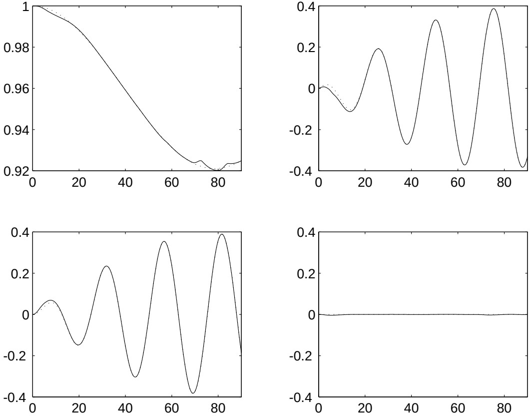
PD Controller – Actual (solid) and reference (dotted) quaternion components

- The state feedback controller is capable to track the reference. The control effort (norm of u_g) similar to the PD case



State Controller – Actual (solid) and reference (dotted) quaternion components

- Analogous result for the output feedback controller. η and ψ are well estimated



Output Controller – Actual (solid) and reference (dotted) quaternion components

Conclusions

- Control laws for rigid/flexible spacecraft can be determined with the Lyapunov approach
- When elastic variables are not known, dynamics in the controller ensure stability
- The absence of measurements of the modal variables is a clear advantage for practical implementations
- The method can be extended to the case of ω not measured
- Extensions are also possible with estimation of perturbations
- The knowledge of system parameters, in particular those describing the elastic motion (natural frequencies and damping ratios), is an obvious limitation, since they are not usually known accurately
- Adaptive robust controllers can avoid this drawback

Thank You!

- See details in
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