Systems & Control Letters 20 (1993) 57–65
North-Holland

Nonlinear regulation for a class of discrete-time systems

B. Castillo

Centro de Investigación y de Estudios Avanzados del I.P.N., Dept. de Ingeniería Eléctrica, Sección de Control Automático Apdo. Postal 14-740, 07000 México D.F., Mexico

S. Di Gennaro

Dipartimento di Ingegneria Elettrica, Università degli Studi di L'Aquila, 67040 Poggio di Roio, L'Aquila, Italy

S. Monaco

Dipartimento di Informatica e Sistemistica, Università di Roma 'La Sapienza', via Eudossiana 18, 00184 Roma, Italy

D. Normand-Cyrot

Laboratoire des Signaux et Systèmes, CNRS-ESE, Plateau de Moulon, 91192 Gif-sur-Yvette, France

Received 23 May 1992
Revised 30 July 1992

Abstract: This paper deals with nonlinear discrete-time regulation for multi-input, multi-output plants. Conditions involving solvability of nonlinear transcendental equations are set. Following the approach recently developped by Isidori and Byrnes (1990) for continuous time systems, the existence of solutions to the posed problem is proved by investigating the properties of the zero dynamics. Finally a condition is given for the existence of an arbitrary approximation of the nonlinear solution.

Keywords: Regulation problem; nonlinear discrete-time systems; approximated solutions.

Correspondence to: Prof. S. Monaco, Dipartimento di Informatica e Sistemistica, Università di Roma 'La Sapienza', via Eudossiana 18, 00184 Roma, Italy.

Introduction

During the last twenty years the regulator problem was extensively studied by several authors. A complete solution for linear systems in state-space form is given in [3,4,5]. More recently in [8] a solution to the problem was proposed in a nonlinear context; the results therein contained represent a significant improvement in the development of nonlinear control theory. A first attempt for solving the same problem in a nonlinear discrete time context is performed in [2] where a regulation scheme is used to get asymptotic output tracking for a SISO process.

In the present paper the general MIMO case is studied making use of the zero output constrained algorithm proposed in [10] for discrete time nonlinear systems. More precisely, following [8] it is shown that the solution of the regulator problem reduces to the solution of transcendental nonlinear equations which represent the discrete-time counterpart of the differential and transcendental equations found for the continuous-time systems. Moreover, their solvability is proved to hold under suitable properties of the zero dynamics of the plant. Since solving the regulator equations may be a difficult task, conditions and methods for computing approximated solutions are given.

The paper is organized as follows. In Section 1 the regulator problem is set in a nonlinear discrete time context. Sufficient conditions for the existence of the solution are stated in Section 2 in terms of solving a set of nonlinear transcendental equations (the regulator equations). In Section 3 the zero output constrained algorithm is presented and sufficient conditions for the solvability of the regulator equations are given, in terms of local properties of the resulting zero dynamics. Section 4 is devoted to study approximated solutions. Finally, an example which illustrates the given results is presented.

1. Problem statement

Let us consider the following multivariable nonlinear discrete-time system:

$$x(k+1) = f(x(k), u(k), w(k)), \tag{1.1a}$$

$$w(k+1) = s(w(k)), \tag{1.1b}$$

$$e(k) = h(x(k)) - r(w(k)), \tag{1.1c}$$

where equations (1.1a) describe the *plant*, with state x, defined in a neighborhood X of the origin of $\mathbb{R}^n$, control $u \in \mathbb{R}^m$, and the disturbance and reference signals w belonging to a neighborhood W of the origin of $\mathbb{R}^q$. Equations (1.1b) represent the external signal generator, the so-called *exosystem*, which models disturbance and reference signals. Finally, (1.1c) describe the *tracking output error* $e \in \mathbb{R}^p$, that is the difference between the plant output $h(x(k))$ and the reference $y_d(k) = r(w(k))$.

Assuming that the vectors $f(x, u, w)$, $s(w)$, $h(x)$ and $r(w)$ are smooth functions on $X \times \mathbb{R}^m \times W$, W, X and W respectively and that $(x, u, w) = (0, 0, 0)$ is an equilibrium point for (1.1), i.e.

$$0 = f(0, 0, 0),$$

$$0 = s(0),$$

$$0 = h(0) = r(0),$$

one defines from (1.1) the following systems:

$$\Sigma: \begin{cases} x(k+1) = f(x(k), u(k), 0) \\ \qquad := f_d(x(k), u(k)), & (1.2a) \\ y(k) = h(x(k)), & (1.2b) \end{cases}$$

$$\Sigma_e: \begin{cases} x_e(k+1) = f_e(x_e(k), u(k)), & (1.3a) \\ e(k) = h_e(x_e(k)), & (1.3b) \end{cases}$$

where $x_e = \mathrm{col}(x, w) \in X \times W$,

$f_e = \mathrm{col}(f(x, u, w), s(w))$

and $h_e = h(x) - r(w)$.

With this in mind, the problem of asymptotically tracking a reference trajectory and rejecting an external disturbance can be formulated in the nonlinear discrete-time context as follows:

State Feedback Discrete Time Regulator Problem (SFDRP): Given the extended system Σ_e, find conditions under which there exists a discrete-time state feedback control law such that the output $e(k)$ goes to zero as k increases and the whole system is asymptotically stable.

More precisely, find a controller of the form

$$u(k) = \gamma(x(k), w(k)), \tag{1.4}$$

where $\gamma(\cdot, \cdot)$ is a smooth mapping defined on $X \times W$, with $\gamma(0, 0) = 0$ such that the following two requirements be satisfied:

(S) The equilibrium $x = 0$ of the dynamics

$$x(k+1) = f_d(x(k), \gamma(x(k), 0)) \tag{1.5}$$

is locally exponentially stable, (i.e. it is asymptotically stable in the first approximation).

(R) There exists a neighborhood U of the origin of $X \times W$ such that, for each initial condition $(x(0), w(0)) \in U$, the solution of the system

$$x(k+1) = f(x(k), \gamma(x(k), w(k)), w(k)), \tag{1.6a}$$

$$w(k+1) = s(w(k)), \tag{1.6b}$$

satisfies the error condition

$$\lim_{k \to \infty} (h(x(k)) - r(w(k))) = 0.$$

In the next section sufficient conditions for the existence of a solution to the SFDRP are given.

Remark 1. (S) and (R) represent the obvious stabilization and regulation requirements respectively. Local exponential stability in absence of disturbances (i.e. $w = 0$) qualifies the stabilizability condition; convergence to zero of the error for the initial states $\binom{x(0)}{w(0)}$ sufficiently close to the origin qualifies the regulation statement.

2. Regulator equations

In order to obtain conditions which ensure the solvability of the regulator problem, the following basic assumptions are stated.

(A1) The equilibrium $w = 0$ of the exosystem is stable in the sense of Lyapunov and all the eigenvalues of the first approximation of the exosystem lie on the unitary circle.

(**A2**) There exists a smooth function $u(k) = v(x(k))$, with $v(0) = 0$ such that the closed loop system

$$x(k+1) = f_{\mathrm{d}}(x(k), v(x(k)))$$

is locally exponentially stable.

On this basis the following result can be set.

Theorem 1. *Assume that* (A1) *and* (A2) *hold. Then the* SFDRP *is locally solvable if there exist two* C^h $(h \geqslant 2)$ *mappings* $\pi(w)$ *and* $c(w)$, *with* $\pi(0) = 0$, $c(0) = 0$, *satisfying*

$$\pi(s(w(k))) = f(\pi(w(k)), c(w(k)), w(k)), \tag{2.1a}$$

$$0 = h(\pi(w(k))) - r(w(k)). \tag{2.1b}$$

(2.1a) and (2.1b) will be called the *discrete-time regulator equations*.

Proof. It is straightforward to verify that the choice of the following controller:

$$\gamma(x(k), w(k)) = c(w(k)) + v(x(k) - \pi(w(k)))$$

satisfies both (S) and (R). For, note that by hypothesis the function $\gamma(x(k), 0) := v(x(k))$ stabilizes the linear approximation of $f_{\mathrm{d}}(x(k), v(x(k)))$, so that (S) is satisfied. Moreover, if (2.1a) holds true, the graph of the mapping $x = \pi(w)$ is a center manifold for the closed-loop dynamics

$$x(k+1) = f(x, \gamma(x, w), w),$$

where by construction $c(w) = \gamma(\pi(w), w)$. By a fundamental property of center manifolds for discrete-time systems [1], the manifold $\pi(w)$ is locally attractive; i.e. for sufficiently small $x(0)$, $w(0)$, it satisfies the condition

$$\| x(k) - \pi(w(k)) \| \leqslant K\beta^k, \quad \forall k, \tag{2.2}$$

where K and β are positive constants and $\beta < 1$. Now for (2.1b) one has

$$e(k) = h(x(k)) - h(\pi(w(k))).$$

Since the exosystem is stable, the term $\pi(w(k))$ is bounded, and therefore, by (2.8), the error

$$\tilde{x}(k) := x(k) - \pi(w(k))$$

is bounded. Hence the function

$$\begin{aligned} e(k) &= \phi(\tilde{x}(k), w(k)) \\ &= h(\tilde{x}(k) + \pi(w(k))) - h(\pi(w(k))) \end{aligned}$$

is an uniformly continuous function over the compact domain of variation of $\tilde{x}(k)$ and therefore, from the fact that $\tilde{x}(k) \to 0$ as $k \to 0$, condition (R) holds true. □

Theorem 1 gives sufficient conditions which assure the solvability of the posed problem. Some more technical details might be clarified in order to prove the necessity under some extra hypotheses.

Remark 2. Assumption (A2) could be replaced by the more general situation of allowing output feedback. In this case, it should be assumed that there exists a function $v(y(k))$ such that the closed loop system

$$x(k+1) = f_{\mathrm{d}}(x(k), v(h(x(k))))$$

is locally exponentially stable, in which case, the controller that satisfies (R) and (S) would have the form

$$\gamma(y(k), w(k)) = c(w(k)) + v(y(k) - h(\pi(w(k)))).$$

3. Zero dynamics and solvability of regulator equations

Solvability of the discrete-time regulator equations (2.1) can be expressed in terms of properties of the zero dynamics of the extended system (1.3). For, let us recall hereafter the discrete-time zero output algorithm proposed in [10]. As pointed out therein it generalizes, to a nonlinear discrete-time context, the inversion algorithm proposed in [12] and used in [8] to solve the continuous time regulator problem.

Definition 1. A given set M in $\mathbb{R}^n$ is said to be *controlled invariant* with respect to the dynamics

(1.2a) if there exists a discrete-time state feedback $u^*(x)$ defined on it such that for any x in M, $f_d(x(k), u^*(x(k)))$ is in M too. □

Definition 2. M is an *output nulling set* for the system (1.2) if:
(i) $h(x(k)) = 0$ for each $x(k) \in M$,
(ii) M is controlled invariant.

Even if in general it is difficult to determine whether or not a maximal output nulling surface does exist, under some regularity conditions, the so-called *zero output constrained dynamic algorithm* proposed in [10] allows the computation of a locally maximal output nulling surface (see Appendix).

To set sufficient conditions for the existence of a solution to the regulator equation (2.1), the following is assumed.

(A3) The zero output constrained algorithm converges for system Σ_e.

Remark 3. Assumption (A3) implies that the zero output constrained algorithm converges for the system Σ. It is enough to choose the coordinates change $z_i(x) = z_{ei}(x, 0)$ once the algorithm has been applied to Σ_e.

Theorem 2. *Under assumptions* (A1), (A2) *and* (A3), *the* SFDRP *is locally solvable if the zero dynamics of* Σ *is hyperbolic* (*i.e. the eigenvalues of its Jacobian linearization do not lie on the unit circle*).

Proof. Application of the zero output constrained dynamic algorithm to the system Σ_e gives the following representation in the new of coordinates:

$$z_{e0}(k+1) = f_{e0}(z_{e0}, z_{e1}, \ldots, z_{ek^*}, \bar{z}_{ek^*}, u),$$
$$z_{e1}(k+1) = f_{e1}(z_{e0}, z_{e1}, \ldots, z_{ek^*}, \bar{z}_{ek^*}, u),$$
$$\vdots \tag{3.1a}$$
$$z_{ek^*}(k+1) = f_{ek^*}(z_{e0}, z_{e1}, \ldots, z_{ek^*}, \bar{z}_{ek^*}, u),$$
$$\bar{z}_{ek^*}(k+1) = \bar{f}_{ek^*}(z_{e0}, z_{e1}, \ldots, z_{ek^*}, \bar{z}_{ek^*}, u). \tag{3.1b}$$

It is easy to verify that the coordinates z_{ei}, $i = 0, \ldots, k^*$, can be chosen so that $\bar{z}_{ek^*}$ has the form $\bar{z}_{ek^*} = (\eta, w)^{\mathrm{T}}$, where $\dim \eta = n - (\sum_{i=0}^{k^*} \rho_i)$. Equations (3.1b) can now be written as

$$\eta(k+1) = q(z_{e0}, z_{e1}, \ldots, z_{ek^*}, \eta, w, u), \tag{3.2}$$
$$w(k+1) = s(w(k)). \tag{3.3}$$

Now $e(k) = 0$ for any $k \geqslant 0$ follows from $z_{ei}(k+1) = 0$, $i = 0, \ldots, k^*$, once that $z_{ei}(k) = 0$. The output nulling feedback u^* can now be computed from (3.1a) with $z_{ei}(k) = 0$, $i = 0, \ldots, k^*$. It is immediately verified that for $w = 0$, the dynamics

$$\eta(k+1) = q(0, 0, \ldots, 0, \eta, 0, u^*) \tag{3.4}$$

coincides with the zero dynamics of system Σ. As noted before, u^* in (3.4) denotes in general situations a solution of the form

$$u = (\bar{u}, \hat{u}) = (\alpha(\eta, w), \beta(\eta, w))$$

with β ensuring that the equilibrium of (3.4) for $w = 0$ be hyperbolic. It follows from the Center Manifold Theorem that there exists a C^h ($h \geqslant 2$) map $\eta(k) = \pi_\eta(w(k))$ which satisfies the equation

$$\pi_\eta(s(w)) = q\big(0, 0, \ldots, 0, \pi_\eta(w), w, \alpha\big(\pi_\eta(w), \beta\big(\pi_\eta(w), w\big)\big)\big). \tag{3.5}$$

The solution to the SFDRP is so given by

$$\pi(w) = \mathrm{col}\big(0, \ldots, 0, \pi_\eta(w)\big),$$
$$c(w) = \mathrm{col}\big(\alpha\big(\pi_\eta(w), w\big), \beta\big(\pi_\eta(w), w\big)\big).$$

Thus the proof is complete. □

It is worthwhile to note that in the original coordinates the solution is computed as follows. From

$$\Phi(x, w) = \mathrm{col}(z_{e0}, z_{e1}, \ldots, z_{ek^*}, \eta, w),$$

since $\partial/\partial x \Phi(x, 0)$ has rank n, one computes

$$x = \phi(z_{e0}, z_{e1}, \ldots, z_{ek^*}, \eta, w),$$

so that the projection in the x-coordinates is given by

$$\pi(w) = \phi\big(0, 0, \ldots, 0, \pi_\eta(w), w\big).$$

4. The approximated solution

Except in very few cases, to find a closed solution to equation (3.5) is a very difficult problem, since it requires to solve a set of nonlinear difference equations. However, in general, approximated solutions are sufficient in practice. Such approximations may be defined on the basis of the expansion with respect to w of the exact solution $\pi_\eta(w)$, namely let

$$\pi_\eta(w(k)) = \pi_\eta^r(w(k)) + \mathrm{O}(w^r), \tag{4.1}$$

where $\pi_\eta^r(w)$ represents a *polynomial approximation of order* r with respect to w of the exact solution and $\mathrm{O}(w^r)$ the remaining terms of order greater than r.

The next result gives a necessary and sufficient condition for the existence of an approximated solution for any arbitrary order r and provides a constructive way to obtain it.

Theorem 3. *Denoting by Q and S the Jacobian matrices*

$$Q = \frac{\partial}{\partial\eta}\left[q(0, 0, \ldots, 0, \eta, w, \alpha(\eta, w), \beta(\eta, w))\right]_{\substack{\eta=0\\ w=0}}$$

and

$$S = \frac{\partial}{\partial w}\left[s(w)\right]_{w=0},$$

for any fixed integer r there exists an approximated solution $\pi_\eta^r(w(k))$ if and only if the eigenvalues of Q do not coincide with $\lambda = \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n}$, $n = 1, \ldots, r$, where λ_{i_n}, $i_n = 1, \ldots, q$, denote the eigenvalues of S.

Proof. The general idea consists in firstly writing the functions $s(w)$, $\pi_\eta(w)$ and $q(w)$ appearing in (3.5) as series expansions with respect to their arguments; then, adequately regrouping and equating the terms of the same power, computing the successive approximated solutions. With this in mind, let

$$M^{(0)} = 1, \qquad M^{(1)} = M,$$
$$M^{(i)} = M \otimes \cdots \otimes M \quad (i \text{ factors}).$$

Under these notations one can rewrite the functions $s(w)$, $\pi_\eta(w)$ and $q(w)$ as follows:

$$s(w) = \sum_{i\geq 1} S_i w^{(i)}, \tag{4.2}$$

$$\pi_\eta(w) = \sum_{n\geq 1} P_n w^{(n)}, \tag{4.3}$$

$$q(0, 0, \ldots, 0, \eta, w, \alpha(\eta, w), \beta(\eta, w)) = \sum_{i,j\geq 0} Q_{ij}\eta^{(i)} \otimes w^{(j)}, \tag{4.4}$$

with $S_1 = S$, and $Q_{10} = Q$.

The problem is to find the coefficients P_n, $n \geq 1$, in such a way that (3.5) holds.

Substituting (4.2), (4.3) and (4.4) in (3.5), after some tedious manipulations the following equality is obtained:

$$\sum_{n\geq 1}\left[\sum_{i=1}^{n} P_i\left(\sum_{\substack{r_1+\cdots+r_i=n\\ r_1,\ldots,r_i\geq 1}} S_{r_1}\otimes\cdots\otimes S_{r_i}\right)\right]w^{(n)}$$
$$= \sum_{n\geq 1}\left[Q_{0n} + Q_{10}P_n + \sum_{\substack{i+j=2\\ i\geq 1; j\geq 0}}^{n} Q_{ij}\left(\sum_{\substack{r_1+\cdots+r_i=n-j\\ r_1,\ldots,r_i\geq 1}} P_{r_1}\otimes\cdots\otimes P_{r_i}\right)\right]w^{(n)}.$$

Equating the coefficients having the same power on w one obtains

$$P_n S_1^{(n)} - Q_{10}P_n = M_n, \quad n \geq 1, \tag{4.5}$$

with

$$M_n = Q_{0n} + \sum_{\substack{i+j=2\\ i\geq 1; j\geq 0}}^{n} Q_{ij}\left(\sum_{\substack{r_1+\cdots+r_i=n-j\\ r_1,\ldots,r_i\geq 1}} P_{r_1}\otimes\cdots\otimes P_{r_i}\right) - \sum_{i=1}^{n-1} P_i\left(\sum_{\substack{r_1+\cdots+r_i=n\\ r_1,\ldots,r_i\geq 1}} S_{r_1}\otimes\cdots\otimes S_{r_i}\right),$$

for $n = 1, \ldots, p$. The terms P_n which define the approximated solution

$$\pi_\eta^r(w(k)) = \sum_{n=1}^{r} P_n w^{(n)}$$

are determined by solving equation (4.5) for each n. By a known result [9], this equation admits a unique solution if and only if the eigenvalues of Q_{10} do not coincide with those of $S_1^{(n)}$. Moreover it can be shown that the eigenvalues of $S_1^{(n)}$ are equal to the products $\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n}$ of the eigenvalues of S_1. For, let us recall that S_1 can be put in normal Jordan form via a transformation matrix, i.e. $S_J = TS_1T^{-1}$, so that

$$S_1^{(n)} = \left(T^{-1}S_JT\right)^{(n)},$$

and using a property of the Kronecker product [9],

$$S_1^{(n)} = (T^{-1})^{(n)}S_J^{(n)}T^{(n)},$$

so the eigenvalues of $S_1^{(n)}$ are the same of those of $S_J^{(n)}$, which are equal to $\lambda = \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n}$. □

The following result gives a sufficient condition for the existence of an approximated solution to the discrete regulator problem.

Corollary 1. *Under the same assumptions as Theorem* 3, *there exists an approximated solution to equation* (3.5) *if the eigenvalues of Q do not lie on the unit circle.*

Proof. The proof is an immediate consequence of assumption (A1) in Section 2. In fact writing the eigenvalues of S_1 in polar form as $\hat{\lambda} = \gamma e^{j\theta}$, since the eigenvalues of S_1 lie on the unitary circle (assumption A1), then $\gamma = 1$. It follows that

$$|\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}| = 1,$$

i.e. the eigenvalues of $S_1^{(n)}$ lie on the unitary circle too and do not coincide with those of Q. □

5. Example

Let us consider the system

$$x_1(k+1) = -x_4 + x_1x_2u_1 + w_1,$$
$$x_2(k+1) = u_1,$$
$$x_3(k+1) = u_2 + x_4 + 2w_1w_2^2,$$
$$x_4(k+1) = u_2 + x_1x_2 - x_3,$$
$$w_1(k+1) = w_1,$$
$$w_2(k+1) = -w_2,$$
$$e_1(k) = x_1 - w_1,$$
$$e_2(k) = x_2 - w_2.$$

The application of the zero output constrained dynamics algorithm gives

$$M_0 = \{(x, w) \mid z_{01} := x_1 - w_1 = 0; \quad z_{02} := x_2 - w_2 = 0\}.$$

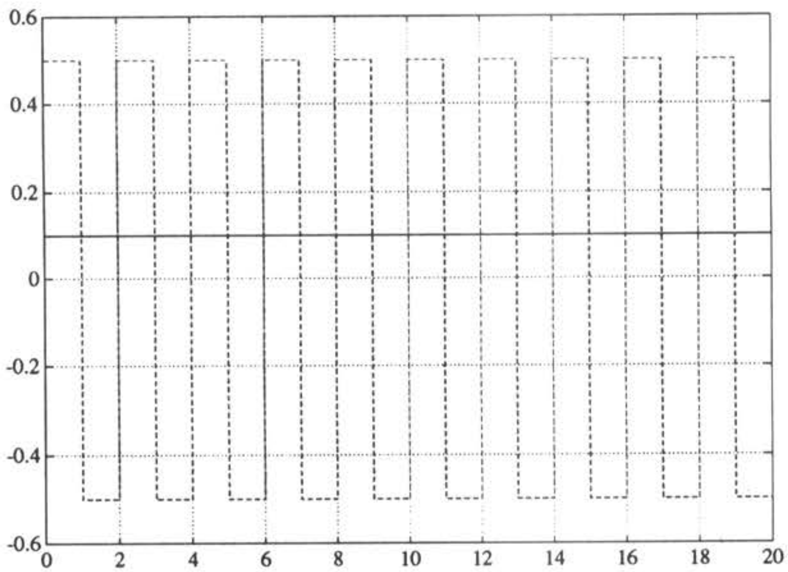

Fig. 1. Reference signals.

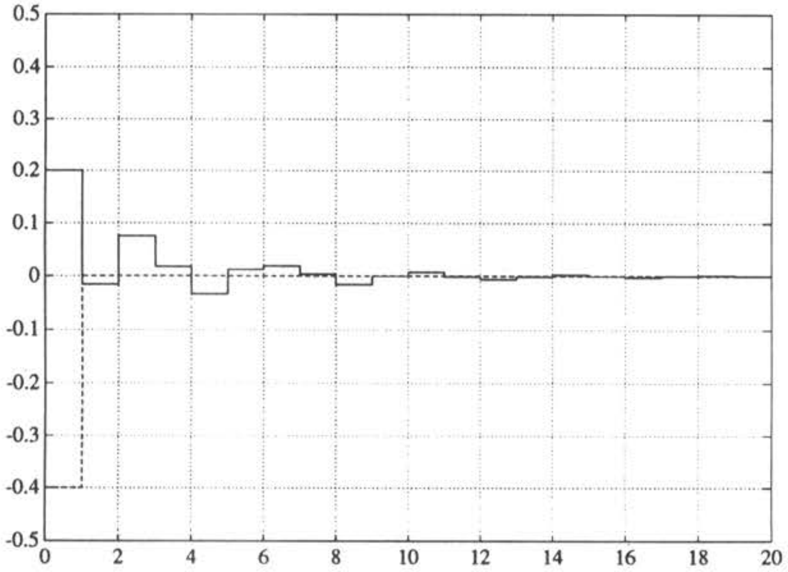

Fig. 2. Outputs errors; solid: e_1, dashed: e_2.

Step 0:

$$R_0^{\bar{z}_0}(z_0) = z_{01}(k) - z_{02}(k) w_1 w_2,$$

$$\lambda_1(\bar{z}_0) = z_{01}(k+1) - z_{02}(k+1) w_1 w_2$$
$$= -x_4 - w_1 w_2^2,$$

$$M_1 = \{(x, w) \mid z_{01} := x_1 - w_1 = 0;$$
$$z_{02} := x_2 - w_2 = 0;$$
$$z_{03} := x_4 + w_1 w_2^2 = 0\}.$$

By solving equation (A.1), one has

$$u_1 = -w_2,$$

$$u_2 = x_3 - w_1 w_2 - w_1 w_2^2,$$

so that the unobservable dynamics are described by

$$x_3(k+1) = x_3 - w_1 w_2 = q(\eta, w).$$

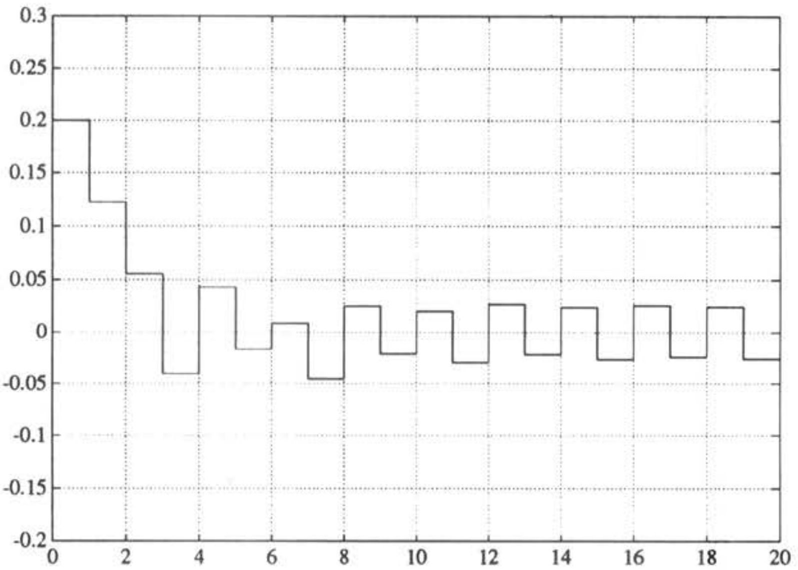

Fig. 3. Unobservable dynamics.

By (3.5) a solution to the SFDRP can be computed, obtaining

$$\pi_1(w) = w_1, \qquad \pi_3(w) = \tfrac{1}{2}w_1w_2,$$

$$\pi_2(w) = w_2, \qquad \pi_4(w) = -w_1w_2^2,$$

$$c_1(w) = w_1, \qquad c_2(w) = -\tfrac{1}{2}w_1w_2 - w_1w_2^2.$$

Simulation results are shown in Figures 1 through 3.

6. Conclusions

Following the approach developped in [8], the regulation problem for a general class of MIMO nonlinear discrete-time systems has been studied. It is shown that the existence of a solution is implied by the solvability of the regulation equations, a set of nonlinear transcendental equations. Generalizing results from the continuous time context, the solvability of these equations is related to the property of the zero dynamics of the system. Moreover, conditions for the existence of an approximated solution are given.

Appendix: Zero output constrained dynamic algorithm

Step 0. Let $M_0 = h^{-1}(0)$ be the $n - \rho_0$ dimensional surface and choose suitable new coordinates $x = \binom{z_0}{\bar z_0}$ such that

$$M_0 = \{x \in \mathbb{R}^n \mid z_0 = 0\}, \quad \dim z_0 = \rho_0,$$

f can be decomposed as

$$f = \begin{pmatrix} f_0(z_0, \bar z_0, u) \\ \bar f_0(z_0, \bar z_0, u) \end{pmatrix}.$$

Assume that

$$\operatorname{rank} \frac{\partial f_0(0, \bar z_0, u)}{\partial u} = r_0$$

is constant on U_0, an open in M_0. It follows from the rank theorem that there exist $\rho_0 - r_0$ analytic functions

$$R_0^{\bar z_0}(z_0) : \mathbb{R}^{\rho_0} \to \mathbb{R}^{\rho_0 - r_0}$$

of full rank at any $\bar z_0$, with $R_0^{\bar z_0}(0) = 0$ and such that

$$\frac{\partial R_0^{\bar z_0}(z_0(k+1))}{\partial u(k)} = \frac{\partial R_0^{\bar z_0}(z_0(k+1))}{\partial z_0(k+1)} \frac{\partial z_0(k+1)}{\partial u(k)} \equiv 0,$$

i.e. $R_0^{\bar z_0}(z_0(k+1))$ does not depend on u.

The output nulling constraint, i.e. $y(k) = 0$ for any $k \geqslant 0$, implies $z_0(k+1) = 0$ when $z_0(k) = 0$. It follows that

$$\lambda_1(\bar z_0) := R_0^{\bar z_0}(f_0(0, \bar z_0, 0)) = R_0^{\bar z_0}(z_0(k+1)) = R_0^{\bar z_0}(0) = 0.$$

Assume that $\operatorname{rank}(\lambda_1) = \rho_1 \leqslant \rho_0 - r_0$ is constant and define on U_0 new coordinates $x = (z_0^{\mathrm T}, z_1^{\mathrm T}, \bar z_1^{\mathrm T})^{\mathrm T}$ such that: $z_1 = \bar\lambda_1(\bar z_0)$, where $\dim z_1 = \rho_1$ and where $\bar\lambda_1$ denotes ρ_1 linearly independent components of $\lambda_1(\bar z_0)$. Denote by M_1 the following surface:

$$M_1 = \{x \in \mathbb{R}^n \mid z_0 = 0, \ z_1 = 0\},$$

with $\dim M_1 = n - \rho_0 - \rho_1$.

The function f can be further decomposed as

$$f = \begin{pmatrix} f_0(z_0, z_1, \bar z_1, u) \\ f_1(z_0, z_1, \bar z_1, u) \\ \bar f_1(z_0, z_1, \bar z_1, u) \end{pmatrix}.$$

The output nulling constraint $y(k) = 0$ for any $k \geqslant 0$ implies, at least on U_1, an open in M_1, that

$$z_0(k+1) = z_1(k+1) = 0$$

when

$$z_0(k) = z_1(k) = 0,$$

which is ensured by solving with respect to u the following set of equations:

$$f_0(0, 0, \bar z_1, u) = 0, \quad f_1(0, 0, \bar z_1, u) = 0.$$

Step $k-1$. At the $(k-1)$-th step one defines the change of coordinates on U_0:

$$x = \left(z_0^{\mathrm T}, z_1^{\mathrm T}, \ldots, z_k^{\mathrm T}, \bar z_k^{\mathrm T}\right)^{\mathrm T},$$

with $z_k = \bar\lambda_k(\bar z_{k-1})$ chosen as ρ_k linearly independent components of $\lambda_k(\bar{z}_{k-1})$ which is assumed to has locally constant rank ρ_k. The $(n - \sum_{i=0}^{k}\rho_i)$-dimensional surface M_k will then be described by

$$M_k = \{x \in \mathbb{R}^n \mid z_0 = 0,\ z_1 = 0, \dots, z_k = 0\}.$$

At this step the function f can be locally expressed as

$$f = \begin{pmatrix} f_0(z_0, z_1, \dots, z_k, \bar{z}_k, u) \\ f_1(z_0, z_1, \dots, z_k, \bar{z}_k, u) \\ \vdots \\ f_k(z_0, z_1, \dots, z_k, \bar{z}_k, u) \\ \bar{f}_k(z_0, z_1, \dots, z_k, \bar{z}_k, u) \end{pmatrix}.$$

The output nulling constraint leads now to solve with respect to u the $(\rho_0 + \cdots + \rho_k)$-dimensional set of equations

$$\begin{aligned} &f_0(0, 0, \dots, 0, \bar{z}_k, u) = 0, \\ &f_1(0, 0, \dots, 0, \bar{z}_k, u) = 0, \\ &\vdots \\ &f_k(0, 0, \dots, 0, \bar{z}_k, u) = 0. \end{aligned} \tag{A.1}$$

Assuming that the functions $\bar{\lambda}_i$, for any $i \geqslant 0$, have locally constant rank, the algorithm stops when there exists an integer k^* such that $M_{k^*} = M_{k^*-1}$. In this case, from the set of equations (A.1), a solution $u^*(\bar{z}_{k^*})$ defined on U_{k^*}, an open of M_{k^*}, can be obtained by means of the implicit function theorem. It must be noted that such a solution might not be unique: in fact, if the rank of equations (A.1), say d, is less than p, the number of outputs, then only d components of the input vector can be obtained, the others being free. Reordering the input vector as $u = (\bar{u}, \hat{u})$, where $\dim \bar{u} = d$, the zero output constrained dynamics are now described by

$$\bar{f}_{k^*}(0, 0, \dots, 0, \bar{z}_{k^*}, \bar{u}^*(\bar{z}_{k^*}), \hat{u}) \tag{A.2}$$

where M_{k^*} denotes the largest surface which is invariant under the modified dynamics and is contained in $h^{-1}(0)$.

Acknowledgment

Mexican CONACYT and Italian M.U.R.S.T. partially supported the present work. The authors are indebted to an anonymous referee for constructive comments and for improving the proof of Theorem 1.

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