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Adaptive output feedback control of synchronous motors

S. DI GENNARO†

This paper solves the problem of position tracking for a permanent magnet synchronous motor when the angular velocity is not measured and all the parameters of the dynamic equations are unknown. The mathematical model is expressed in the reference fixed to the stator. The adaptation algorithm gives a degree of robustness to the control scheme since it enables one to identify the motor's parameters under the hypothesis of persistent excitation, which is always satisfied when the unknown parameters are only the stator resistance and the load torque. The identification is exponential in these cases. Moreover, it is shown that the closed-loop trajectory corresponding to the nominal parameter values is not influenced by variations of stator resistance and load torque. The performances of the proposed controller are compared with those of a controller which solves the tracking problem with an observer for the angular velocity, designed on the basis of the motor's nominal parameters.

1. Introduction

Permanent magnet synchronous motors have many features which render them appealing in many applications including compact structure, high air-gap flux density, high power to inertia ratio and high torque capability. In addition, they can be used as brushless dc motors by using an appropriate electronic commutation, so showing characteristics of a conventional dc motor: speed proportional to the supply voltage, torque proportional to the armature current, start/stall torque higher than the running torque, high efficiency and response, and linearity. The elimination of brushes and commutators also solves the problem associated with contacts and improves reliability and operating life.

There is a trend to replace dc motors by ac synchronous, permanent magnet and switched reluctance ones, which are more difficult to control but offer several advantages. Their characteristics render them suitable as direct drives in various applications, such as in robotics and aerospace engineering, in which motor weight is a crucial factor. The main drawbacks of these motors are that they are non-linear and involve parameters which may vary during operation.

During motor operation, in fact, some of the parameters appearing in the equations describing its dynamics may vary. When a sinusoidal flux distribution is assumed, these equations depend on the stator winding resistance R and inductance L, the torque load C_l, the inertia J, the viscous friction coefficient f and the motor torque constant k_{m}. If these parameters can be considered constant, various control strategies are available. Among them, a feedback linearization technique can be used (Bodson and Chiasson 1989, Zribi and Chiasson 1991, Bodson *et al.* 1993), which guarantees good performances even when not implemented by up-to-date technologies. However, in general these parameters cannot be considered constant or perfectly known; as a consequence, these control strategies fail to be robust with respect to parameter variations. In particular, C_l and R vary during operation. Moreover, the other parameters may be known with some imprecision, and can be identified by off-line algorithms by the designer in each particular application (Bodson *et al.* 1993). This implies the commercial problem of control tuning. A solution to this problem, which gives at the same time a degree of robustness to the control strategy, is given by adaptive algorithms which result from recent developments in non-linear adaptive control (Kokotovic 1991, Kannellakopoulos *et al.* 1991, Maino *et al.* 1993, 1995, Seto *et al.* 1994). Computational complexity has been an important problem to solve in these cases, but powerful digital signal processors (DSP) are now available at low cost. They allow the implementation of sophisticated and complex non-linear adaptive algorithms designed to meet increasingly more demanding performance requirements in applications.

Another aspect to take into account is the availability of state variables. While the other state variables may be measurable in permanent magnet motors, the angular velocity is sometimes not directly measured but is estimated from the angular position. Rotor position is in fact critically important in vector control in order to facilitate the electronic commutation. A position sensor must therefore be installed onto the machine. The measurement of the angular speed would require a separate sensor, costly and mechanically impracticable. The common solution is to estimate the speed from position measurements by means of recursive algorithms (Lorenz and Van Patten 1991), whose limitations are the time-lag in the estimation and the introduced

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quantization noise from both sensor and estimation scheme. Digital filters can be used for the latter problem, but they imply the use of old samples, so reducing the accuracy of the speed estimation. Recursive algorithms which are sensor based thus do not provide an accurate estimate with high resolution. On the other hand, the use of state observers, based on simplified linearized dynamics or on the exact non-linear model, requires the knowledge of system parameters. These values can be known only with a certain degree of precision or can vary during the operation of the motor. In this case their application in controlling the system is not workable. The design of dynamic compensators, now implementable with the existing DSP, is a solution to this problem.

Motivated by the previous discussion, in this work we design a dynamic compensator which solves the problem of position tracking when the angular position is not measured and when the system parameters are slowly varying with respect to a nominal value or are unknown. The so-called 'direct' component of the current is also forced to track a desired reference. The dynamics of the motor are expressed in the frame fixed with the stator; in fact, as clarified and justified by the digital implementation of the control laws (Georgiou *et al.* 1992, Chelouah *et al.* 1993), this choise turns out to be more appropriate.

Applications of non-linear control theory (Isidori 1993, Khalil 1992) to the control of synchronous motors can be found in Bodson and Chiasson (1989), Zribi and Chiasson (1991) and Bodson *et al.* (1993), on the basis of perfect parameter knowledge, whereas in Grčar *et al.* (1991) and in Marino *et al.* (1995) non-linear adaptive controllers are designed when all the state is available for measurements. The present paper is the extension of the approach illustrated in Marino *et al.* (1995) to the case of partial state measurement.

The paper is organized as follows. We recall the mathematical model of a PM synchronous motor in the next section, and the state-feedback stabilizing and adaptive controls in §3. In §4 we present the main result of this paper, consisting of an adaptive controller based on measurements of the angular position and the currents, together with a sensitivity analysis of the nominal closed-loop trajectories in presence of parameter variations. In §5 simulation results are shown. Some comments conclude the paper.

2. Mathematical model

In this work we assume that the motor under study satisfies the usual hypotheses (linear magnetic materials, symmetry of the rotor and between the two phases, nonlinear flux density distribution due to air gap geometry only, negligible magnetic hysteresis and Foucault currents) under which it can be satisfactorily modelled by means of the usual simplified equations written in the stator reference frame (α, β). Moreover, we suppose that the motor has no saliency and a sinusoidal flux density distribution.

Let ϑ be the rotor angle, ω the rotor speed,

$$v = \begin{pmatrix} v_\alpha \\ v_\beta \end{pmatrix}, \quad i = \begin{pmatrix} i_\alpha \\ i_\beta \end{pmatrix}$$

the stator voltage and current vectors. The equations describing the dynamics of a permanent magnet synchronous motor in the (α, β) frame are (Leonard 1985)

$$\left.\begin{aligned} \dot{\vartheta} &= \omega \\ \dot{\omega} &= \frac{k_m}{J}(-i_\alpha \sin p\vartheta + i_\beta \cos p\vartheta) - \frac{f}{J}\omega - \frac{C_l}{J} \\ &= \frac{1}{\alpha_1}(-i_\alpha q_1 + i_\beta q_0) - \alpha_2\omega - c \\ \frac{di_\alpha}{dt} &= -\frac{R}{L}i_\alpha + \frac{k_m}{L}\omega \sin p\vartheta + \frac{1}{L}v_\alpha \\ &= -\alpha_3 i_\alpha + \alpha_4 q_1 \omega + \frac{1}{\alpha_5}v_\alpha \\ \frac{di_\beta}{dt} &= -\frac{R}{L}i_\beta + \frac{k_m}{L}\omega \cos p\vartheta + \frac{1}{L}v_\beta \\ &= -\alpha_3 i_\beta - \alpha_4 q_0 \omega + \frac{1}{\alpha_5}v_\beta \end{aligned}\right\} \quad (1)$$

where q_0, q_1 indicate the quantities

$$q_0 = \cos p\vartheta, \quad q_1 = \sin p\vartheta$$

C_l is the resistant load torque, R the stator resistance, L the inductance, p the pole pair number, J the inertia, f the friction coefficient, $k_m = p\Phi$ the motor torque constant, Φ the flux produced by the permanent magnets, and

$$c = \frac{C_l}{J}, \quad \alpha_1 = \frac{J}{k_m}, \quad \alpha_3 = \frac{R}{L}, \quad \alpha_5 = L$$
$$\alpha_2 = \frac{f}{J}, \quad \alpha_4 = \frac{k_m}{L} \quad k_m = p\Phi, \quad \alpha_6 = \frac{k_m}{J} = \frac{1}{\alpha_1}$$

Note that the dynamics of the variables q_0, q_1 are

$$\dot{q}_0 = -p\omega q_1, \quad \dot{q}_1 = p\omega q_0 \quad (2)$$

A useful represetation of a PM synchronous motor is also that in the fixed rotor frame (d, q), introduced by Park (Park 1929, Leonard 1985) and based on the transformation given by the orthogonal matrix

$$T(\vartheta) = \begin{pmatrix} q_0 & q_1 \\ -q_1 & q_0 \end{pmatrix}$$

With $T(\vartheta)$ the vectors i and v, expressed in the (α, β) frame, are transformed into vectors in the (d, q) frame

$$\left.\begin{aligned} i_d &= i_\alpha q_0 + i_\beta q_1 & v_d &= v_\alpha q_0 + v_\beta q_1 \\ i_q &= -i_\alpha q_1 + i_\beta q_0 & v_q &= -v_\alpha q_1 + v_\beta q_0 \end{aligned}\right\} \quad (3)$$

From (1) it is clear that $k_m i_q = k_m(-i_\alpha q_1 + i_\beta q_0)$ represents the electromagnetic torque generated by the motor. A common control objective is to track a desired reference trajectory ω_r, imposing at the same time a reference i_{dr} for the 'direct component' i_d of the current. This allows one to realize the so-called 'field weakening' at high speed. Hence, i_{dr} is a function of the reference velocity ω_r: when $\omega_r \le \omega_n$ (ω_n is the nominal speed), then $i_{dr} = 0$, so maximizing the produced torque, whereas if $\omega_r > \omega_n$, then $i_{dr} = i_{dr}(\dot\vartheta_r)$.

Note that if $\psi = (k_m/J) i_{qr} = (k_m/J)(-i_{\alpha r} q_1 + i_{\beta r} q_0)$ is a desired value for the angular acceleration, and $i_{dr} = i_{\alpha r} q_0 + i_{\beta r} q_1$ is the reference for i_d, then the references for the α, β components of the current are such that

$$\begin{pmatrix} q_0 & q_1 \\ -q_1 & q_0 \end{pmatrix} \begin{pmatrix} i_{\alpha r} \\ i_{\beta r} \end{pmatrix} = \begin{pmatrix} i_{dr} \\ \alpha_1 \psi \end{pmatrix}$$

i.e.

$$i_{\alpha r} = i_{dr} q_0 - \alpha_1 q_1 \psi, \quad i_{\beta r} = i_{dr} q_1 + \alpha_1 q_0 \psi \quad (4)$$

The control objective is to determine a dynamic controller which ensures the tracking of the angular position ϑ_r and the reference i_{dr} while i_α, i_β and ϑ are measured, when the parameters α_i, $i = 1, \ldots, 6$, and the load C_l are unknown or approximately known.

3. Recalls on state-feedback stabilization and adaptive control

In this section we briefly recall some facts known from the literature (Zribi and Chiasson 1991, Bodson *et al.* 1993, Marino *et al.* 1995). Some of them, such as the adaptive state-feedback control (Marino *et al.* 1995), are re-expressed in the (α, β) reference since a slight simplification can be obtained and also to allow a clear comparison with the new result presented in the following section.

3.1. *Feedback stabilization—perfect parameter knowledge*

When the state variables, the parameters and the load torque are known, a simple state feedback controller can be designed in order to obtain exponential tracking of a desired angular trajectory ϑ_r. We use here a backstepping procedure (Seto *et al.* 1994). Denoting by

$$\vartheta_e = \vartheta - \vartheta_r$$

the tracking error, and defining by

$$\omega_r = \dot\vartheta_r - k_1 \vartheta_e = -k_1 \vartheta + (\dot\vartheta_r + k_1 \vartheta_r) \quad (5)$$

the angular reference and by

$$\omega_e = \omega - \omega_r$$

the tracking error the ω, one obtains

$$\left.\begin{aligned} \dot\vartheta_e &= -k_1 \vartheta_e + \omega_e \\ \dot\omega_e &= \frac{1}{\alpha_1}(-i_\alpha q_1 + i_\beta q_0) - \alpha_2 \omega - c - \dot\omega_r \\ &= \frac{1}{\alpha_1}(-i_{\alpha e} q_1 + i_{\beta e} q_0) + \frac{1}{\alpha_1}(-i_{\alpha r} q_1 + i_{\beta r} q_0) \\ &\quad - \alpha_2 \omega - c - \dot\omega_r \end{aligned}\right\} \quad (6)$$

with $\dot\omega_r = -k_1 \omega + (\ddot\vartheta_r + k_1 \dot\vartheta_r)$, $i_{\alpha r}$, $i_{\beta r}$ the references defined in (4), where

$$i_{dr} = i_{dr}(\dot\vartheta_r), \quad \psi = -\vartheta_e - k_2 \omega_e + \alpha_2 \omega + c + \dot\omega_r$$

are the desired i_d and angular acceleration references. Hence

$$\dot\omega_e = -\vartheta_e - k_2 \omega_e + \alpha_6(-i_{\alpha e} q_1 + i_{\beta e} q_0)$$

where $\alpha_6 = 1/\alpha_1$. Moreover, one calculates the error dynamics

$$\left.\begin{aligned} \frac{di_{\beta e}}{dt} &= -\alpha_3 i_\beta - \alpha_4 q_0 \omega + \frac{1}{\alpha_5} v_\beta - \frac{di_{\beta r}}{dt} \\ \frac{di_{\alpha e}}{dt} &= -\alpha_3 i_\alpha + \alpha_4 q_1 \omega + \frac{1}{\alpha_5} v_\alpha - \frac{di_{\alpha r}}{dt} \end{aligned}\right\} \quad (7)$$

Therefore, the controls

$$\left.\begin{aligned} v_\beta &= -\alpha_5 \varrho_\beta, \quad \varrho_\beta = \alpha_6 q_0 \omega_e + k_3 i_{\beta e} - \alpha_3 i_\beta \\ &\qquad - \alpha_4 q_0 \omega - \frac{di_{\beta r}}{dt} \\ v_\alpha &= -\alpha_5 \varrho_\alpha, \quad \varrho_\alpha = -\alpha_6 q_1 \omega_e + k_4 i_{\alpha e} - \alpha_3 i_\alpha \\ &\qquad + \alpha_4 q_1 \omega - \frac{di_{\alpha r}}{dt} \end{aligned}\right\} \quad (8)$$

are such that

$$\frac{di_{\beta e}}{dt} = \alpha_6 q_1 \omega_e - k_3 i_{\beta e}, \quad \frac{di_{\alpha e}}{dt} = -\alpha_6 q_0 \omega_e - k_4 i_{\alpha e}$$

Denoting by $x_e = (\vartheta_e \quad \omega_e \quad i_{\beta e} \quad i_{\alpha e})^T$ the error variable vector, one obtains the time-varying linear system

$$\left.\begin{aligned} \dot x_e &= A(t) x_e \\ A(t) &= \begin{pmatrix} -k_1 & 1 & 0 & 0 \\ -1 & -k_2 & \alpha_6 q_0 & -\alpha_6 q_1 \\ 0 & -\alpha_6 q_0 & -k_3 & 0 \\ 0 & \alpha_6 q_1 & 0 & -k_4 \end{pmatrix} \end{aligned}\right\} \quad (9)$$

ensuring exponential tracking of the angular position. This can be seen by taking the time-derivative of the radially unbounded Lyapunov function $V = \frac{1}{2} x_e^T x_e$ along the trajectories of (9), obtaining

$$\dot{V} = -x_e^T Q x_e < 0, \quad Q = \begin{pmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}$$

where $k_1, \ldots, k_4 > 0$, since $(A + A^T)/2 = -Q$. Hence, $x_e = 0$ is a globally exponentially stable equilibrium point (Khalil 1992).

3.2. *Adaptive state-feedback control*

When the parameters and the load torque in (1) are unknown, or known up to a certain degree of precision, the control (8) cannot be implemented. If the whole state is measurable, an adaptive version of this control, proposed by Marino *et al.* (1995), is here briefly recalled. This approach is re-formulated in the (α, β) frame and a little simplification is introduced with respect to that in Marino *et al.* (1995). Defining the angular reference ϑ_r, the tracking error ϑ_e and the error ω_e as in §3.1, one obtains equations (6), where the reference signals $i_{\alpha r}$, $i_{\beta r}$ for the currents i_α, i_β are now given by

$$i_{\beta r} = i_{dr} q_1 + \hat{\alpha}_1 q_0 \psi \tag{10}$$

$$i_{\alpha r} = i_{dr} q_0 - \hat{\alpha}_1 q_1 \psi \tag{11}$$

$$\psi = -\vartheta_e - k_2 \omega_e + \hat{\alpha}_2 \omega + \hat{c} + \dot{\omega}_r = \psi_0 + \psi_1 \omega$$

where by $\hat{\alpha}_i$ we denote the estimates of α_i, $i = 1, \ldots, 6$. In the estimated angular acceleration ψ the explicit dependence on ω has been put in evidence. Therefore

$$\dot{\omega}_e = -\vartheta_e - k_2 \omega_e + \alpha_6(-i_{\alpha e} q_1 + i_{\beta e} q_0) - \tilde{c} - \frac{\tilde{\alpha}_1}{\alpha_1}\psi - \tilde{\alpha}_2 \omega$$

where the relationship $\hat{\alpha}_1 = \alpha_1 - \tilde{\alpha}_1$ has been used.

As far as the dynamics of the errors $i_{\alpha e}$, $i_{\beta e}$ are concerned, one obtains again equations (7), where

$$\left.\begin{aligned} \frac{di_{\beta r}}{dt} &= \frac{di_{dr}}{dt} q_1 + i_{dr}\dot{q}_1 + \dot{\hat{\alpha}}_1 q_0 \psi + \hat{\alpha}_1 \dot{q}_0 \psi + \hat{\alpha}_1 q_0 \dot{\psi} \\ \frac{di_{\alpha r}}{dt} &= \frac{di_{dr}}{dt} q_0 + i_{dr}\dot{q}_0 - \dot{\hat{\alpha}}_1 q_1 \psi - \hat{\alpha}_1 \dot{q}_1 \psi - \hat{\alpha}_1 q_1 \dot{\psi} \end{aligned}\right\} \tag{12}$$

with $\dot{q}_0$, $\dot{q}_1$ given by (2) and $\dot{\psi} = \dot{\psi}_0 + \dot{\psi}_1 \omega + \psi_1[\alpha_6(-i_\alpha q_1 + i_\beta q_0) - \alpha_2 \omega - c]$. The dynamics of $\hat{\alpha}_1$ have not yet been designed; however, note that they must contain only quantities which can be implemented and therefore cancelled by the control. Hence, the control laws

$$v_\beta = -\hat{\alpha}_5 \varrho_\beta$$

$$v_\alpha = -\hat{\alpha}_5 \varrho_\alpha$$

$$\begin{aligned} \varrho_\beta = {} & \hat{\alpha}_6 q_0 \omega_e + k_3 i_{\beta e} - \hat{\alpha}_3 i_\beta - \hat{\alpha}_4 q_0 \omega - \frac{di_{dr}}{dt} q_1 \\ & - i_{dr} p q_0 \omega - \dot{\hat{\alpha}}_1 q_0 \psi + \hat{\alpha}_1 p \omega q_1 \psi \\ & - \hat{\alpha}_1 q_0 [\dot{\psi}_0 + \dot{\psi}_1 \omega + \psi_1 [\hat{\alpha}_6(-i_\alpha q_1 + i_\beta q_0) - \hat{\alpha}_2 \omega - \hat{c}]] \end{aligned}$$

$$\begin{aligned} \varrho_\alpha = {} & -\hat{\alpha}_6 q_1 \omega_e + k_4 i_{\alpha e} - \hat{\alpha}_3 i_\alpha + \hat{\alpha}_4 q_1 \omega - \frac{di_{dr}}{dt} q_0 \\ & + i_{dr} p q_1 \omega + \dot{\hat{\alpha}}_1 q_1 \psi + \hat{\alpha}_1 p \omega q_1 \psi \\ & + \hat{\alpha}_1 q_0 [\dot{\psi}_0 + \dot{\psi}_1 \omega + \psi_1 [\hat{\alpha}_6(-i_\alpha q_1 + i_\beta q_0) - \hat{\alpha}_2 \omega - \hat{c}]] \end{aligned}$$

are such that $(\hat{\alpha}_5 = \alpha_5 - \tilde{\alpha}_5, \hat{\alpha}_6 = \alpha_6 - \tilde{\alpha}_6)$

$$\begin{aligned} \frac{di_{\beta e}}{dt} = {} & -\alpha_6 q_0 \omega_e - k_3 i_{\beta e} - \tilde{\alpha}_3 i_\beta - \tilde{\alpha}_4 q_0 \omega + \frac{\tilde{\alpha}_5}{\alpha_5} \varrho_\beta + \tilde{\alpha}_6 q_0 \omega_e \\ & - \hat{\alpha}_1 q_1 \psi_1 [\tilde{\alpha}_6(-i_\alpha q_1 + i_\beta q_0) - \tilde{\alpha}_2 \omega - \tilde{c}] \end{aligned}$$

$$\begin{aligned} \frac{di_{\alpha e}}{dt} = {} & \alpha_6 q_1 \omega_e - k_4 i_{\alpha e} - \tilde{\alpha}_3 i_\alpha + \tilde{\alpha}_4 q_1 \omega + \frac{\tilde{\alpha}_5}{\alpha_5} \varrho_\alpha - \tilde{\alpha}_6 q_1 \omega_e \\ & + \hat{\alpha}_1 q_1 \psi_1 [\tilde{\alpha}_6(-i_\alpha q_1 + i_\beta q_0) - \tilde{\alpha}_2 \omega - \tilde{c}] \end{aligned}$$

Denoting by $\tilde{\alpha} = (\tilde{c} \quad \tilde{\alpha}_1 \quad \ldots \quad \tilde{\alpha}_6)^T$ the estimate error vector, the dynamics of the error vector x_e, defined in §3.1, are given by

$$\dot{x}_e = A(t) x_e + \Gamma(t) D \tilde{\alpha}$$

where (see below)

$$A(t) = \begin{pmatrix} -k_1 & 1 & 0 & 0 \\ -1 & -k_2 & \alpha_6 q_0 & -\alpha_6 q_1 \\ 0 & -\alpha_6 q_0 & -k_3 & 0 \\ 0 & \alpha_6 q_1 & 0 & -k_4 \end{pmatrix}, \qquad D = \text{diag}\left\{1, \frac{1}{\alpha_1}, 1, 1, 1, \frac{1}{\alpha_5}, 1\right\}$$

$$\Gamma(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -\psi & -\omega & 0 & 0 & 0 & 0 \\ \hat{\alpha}_1 q_0 \psi_1 & 0 & \hat{\alpha}_1 q_0 \psi_1 \omega & -i_\beta & -q_0 \omega & \varrho_\beta & q_0 \omega_e - \hat{\alpha}_1 q_0 \psi_1 (-i_\alpha q_1 + i_\beta q_0) \\ -\hat{\alpha}_1 q_1 \psi_1 & 0 & -\hat{\alpha}_1 q_1 \psi_1 \omega & -i_\alpha & q_1 \omega & \varrho_\alpha & -q_1 \omega_e + \hat{\alpha}_1 q_1 \psi_1 (-i_\alpha q_1 + i_\beta q_0) \end{pmatrix}$$

The updating laws $\dot{\hat{c}}$, $\dot{\hat{\alpha}}_i$, $i = 1, \ldots, 6$, can be designed such that the Lyapunov function

$$V = \tfrac{1}{2}x_e^T x_e + \tfrac{1}{2}\tilde{\alpha}^T D \Lambda^{-1} \tilde{\alpha}$$

with $\Lambda = \text{diag}\{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\}$, has time derivative semi-positive definite

$$\begin{aligned}\dot{V} &= x_e^T(A x_e + \Gamma D \tilde{\alpha}) - \tilde{\alpha}^T D \Lambda^{-1} \dot{\hat{\alpha}} \\ &= -x_e^T Q x_e + x_e^T \Gamma D \tilde{\alpha} - \tilde{\alpha}^T D \Lambda^{-1} \dot{\hat{\alpha}} \\ &= -x_e^T Q x_e + \tilde{\alpha}^T D(\Gamma^T x_e - \Lambda^{-1}\dot{\hat{\alpha}})\end{aligned}$$

where $\hat{\alpha} = (\hat{c} \quad \hat{\alpha}_1 \quad \ldots \quad \hat{\alpha}_6)^T$, and Q is as in §3.1. Setting

$$\dot{\hat{\alpha}} = \Lambda \Gamma^T x_e$$

one has $\dot{V} = -x_e^T Q x_e \leq 0$. Therefore, $x_e \in L_2 \cap L_\infty$; moreover, it is easy to check that $\dot{x} \in L_\infty$. Applying Barbalat's lemma, it follows that $\lim_{t\to\infty} x_e = 0$. Finally, under the persistent excitation condition (Narendra and Annaswamy 1989, Marino and Tomei 1995), i.e. if there exists a positive constant T such that

$$\int_{t_0}^{t_0+T} \Gamma^T(\tau)\Gamma(\tau)\,d\tau \tag{13}$$

is positive-definite for any $t_0 \geq 0$, then for the linear time-varying system

$$\begin{aligned}\dot{x}_e &= A(t)x_e + \Gamma(t) D \tilde{\alpha} \\ \dot{\tilde{\alpha}} &= -\Lambda \Gamma^T(t) x_e\end{aligned}$$

$\begin{pmatrix} x_e \\ \tilde{\alpha} \end{pmatrix} = 0$ is a globally exponentially stable equilibrium point (Narendra and Annaswamy 1989).

4. Output-feedback adaptive control

In this section we present the main contribution of this paper. We suppose that the angular velocity ω is not measurable. Moreover, C_l and the parameters appearing in (1) are also unknown. We derive hereinafter a dynamic controller, ensuring asymptotic tracking of the references ϑ_r, i_{dr}. We show that the parameter c is always identified and that α_3 is identified in all physical situations of importance; we also note that, under the hypothesis of persistence of excitation, the tracking and the parameter identification are exponential. Finally, we carry out a sensitivity analysis of the nominal closed-loop trajectory in presence of parameter variations.

4.1. *Control synthesis*

We use the notations introduced in the previous sections. The dynamics of ϑ_e and ω_e are given by equations (6), with $i_{\alpha r}$, $i_{\beta r}$ the α and β reference currents given by (10), (11), where the estimated angular acceleration is now

$$\begin{aligned}\psi &= -\vartheta_e - k_2(\hat{\omega} - \omega_r) + \hat{\alpha}_2\hat{\omega} + \hat{c} - k_1\hat{\omega} \\ &\quad + \hat{c} - k_1\hat{\omega} + (\ddot{\vartheta}_r + k_1\dot{\vartheta}_r)\end{aligned} \tag{14}$$

Note that ψ is a known quantity. From (6) one works out

$$\begin{aligned}\dot{\omega}_e &= -\vartheta_e - k_2\omega_e + \frac{1}{\alpha_1}(-i_{\alpha e}q_1 + i_{\beta e}q_0) \\ &\quad + (k_1 + k_2 - \alpha_2)\tilde{\omega} - \tilde{c} - \frac{\tilde{\alpha}_1}{\alpha_1}\psi - \tilde{\alpha}_2\hat{\omega}\end{aligned} \tag{15}$$

where we have indicated by $\tilde{\omega} = \omega - \hat{\omega}$ the estimation error on the angular velocity and where we have used the following relationships

$$\alpha_2\omega - \hat{\alpha}_2\hat{\omega} = \alpha_2\tilde{\omega} + \tilde{\alpha}_2\hat{\omega}, \quad \hat{\omega} - \omega_r = \omega_e - \tilde{\omega}$$

The dynamic controller will be derived making use of the Lyapunov function

$$\begin{aligned}V = \frac{1}{2}\bigg[&\vartheta_e^2 + \omega_e^2 + \tilde{\omega}^2 + i_{\alpha e}^2 + i_{\beta e}^2 + \frac{1}{\lambda_0}\tilde{c}^2 + \frac{1}{\lambda_1}\frac{\tilde{\alpha}_1^2}{\alpha_1} \\ &+ \frac{1}{\lambda_2}\tilde{\alpha}_2^2 + \frac{1}{\lambda_3}\tilde{\alpha}_3^2 + \frac{1}{\lambda_4}\tilde{\alpha}_4^2 + \frac{1}{\lambda_5}\frac{\tilde{\alpha}_5^2}{\alpha_5} + \frac{1}{\lambda_6}\tilde{\alpha}_6^2\bigg]\end{aligned} \tag{16}$$

where $\lambda_i > 0$, $i = 0, 1, \ldots, 6$, are weighting positive constants. Referring to the Appendix for the proof, given the controller

$$\left.\begin{aligned}\dot{\xi}_1 &= -(\bar{k}_1 + \hat{\alpha}_2)\xi_1 - \xi_2 \\ &\quad + (2\lambda_0 - (\bar{k}_1 + \hat{\alpha}_2)\bar{k}_1)\vartheta \\ &\quad + \psi + W_\alpha i_{\alpha e} + W_\beta i_{\beta e} \\ \dot{\xi}_2 &= 2\lambda_0\xi_1 + 2\lambda_0\bar{k}_1\vartheta - \lambda_0(\hat{\omega} - \omega_r) \\ \dot{\zeta}_1 &= 2\left(\dot{a}_0\vartheta + \dot{a}_1\frac{\vartheta^2}{2} + \dot{a}_2\frac{\vartheta^3}{3}\right) + (\omega_r + \hat{\omega})\psi \\ \dot{\zeta}_2 &= 2\dot{\xi}_1\vartheta + \hat{\omega}(\omega_r + \hat{\omega}) \\ \dot{\hat{\alpha}}_3 &= \lambda_3(-i_{\alpha e}i_\alpha - i_{\beta e}i_\beta) \\ \dot{\hat{\alpha}}_4 &= \lambda_4(i_{\alpha e}q_1 - i_{\beta e}q_0)\hat{\omega} \\ \dot{\hat{\alpha}}_5 &= \lambda_5(i_{\alpha e}\varrho_\alpha + i_{\beta e}\varrho_\beta) \\ \dot{\hat{\alpha}}_6 &= \lambda_6(i_{\alpha e}q_1 - i_{\beta e}q_0)(\omega_r - \hat{\omega}) \\ \hat{\omega} &= \xi_1 + \bar{k}_1\vartheta \\ \hat{c} &= \xi_2 - 2\lambda_0\vartheta \\ \hat{\alpha}_1 &= \lambda_1\left[\zeta_1 - 2\left(a_0\vartheta + a_1\frac{\vartheta^2}{2} + a_2\frac{\vartheta^3}{3} + a_3\frac{\vartheta^4}{4}\right)\right] \\ \hat{\alpha}_2 &= \lambda_2(\zeta_2 - 2\xi_1\vartheta - \bar{k}_1\vartheta^2)\end{aligned}\right|$$

$$\left.\begin{aligned}
v_\beta &= -\hat{\alpha}_5\Big[-\hat{\alpha}_6 q_1(\hat{\omega}-\omega_{\mathrm{r}}) + (k_4+\delta)i_{\alpha\mathrm{e}} \\
&\quad - \hat{\alpha}_3 i_\alpha + \hat{\alpha}_4 q_1\hat{\omega} - \frac{\mathrm{d}i_{d\mathrm{r}}}{\mathrm{d}t}q_0 + i_{d\mathrm{r}}pq_1\hat{\omega} \\
&\quad - \lambda_1(\hat{\omega}-\omega_r)q_1\psi^2 + \hat{\alpha}_1 pq_0\psi\hat{\omega} \\
&\quad + \hat{\alpha}_1 q_1(\gamma_0+\gamma_1\hat{\omega})\Big] \\
v_\alpha &= -\hat{\alpha}_5\Big[\hat{\alpha}_6 q_0(\hat{\omega}-\omega_{\mathrm{r}}) + (k_3+\delta)i_{\beta\mathrm{e}} \\
&\quad - \hat{\alpha}_3 i_\beta - \hat{\alpha}_4 q_0\hat{\omega} - \frac{\mathrm{d}i_{d\mathrm{r}}}{\mathrm{d}t}q_1 - i_{d\mathrm{r}}pq_1\hat{\omega} \\
&\quad + \lambda_1(\hat{\omega}-\omega_{\mathrm{r}})q_0\psi^2 + \hat{\alpha}_1 pq_1\psi\hat{\omega} \\
&\quad - \hat{\alpha}_1 q_0(\gamma_0+\gamma_1\hat{\omega})\Big]
\end{aligned}\right\} \quad (17)$$

where $\bar{k}_1, k_i > 0, i = 1, \ldots, 4, \delta > \frac{1}{2}k_{\mathrm{m}}|1/L - 2/J|$, ψ as in (14), and

$$\left.\begin{aligned}
W_\alpha &= q_0(p\hat{\alpha}_1\psi) + q_1(-2\lambda_1\psi^2 + \hat{\alpha}_1\gamma_1 + pi_{dr}) \\
W_\beta &= -q_0(-2\lambda_1\psi^2 + \hat{\alpha}_1\gamma_1 + pi_{dr}) + q_1(p\hat{\alpha}_1\psi) \\
a_0(t) &= \lambda_2\xi_1\zeta_2 + \xi_2 - (k_1+k_2)\xi_1 + \ddot{\vartheta}_{\mathrm{r}} \\
&\quad + (k_1+k_2)\dot{\vartheta}_{\mathrm{r}} + (k_1k_2+1)\vartheta_{\mathrm{r}} \\
a_1(t) &= \lambda_2(\bar{k}_1\zeta_2 - \xi_1^2) - 2\lambda_0 \\
&\quad - (k_1+k_2)\bar{k}_1 - (k_1k_2+1) \\
a_2(t) &= -\tfrac{3}{2}\lambda_2\bar{k}_1\xi_1 \\
a_3 &= -\lambda_2\frac{\bar{k}_1^2}{2}
\end{aligned}\right\} \quad (18)$$

the time-derivative along the trajectories of the system (1), controlled by (17), is given by

$$\dot{V} \leq -k_1\vartheta_{\mathrm{e}}^2 - (\omega_{\mathrm{e}} \quad \tilde{\omega})\tilde{Q}\begin{pmatrix}\omega_{\mathrm{e}} \\ \tilde{\omega}\end{pmatrix} - k_3 i_{\beta\mathrm{e}}^2 - k_4 i_{\alpha\mathrm{e}}^2 \quad (19)$$

If $\alpha_{2\max}$ is the upper-bound of α_2, the matrix $\tilde{Q}$ is given by

$$\tilde{Q} = \begin{pmatrix} k_2 - \dfrac{\alpha_{2\max}}{2} & -\dfrac{k_1+k_2}{2} \\ -\dfrac{k_1+k_2}{2} & \bar{k}_1 + \dfrac{\alpha_2}{2} - 2\delta \end{pmatrix}$$

and is positive definite for an appropriate choice of the (positive) gains $k_1, k_2, \bar{k}_1$. This happens for instance with

$$k_2 > \frac{\alpha_{2\max}}{2}$$

$$\bar{k}_1 > 2\delta - \frac{(k_1+k_2)^2}{2(2k_2 - \alpha_{2\max})}$$

Hence $\dot{V} \leq 0$ and, assuming that the desired reference ϑ_{r} and its derivatives $\dot{\vartheta}_{\mathrm{r}}, \ddot{\vartheta}_{\mathrm{r}}, \dddot{\vartheta}_{\mathrm{r}}$ are bounded, we deduce that the vector

$$x_{\mathrm{e}} = (\vartheta_{\mathrm{e}} \quad \omega_{\mathrm{e}} \quad i_{\beta\mathrm{e}} \quad i_{\alpha_{\mathrm{e}}} \quad \tilde{\omega})^{\mathrm{T}}$$

belongs to $L_2 \cap L_\infty$, while $\tilde{c}$ and $\tilde{\alpha}_i, i = 1, \ldots, 6$, are bounded. This also ensures that $\dot{x}_e \in L_\infty$. Therefore, the application of Barbalat's lemma ensures that $\lim_{t\to\infty} x_e = 0$. In particular, this means that $\lim_{t\to\infty}\hat{\omega} = \omega$. Moreover, in the Appendix it is shown that the closed-loop dynamics are in the form

$$\left.\begin{aligned}
\dot{x}_e &= A(t)x_{\mathrm{e}} + \Gamma(t)D\tilde{\alpha} \\
\dot{\tilde{\alpha}} &= -\Lambda\Gamma^T(t)x_{\mathrm{e}}
\end{aligned}\right\} \quad (20)$$

with

$$A(t) = \begin{pmatrix}
-k_1 & 1 & 0 & 0 & 0 \\
-1 & -k_2 & \alpha_6 q_0 & -\alpha_6 q_1 & k_1+k_2-\alpha_2 \\
0 & -\alpha_6 q_0 & -(k_3+\delta) & 0 & W_\beta - q_0(\alpha_4-\alpha_6) \\
0 & \alpha_6 q_1 & 0 & -(k_4+\delta) & W_\alpha + q_1(\alpha_4-\alpha_6) \\
0 & 0 & -(W_\beta - \alpha_6 q_0) & -(W_\alpha + \alpha_6 q_1) & -(\bar{k}_1+\alpha_2)
\end{pmatrix}$$

$$\Gamma(t) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -\psi & -\hat{\omega} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -i_\beta & -q_0\hat{\omega} & \varrho_\beta & -(\omega_{\mathrm{r}}-\hat{\omega})q_0 \\
0 & 0 & 0 & -i_\alpha & q_1\hat{\omega} & \varrho_\alpha & (\omega_{\mathrm{r}}-\hat{\omega})q_1 \\
-1 & -\psi & -\hat{\omega} & 0 & 0 & 0 & 0
\end{pmatrix}$$

and Λ, D defined as in §3.2. Under the hypothesis of boundedness of $\mathrm{d}^j i_{d\mathrm{r}}/\mathrm{d}t^j$, $j = 0, 1$, and $\mathrm{d}^j\vartheta_{\mathrm{r}}/\mathrm{d}t^j$, $j = 0, \ldots, 3$, it is possible to check that $\Gamma(t), \dot{\Gamma}(t) \in L_\infty$, $A(t)$, $\dot{A}(t) \in L_\infty$, since in particular ψ, given by (14), is bounded as well as the functions $a_0(t), a_1(t), a_2(t)$ (see equation (29)). Therefore, we can also say that

$$\ddot{x}_{\mathrm{e}} = \dot{A}(t)x_{\mathrm{e}} + A(t)\dot{x}_{\mathrm{e}} + \dot{\Gamma}(t)D\tilde{\alpha} + \Gamma(t)D\dot{\tilde{\alpha}} \in L_\infty$$

so that $\dot{x}_{\mathrm{e}}$ is a uniformly continuous function. Moreover

$$\int_0^\infty \dot{x}_{\mathrm{e}}(\tau)\,\mathrm{d}\tau = x_{\mathrm{e}}(\infty) - x_{\mathrm{e}}(0) = -x_{\mathrm{e}}(0) < \infty$$

since $x_{\mathrm{e}}(\infty) = 0$, and, by using again Barbalat's lemma, one can state that as $t \to \infty$

$$\dot{x}_{\mathrm{e}} = 0 = A(t)x_{\mathrm{e}} + \Gamma(t)D\tilde{\alpha} = \Gamma(t)D\tilde{\alpha}$$

Since $\Gamma(t)$ is not a full-rank matrix, we cannot infer that $\tilde{\alpha} \to 0$. Nevertheless, if the persistent excitation condition (13) holds true, x_{e} and all the parameter errors converge exponentially to zero (Narendra and Annaswamy 1989).

When all the parameters are known, except the load torque C_l and the resistance R ($\tilde{\alpha}_i = 0, i \neq 3$), the persistence of excitation condition (13) is always verified; in fact, it assumes an interesting form, as is shown in the case of static feedback in Marino *et al.* (1995)

$$\int_{t_0}^{t_0+T} [i_\alpha^2(\tau) + i_\beta^2(\tau)]\, d\tau > 0$$

for a $T > 0$ and for all $t_0 \geq 0$, since $\Gamma^T\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & i_\alpha^2 + i_\beta^2 \end{pmatrix}$. This is true in all physical situations.

4.2. *Sensitivity analysis*

The system (20) is in the form

$$\dot{z} = f(t, z, p), \quad z = \begin{pmatrix} x_e \\ \tilde{\alpha} \end{pmatrix}, \quad p = \begin{pmatrix} \alpha_1 \\ \alpha_4 \\ \alpha_6 \end{pmatrix}$$

where the dependence on the parameters $\alpha_1, \alpha_4, \alpha_6$ is through $A(t)$ and where $z(t_0) = z_0$ does not depend on p. It is possible to perform an estimate of the effects of the parameter variations on the closed-loop system trajectories as follows (Khalil 1992). Let

$$p_0 = \begin{pmatrix} \alpha_{10} \\ \alpha_{40} \\ \alpha_{60} \end{pmatrix}$$

be the nominal parameter values. Avoiding technical details, the solution

$$z(t, p) = z_0 + \int_{t_0}^{t} f(\tau, z(\tau, p), p)\, d\tau$$

of the original differential equation is close to the solution $z(t, p_0)$ of the equation

$$\dot{z} = f(t, z, p_0), \quad z(t_0) = z_0$$

if the difference $\| p - p_0 \|$ is sufficiently small. Moreover, in a neighbourhood of p_0 the solution $z(t, p)$ can be differentiated with respect to p

$$\frac{\partial z(t,p)}{\partial p} = z_p(t,p) = \int_{t_0}^{t} \left[\frac{\partial f(\tau, z(\tau, p), p)}{\partial z} \frac{\partial z(t,p)}{\partial p} + \frac{\partial f(\tau, z(\tau,p), p)}{\partial p} \right] d\tau$$

so that $z_p(t, p)$ satisfies the following differential equation

$$\left.\begin{aligned} \dot{z}_p(t,p) &= \mathcal{A} z_p(t,p) + \mathcal{B} \\ \mathcal{A} &= \left.\frac{\partial f(t,z,p)}{\partial z}\right|_{z=z(t,p)} \\ \mathcal{B} &= \left.\frac{\partial f(t,z,p)}{\partial p}\right|_{z=z(t,p)} \end{aligned}\right\} \qquad (21)$$

with $\partial z(t_0)/\partial p = z_p(t_0) = 0$. The quantity $z_p(t, p_0) = \partial z(t, p_0)/\partial p$, which gives an estimate of the effect of the variations of p on the solution $z(t, p_0)$, is the so-called sensitivity function and can be determined by solving (numerically) equation (21) along with equations (20)

$$\begin{pmatrix} \dot{x}_e \\ \dot{\tilde{\alpha}} \end{pmatrix} = \begin{pmatrix} A(t)x_e + \Gamma(t) D \tilde{\alpha} \\ -\Lambda \Gamma^T(t) x_e \end{pmatrix}$$

$$\dot{z}_p(t,p) = \mathcal{A} z_p(t,p) + \mathcal{B}$$

for $p = p_0$, with $\begin{pmatrix} \dot{x}_e(t_0) \\ \dot{\tilde{\alpha}}(t_0) \end{pmatrix} = \begin{pmatrix} x_{e_0} \\ \tilde{\alpha}_0 \end{pmatrix}, z_p(t_0) = 0$. In the present case the matrices to be used in (21) are

$$\mathcal{A}|_{p=p_0} = \begin{pmatrix} A(t)|_{p=p_0} & \Gamma(t)D \\ -\Lambda\Gamma^T(t) & 0 \end{pmatrix}$$

$$\mathcal{B}|_{p=p_0} = \begin{pmatrix} 0 & 0 & 0 \\ -\tilde{\omega} & 0 & q_0 i_{\beta e} - q_1 i_{\alpha e} \\ 0 & -q_0\tilde{\omega} & q_0(-\omega_e + \tilde{\omega}) \\ 0 & q_1\tilde{\omega} & q_1(\omega_e - \tilde{\omega}) \\ -\tilde{\omega} & 0 & q_0 i_{\beta e} - q_1 i_{\alpha e} \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix}$$

It is worth noting that the closed-loop system (20) is not influenced by variations of c, α_2, α_3, α_5, since $A(t)$ does not depend on these parameters. This means in particular that variations of the resistant torque C_l and of the stator resistance R influence the closed-loop state evolution only through the initial conditions, but not directly the dynamics (20). This can be considered a further degree of robustness of the control system against these parameter variations.

5. Simulation results

The dynamic controller (17), proposed in §4, has been applied to a permanent magnet synchronous motor with $p = 6$ and characterized by the nominal parameters reported in table 1. The results are

R_0	$3\,\Omega$	J_0	$0.01\ \mathrm{kg\,m^2}$
L_0	$0.006\ \mathrm{H}$	f_0	$0.0014\ \mathrm{N\,m\,s}$
k_{m0}	$2\ \mathrm{N\,m\,A^{-1}}$	C_{l0}	$0.5\ \mathrm{N\,m\,s}$

Table 1. Nominal parameters of the PM synchronous motor.

compared with those of a non-linear controller based on the same nominal values; a reduced-order observer is used in this second case for estimating the angular velocity ω (Bodson *et al.* 1993). Incidentally, we note that the adaptive controllers proposed in Grčar *et al.* (1991) and in Marino *et al.* (1995) cannot be applied here since they are based on the measurement of the whole state. Finally, in order to render the simulations more realistic, the controller (17) has been tested considering the effects of stochastic disturbances on the motor parameters and of noises on the measured variables.

When the non-linear controller (Bodson *et al.* 1993) is applied, the only parameter subject to variations is the rotor resistance, whose real value is $R = 6\,\Omega$ (+100% w.r.t. R_0), whereas the others are equal to their nominal values. In the case of the controller (17) the same value for R has been considered; moreover, a variation with respect to the nominal value is supposed for C_l; more precisely, it is supposed that C_l is zero until $t = 0.4$ s (−100% w.r.t. C_{l0}) and equal to 2 N m for $t \geq 0.4$ s (+400% w.r.t. C_{l0}). The other parameters are supposed equal to L_0, k_{m0}, J_0 and f_0. With this choice we have

$$c = \begin{cases} 0 & \text{for } t < 0.4, \quad \alpha_1 = 0.005, \quad \alpha_2 = 0.14, \quad \alpha_3 = 1000 \\ 200 & \text{for } t \geq 0.4, \quad \alpha_4 = 333.33, \quad \alpha_5 = 0.006, \quad \alpha_6 = 200 \end{cases}$$

Finally, we have considered the same controller (17) with the following disturbances on R and C_l

$$R_d = R + 0.2\,R_0(1 + 0.1\mathcal{N}(t)), \quad C_{ld} = C_l + 0.01\,\mathcal{N}(t)$$

and with the noises

$$\vartheta_m = \vartheta + 0.005\,\mathcal{N}(t)$$
$$i_{\alpha m} = i_\alpha + 0.02\,\mathcal{N}(t)$$
$$i_{\beta m} = i_\beta + 0.02\,\mathcal{N}(t)$$

on the measured variables. Here $\mathcal{N}(t)$ denotes the normal distribution with mean zero and standard deviation one.

Note that the comparison between the non-linear controller and the adaptive controller (17) has been conducted under a condition more favourable to the first one, since in this case we have supposed c known and α_3 the only unknown parameter.

The tuning parameters corresponding to the estimations of R and C_l have been set equal to $\lambda_0 = 10^4$ and $\lambda_3 = 10^4$ $(\lambda_i = 0, i = 1, 2, 4, 5, 6)$, and the controller gains equal to

$$k_1 = 40, \quad k_2 = 40, \quad k_3 = 400, \quad k_4 = 400, \quad \bar{k}_1 = 100$$

The same values (with the exception of λ_0) have been used for the non-linear controller.

The reference trajectories have been chosen equal to $i_{dr} = 0$ and $\vartheta_f = 10$ rad. For the latter, in order to overcome the problems of strong current excitation consequent to a discontinuous reference trajectory, a polinomial reference has been used. Setting

$$\vartheta_r(t) = \sum_{i=0}^{m} c_i t^i$$

m and c_i have been computed so that $\vartheta_r(0) = 0$, $\vartheta_r(t_f) = \vartheta_f$, with the smoothness conditions

$$\dot{\vartheta}_r(0) = 0, \quad \ddot{\vartheta}_r(0) = 0, \quad \dot{\vartheta}_r(t_f) = 0, \quad \ddot{\vartheta}_r(t_f) = 0$$

thus obtaining

$$c_0 = c_1 = c_2 = c_3 = 0$$
$$c_4 = \frac{35}{t_f^4}\vartheta_f$$
$$c_5 = -\frac{84}{t_f^5}\vartheta_f$$
$$c_6 = \frac{70}{t_f^6}\vartheta_f$$
$$c_7 = -\frac{20}{t_f^7}\vartheta_f$$

where $t_f = 0.5$ s is the response time at 99% of the steady-state value ϑ_f.

Figures 1–9 summarize the simulation results. Note that NLC, AC and ACN stand for non-linear controller, adaptive controller (17), and adaptive controller (17) with noises and disturbances.

In the case of the non-linear controller (see figures 1–6), the simulations show errors on both transient and steady-state conditions, due to the error on R. In fact, the i_d current (figure 1) and the position tracking error (figures 2 and 5) are significantly different from zero. Even more evident is the error on the angular velocity (figures 3 and 6). The inputs are given in figure 4.

The simulations obtained with the adaptive controller (17) show that the position is well tracked with a small error (figures 2 and 5), as well as the angular velocity (figures 3 and 6), while the i_d current remains close to zero (figure 1). The main errors on these variables are due to the estimation errors at $t = 0$ and at $t = 0.4$ on c and on the parameter α_3 (see figures 8 and 9). Also, the angular rate ω is well estimated (figure 9). Figure 7 gives the errors on the α and β current components with respect to the references, while figure 4 shows the behaviour of the inputs v_α and v_β. We recall that in the present

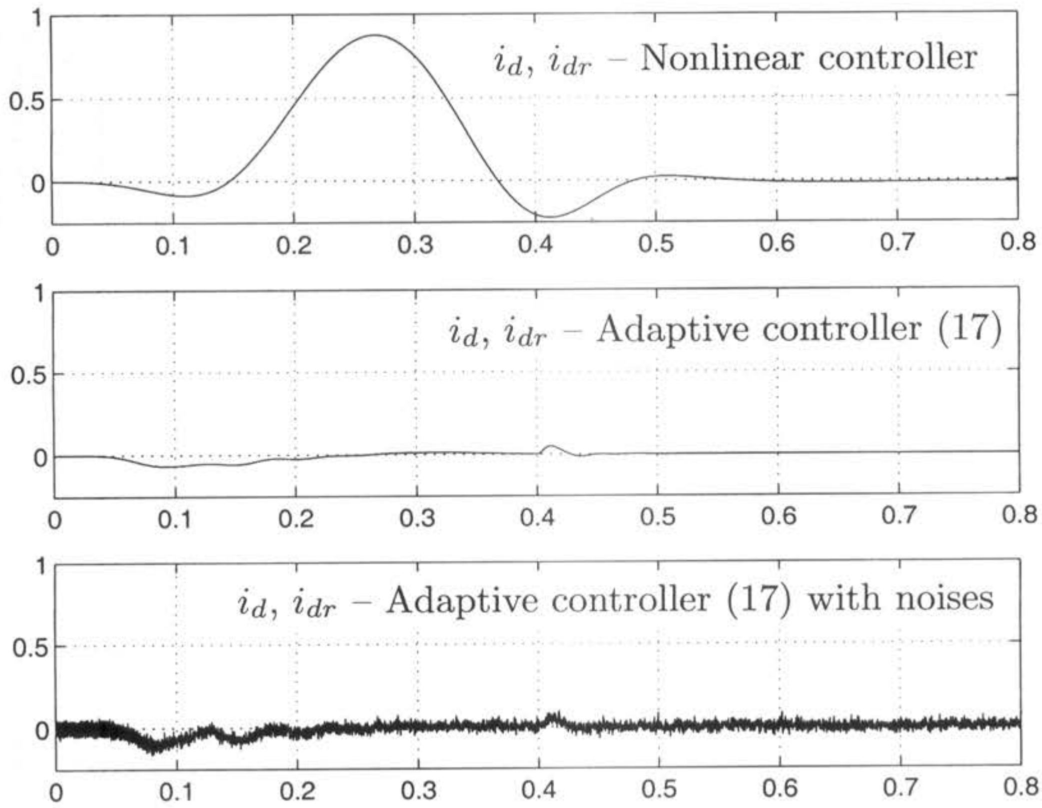

Figure 1. Current i_d (A).

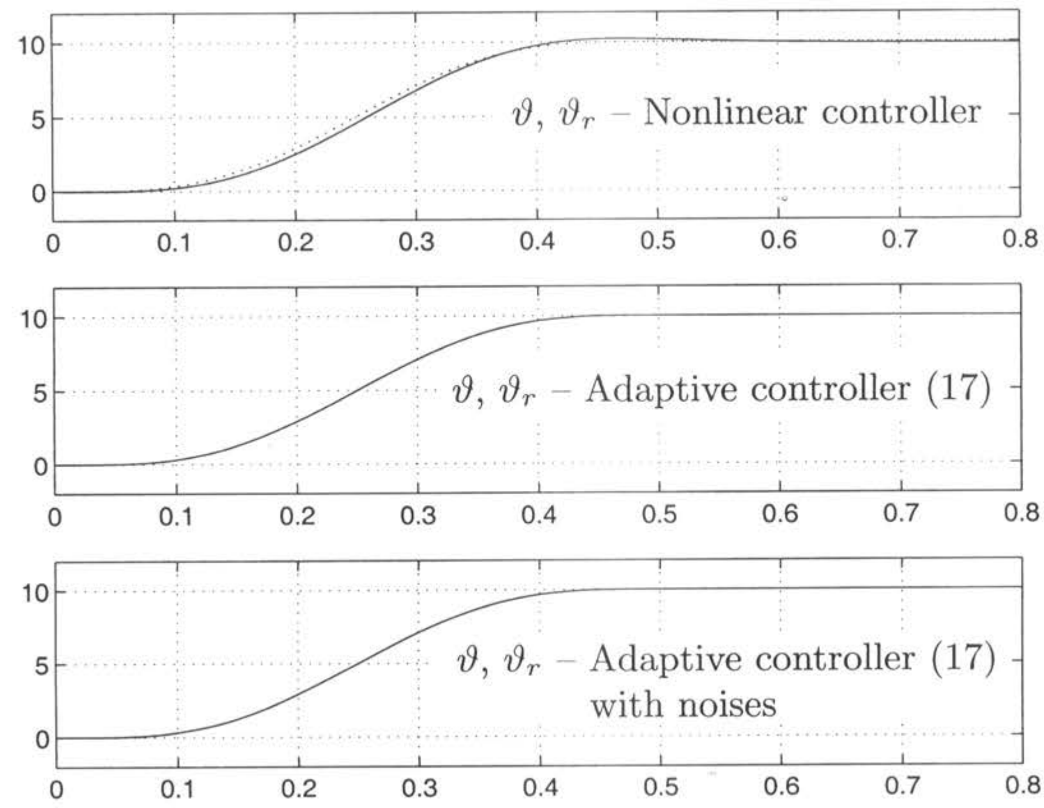

Figure 2. Positions ϑ (solid) and ϑ_r (dotted) (rad).

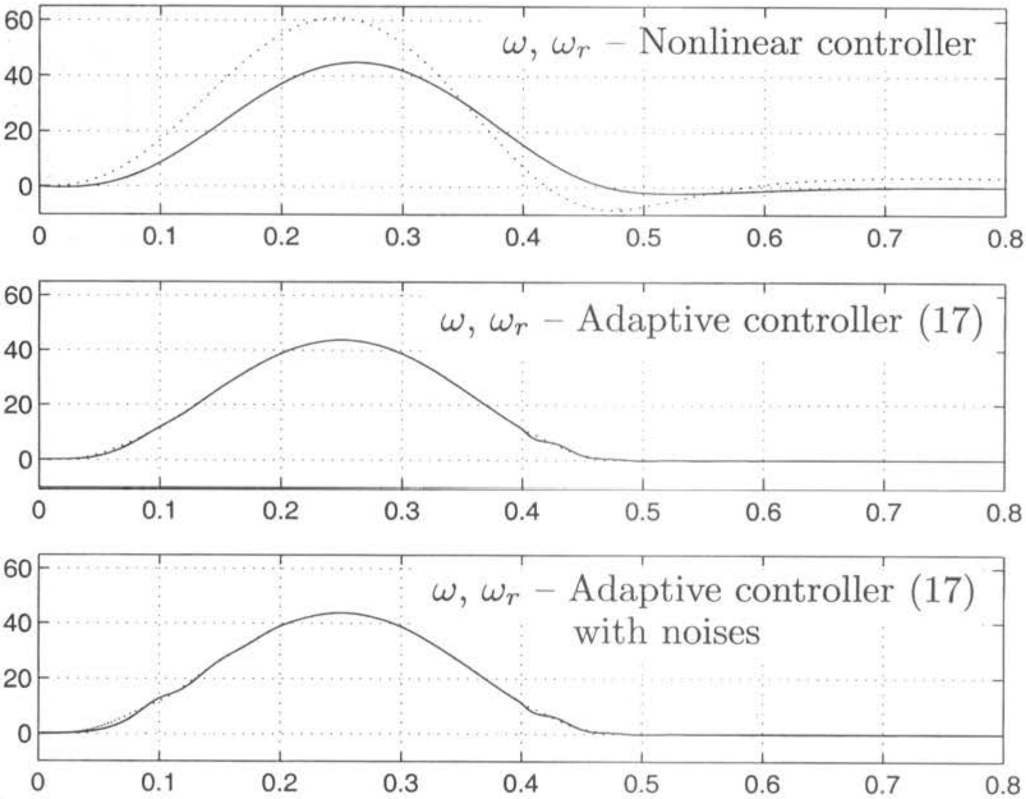

Figure 3. Velocities ω (solid) and ω_r (dotted) (rad/s).

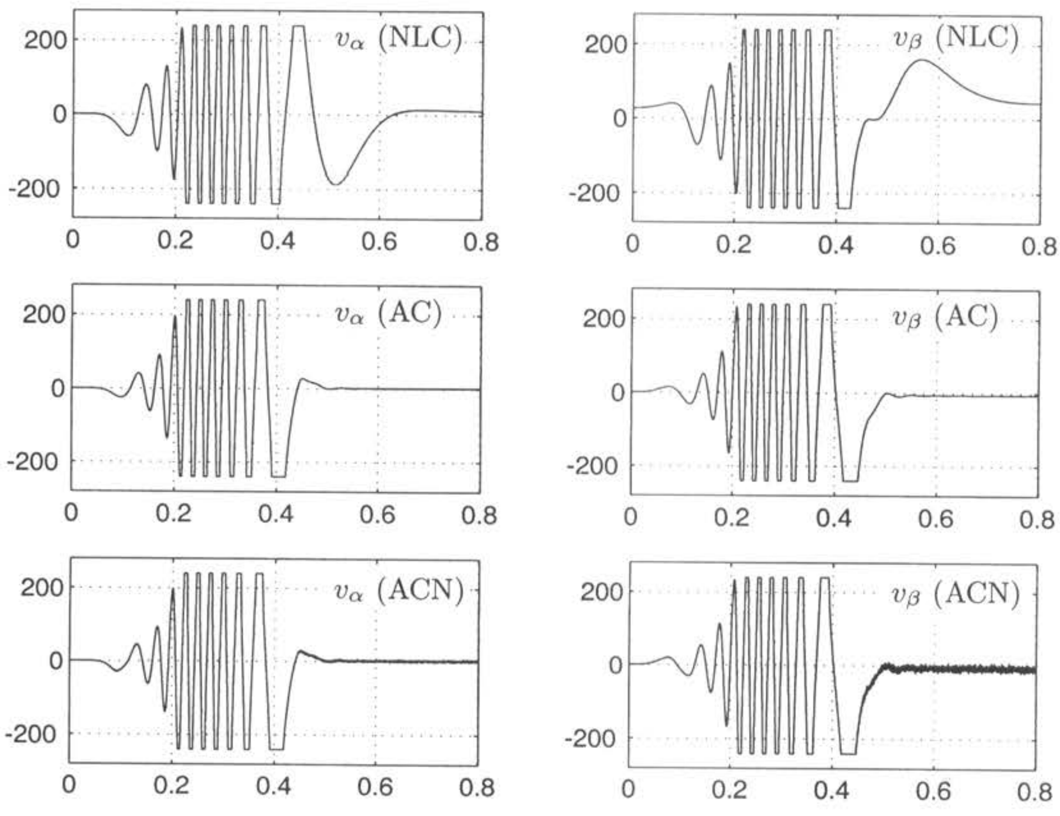

Figure 4. Inputs v_α and v_β (V).

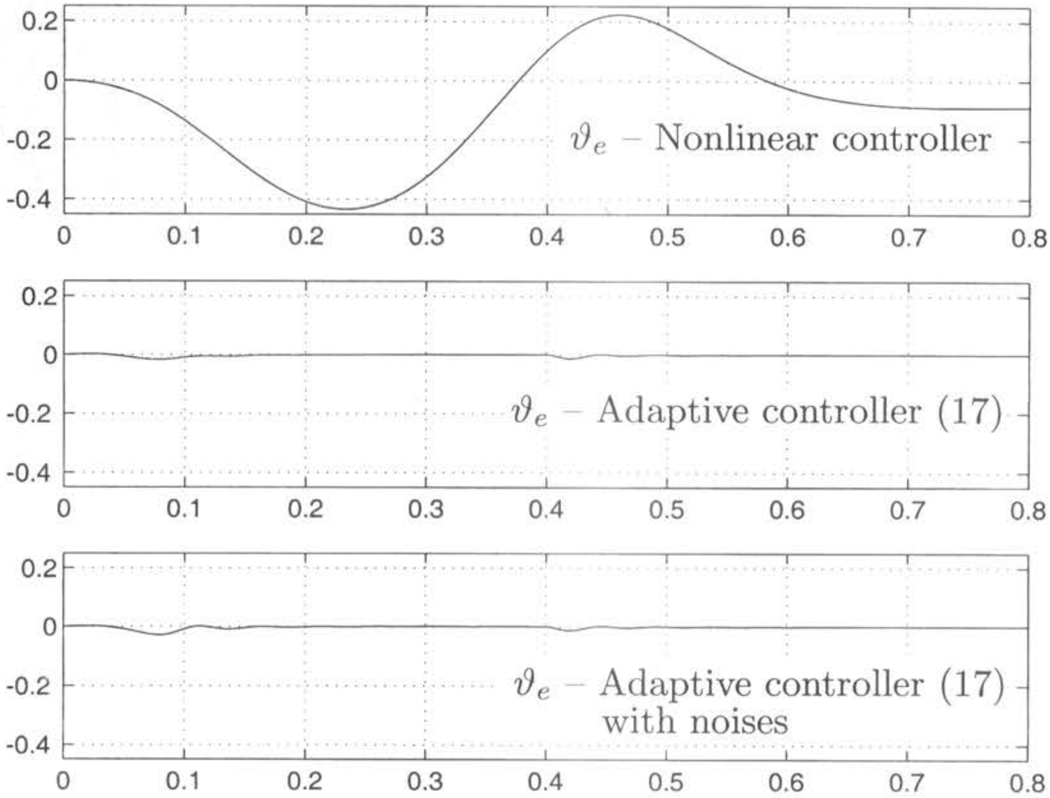

Figure 5. Error ϑ_e (rad).

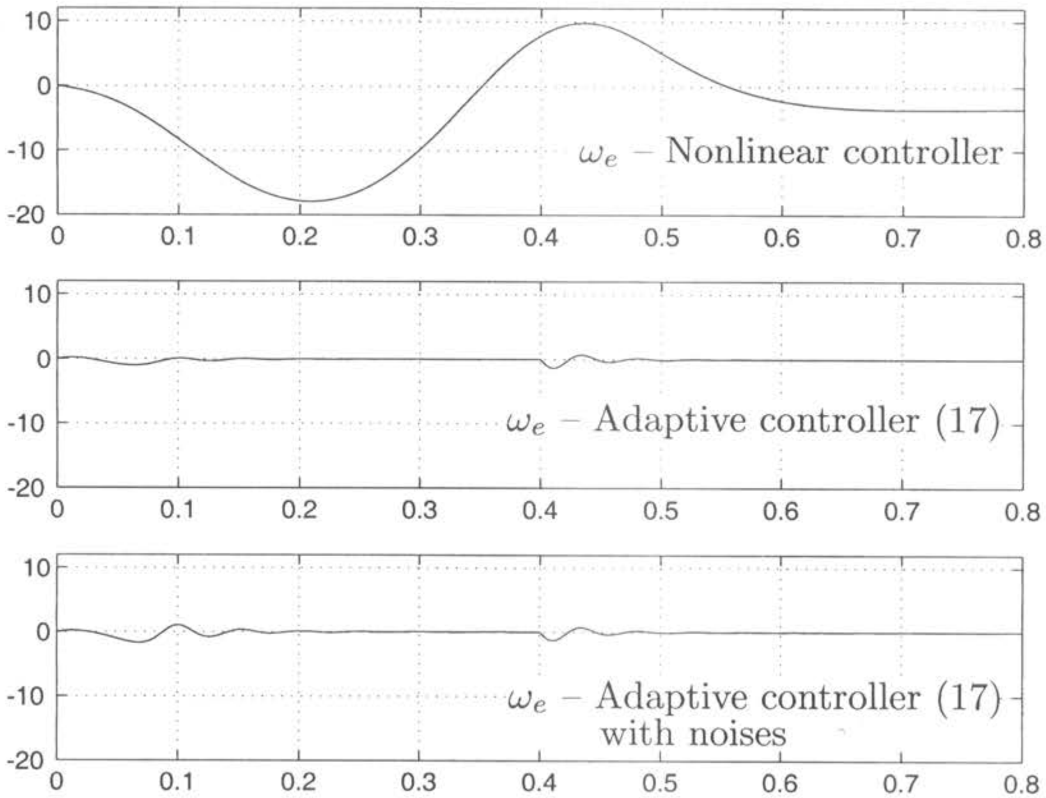

Figure 6. Error ω_e (rad/s).

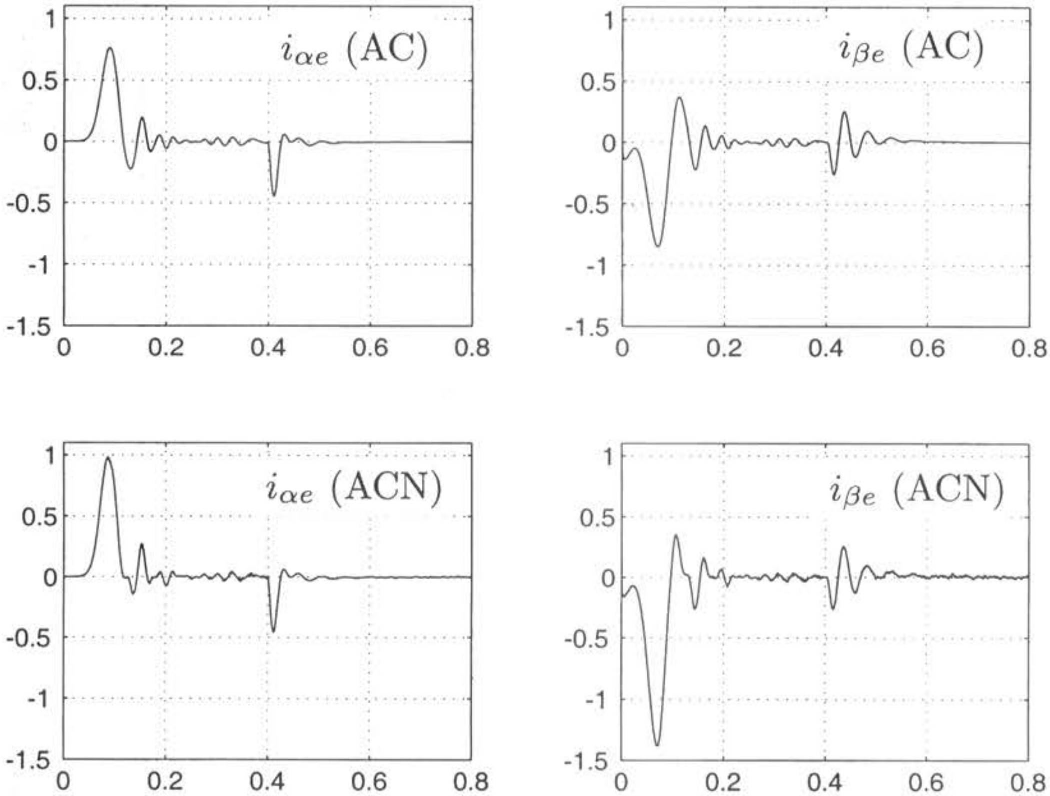

Figure 7. Errors $i_{\alpha e}$, $i_{\beta e}$ (A).

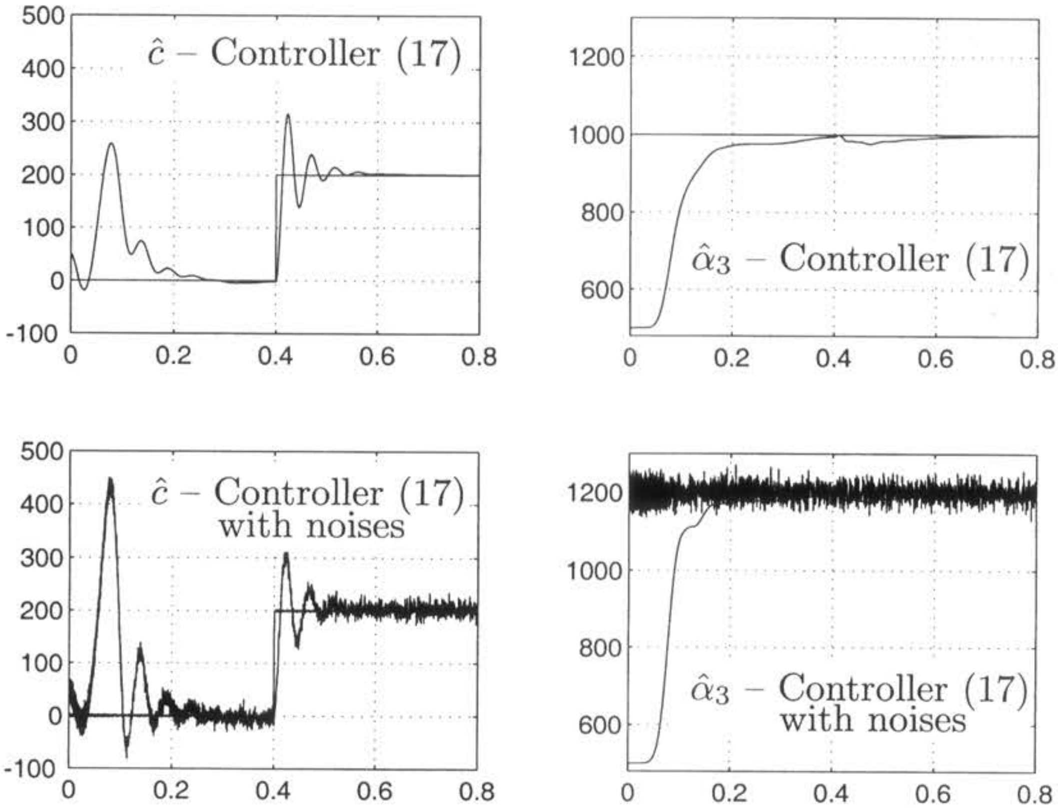

Figure 8. Estimated load torque $\hat{c}$ and coefficient $\hat{\alpha}_3$, and real values.

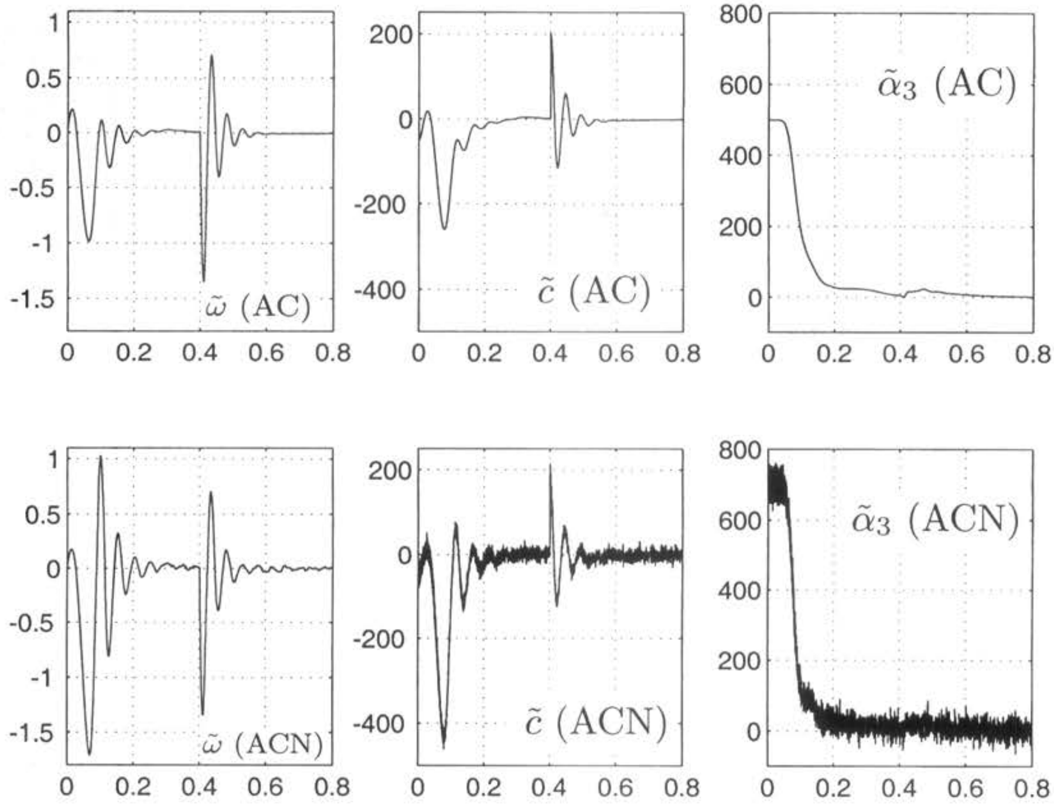

Figure 9. Errors $\tilde{\omega}$, $\tilde{c}$, $\tilde{\alpha}_3$.

situation the closed-loop trajectories are not influenced by variations of R and C_l, as shown in §4.2, whereas they obviously depend on the values λ_0 and λ_3. Finally, when the disturbances and noises are present, the performances of the adaptive controller (17) remain practically the same.

6. Conclusions

In this work we have presented a non-linear adaptive controller which solves the problem of position tracking for a permanent magnet synchronous motor. The angular velocity is supposed unavailable for measurements and motor parameters are considered unknown. If the hypothesis of persistence of excitation is verified, the problem is solved and the whole system is exponentially stable. The parameters are hence exponentially identified. This hypothesis is equivalent to the condition that the current is non-zero, and hence always verified, when the unknown parameters are stator resistance and load torque.

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The author wishes to thank Salvatore Monaco and Dorothée Normand-Cyrot for their suggested ideas. The work was supported by Ministero dell'Università e della Ricerca Scientifica e Tecnologica, National Program 'Teoria dei Sistemi e del Controllo'.

Appendix

In this Appendix the expressions (19) for $\dot{V}$ and (20) for the closed-loop dynamics are derived. Let us consider the function (16). As a first step in the derivation of the dynamic controller, let us introduce the following estimate dynamics

$$\dot{\xi}_1 = -(\bar{k}_1 + \hat{\alpha}_2)\xi_1 - \xi_2 + (\bar{k}_2 - (\bar{k}_1 + \hat{\alpha}_2)\bar{k}_1)\vartheta - \mu_1(t, y, \hat{\omega})$$

$$\dot{\xi}_2 = \bar{k}_2\xi_1 + \bar{k}_1\bar{k}_2\vartheta - \mu_2(t, y, \hat{\omega})$$

$$\hat{\omega} = \xi_1 + \bar{k}_1\vartheta \quad (22)$$

$$\hat{c} = \xi_2 - \bar{k}_2\vartheta \quad (23)$$

with $\mu_i(t, y, \hat{\omega})$, $i = 1, 2$, auxiliary functions depending on the output y, $\bar{k}_1, \bar{k}_2 > 0$. Therefore, the dynamics of $\tilde{\omega}$, $\tilde{c}$ are given by

$$\dot{\tilde{\omega}} = \frac{1}{\alpha_1}(-i_\alpha q_1 + i_\beta q_0) - \alpha_2\omega - c - \dot{\hat{\omega}}$$

$$= \alpha_6(-i_{\alpha e}q_1 + i_{\beta e}q_0) + \frac{1}{\alpha_1}(-i_{\alpha r}q_1 + i_{\beta r}q_0)$$

$$- \alpha_2\omega - c - \dot{\hat{\omega}}$$

$$= \alpha_6(-i_{\alpha e}q_1 + i_{\beta e}q_0) + \left(1 - \frac{\tilde{\alpha}_1}{\alpha_1}\right)\psi - \tilde{c}$$

$$- (\bar{k}_1 + \alpha_2)\tilde{\omega} - \tilde{\alpha}_2\hat{\omega} + \mu_1(t, y, \hat{\omega})$$

$$\dot{\tilde{c}} = -\dot{\hat{c}} = \bar{k}_2\tilde{\omega} + \mu_2(t, y, \hat{\omega})$$

since $\hat{\alpha}_i = \alpha_i - \tilde{\alpha}_i$, $i = 1, 2$, $\xi_1 + \bar{k}_1\vartheta = \hat{\omega}$, $\xi_2 - \bar{k}_2\vartheta = \hat{c}$ and $1/\alpha_1 = \alpha_6$.

Now, once the function (16) has been derived with respect to the time, one obtains

$$\dot{V} = \vartheta_e[-k_1\vartheta_e + \omega_e] + \omega_e\dot{\omega}_e + \tilde{\omega}\dot{\tilde{\omega}} + i_{\alpha e}\frac{di_{\alpha e}}{dt} + i_{\beta e}\frac{di_{\beta e}}{dt}$$

$$+ \frac{1}{\lambda_0}\tilde{c}[\bar{k}_2\tilde{\omega} + \mu_2(t, y, \hat{\omega})]$$

$$- \left[\frac{\tilde{\alpha}_1}{\alpha_1}\frac{\dot{\hat{\alpha}}_1}{\lambda_1} + \tilde{\alpha}_2\frac{\dot{\hat{\alpha}}_2}{\lambda_2} + \tilde{\alpha}_3\frac{\dot{\hat{\alpha}}_3}{\lambda_3} + \tilde{\alpha}_4\frac{\dot{\hat{\alpha}}_4}{\lambda_4} + \frac{\tilde{\alpha}_5}{\alpha_5}\frac{\dot{\hat{\alpha}}_5}{\lambda_5} + \tilde{\alpha}_6\frac{\dot{\hat{\alpha}}_6}{\lambda_6}\right] \tag{24}$$

The term Φ containing $(\omega_e \quad \tilde{\omega})$ can be rewritten as

$$\Phi = -(\omega_e \quad \tilde{\omega})\begin{pmatrix} k_2 & -(k_1 + k_2 - \alpha_2) \\ 0 & \bar{k}_1 + \alpha_2 \end{pmatrix}\begin{pmatrix} \omega_e \\ \tilde{\omega} \end{pmatrix}$$

$$+ \alpha_6(\omega_e + \tilde{\omega})(-i_{\alpha e}q_1 + i_{\beta e}q_0)$$

$$- (\omega_e + \tilde{\omega})\tilde{c} - (\omega_e + \tilde{\omega})\psi\frac{\tilde{\alpha}_1}{\alpha_1} - (\omega_e + \tilde{\omega})\hat{\omega}\tilde{\alpha}_2$$

$$+ \tilde{\omega}[\psi + \mu_1(t, y, \hat{\omega})] - \omega_e\vartheta_e$$

We suppose that an upper-bound is known for the unknown parameter α_2. Let $\alpha_{2\max}$ be this bound. Since

$$-\alpha_2\omega_e\tilde{\omega} \le \alpha_{2\max}\frac{\omega_e^2}{2} + \alpha_2\frac{\tilde{\omega}^2}{2}$$

it follows that

$$\Phi \le -(\omega_e \quad \tilde{\omega})Q\begin{pmatrix} \omega_e \\ \tilde{\omega} \end{pmatrix} + \alpha_6(\omega_e + \tilde{\omega})(-i_{\alpha e}q_1 + i_{\beta e}q_0)$$

$$- (\omega_e + \tilde{\omega})\tilde{c} - (\omega_e + \tilde{\omega})\psi\frac{\tilde{\alpha}_1}{\alpha_1}$$

$$- (\omega_e + \tilde{\omega})\hat{\omega}\tilde{\alpha}_2 + \tilde{\omega}[\psi + \mu_1(t, y, \hat{\omega})] - \omega_e\vartheta_e$$

with

$$Q = \begin{pmatrix} k_2 - \dfrac{\alpha_{2\max}}{2} & -\dfrac{k_1 + k_2}{2} \\ -\dfrac{k_1 + k_2}{2} & \bar{k}_1 + \dfrac{\alpha_2}{2} \end{pmatrix}$$

Hence, (24) yields

$$\dot{V} \le -k_1\vartheta_e^2 - (\omega_e \quad \tilde{\omega})Q\begin{pmatrix} \omega_e \\ \tilde{\omega} \end{pmatrix} + \tilde{\omega}[\psi + \mu_1(t, y, \hat{\omega})]$$

$$+ i_{\alpha e}\left[-\alpha_6(\omega_e + \tilde{\omega})q_1 + \frac{di_{\alpha e}}{dt}\right]$$

$$+ i_{\beta e}\left[\alpha_6(\omega_e + \tilde{\omega})q_0 + \frac{di_{\beta e}}{dt}\right]$$

$$+ \tilde{c}\left[-(\omega_e + \tilde{\omega}) + \frac{\bar{k}_2}{\lambda_0}\tilde{\omega} + \frac{1}{\lambda_0}\mu_2(t, y, \hat{\omega})\right]$$

$$- \frac{\tilde{\alpha}_1}{\alpha_1}\left[(\omega_e + \tilde{\omega})\psi + \frac{\dot{\hat{\alpha}}_1}{\lambda_1}\right] - \tilde{\alpha}_2\left[(\omega_e + \tilde{\omega})\hat{\omega} + \frac{\dot{\hat{\alpha}}_2}{\lambda_2}\right]$$

$$- \tilde{\alpha}_3\frac{\dot{\hat{\alpha}}_3}{\lambda_3} - \tilde{\alpha}_4\frac{\dot{\hat{\alpha}}_4}{\lambda_4} - \frac{\tilde{\alpha}_5}{\alpha_5}\frac{\dot{\hat{\alpha}}_5}{\lambda_5} - \tilde{\alpha}_6\frac{\dot{\hat{\alpha}}_6}{\lambda_6} \tag{25}$$

In (25), the term multiplying $\tilde{c}$ can be eliminated by setting $\bar{k}_2 = 2\lambda_0$ and

$$\mu_2(t, y, \hat{\omega}) = \lambda_0(\hat{\omega} - \omega_r)$$

Hence

$$\dot{\tilde{c}} = \lambda_0(\omega_e + \tilde{\omega}) \tag{26}$$

The term multiplying $\tilde{\alpha}_2$ can be compensated by introducing the following dynamic system

$$\left.\begin{aligned} \dot{\zeta}_2 &= 2\dot{\xi}_1\vartheta + \hat{\omega}(\omega_r + \hat{\omega}) \\ \hat{\alpha}_2 &= \lambda_2(\zeta_2 - 2\xi_1\vartheta - \bar{k}_1\vartheta^2) \end{aligned}\right\} \tag{27}$$

with ξ_1 as in (22), which is such that

$$\dot{\hat{\alpha}}_2 = -\lambda_2(\omega_e + \tilde{\omega})\hat{\omega} \tag{28}$$

Analogously, in order to cancel the term multiplying $\tilde{\alpha}_1$, note first that ψ, given by (14), can be rewritten as

$$\psi = a_0(t) + a_1(t)\vartheta + a_2(t)\vartheta^2 + a_3\vartheta^3 \tag{29}$$

where (22), (23), (27) and (5) were used, and $a_0(t)$, $a_1(t)$, $a_2(t)$, a_3 are as in (18). Hence, introducing the dynamic system

$$\dot{\zeta}_1 = 2\left(\dot{a}_0\vartheta + \dot{a}_1\frac{\vartheta^2}{2} + \dot{a}_2\frac{\vartheta^3}{3}\right) + (\omega_r + \hat{\omega})\psi$$

$$\hat{\alpha}_1 = \lambda_1\left[\zeta_1 - 2\left(a_0\vartheta + a_1\frac{\vartheta^2}{2} + a_2\frac{\vartheta^3}{3} + a_3\frac{\vartheta^4}{4}\right)\right]$$

one works out that

$$\dot{\hat{\alpha}}_1 = \lambda_1[\omega_r\psi - (a_0 + a_1\vartheta + a_2\vartheta^2 + a_3\vartheta^3)\omega]$$

$$= -\lambda_1(\omega_e + \tilde{\omega})\psi \tag{30}$$

Let us now compute the derivatives of $i_{\alpha e}$, $i_{\beta e}$ along the dynamics (1) of the system, given by (7) in which $di_{\alpha r}/dt$, $di_{\beta r}/dt$ have the form (12), with

$$\dot{\psi} = \gamma_0 + \gamma_1 \omega$$
$$\gamma_0 = \dot{a}_0 + \dot{a}_1 \vartheta + \dot{a}_2 \vartheta^2$$
$$\gamma_1 = a_1 + 2a_2\vartheta + 3a_3\vartheta^2$$

Therefore

$$\frac{\mathrm{d}i_{\beta e}}{\mathrm{d}t} = -\alpha_3 i_\beta - \alpha_4 q_0 \omega + \frac{1}{\alpha_5} v_\beta + \lambda_1(\omega_e + \tilde{\omega}) q_0 \psi^2 + p\hat{\alpha}_1 q_1 \psi\omega - \hat{\alpha}_1 q_0(\gamma_0 + \gamma_1\omega) - \frac{\mathrm{d}i_{dr}}{\mathrm{d}t} q_1 - i_{dr} p\omega q_0$$

$$\frac{\mathrm{d}i_{\alpha e}}{\mathrm{d}t} = -\alpha_3 i_\alpha + \alpha_4 q_1 \omega + \frac{1}{\alpha_5} v_\alpha - \lambda_1(\omega_e + \tilde{\omega}) q_1 \psi^2 + p\hat{\alpha}_1 q_0 \psi\omega + \hat{\alpha}_1 q_1(\gamma_0 + \gamma_1\omega) - \frac{\mathrm{d}i_{dr}}{\mathrm{d}t} q_0 + i_{dr} p\omega q_1$$

Using the controls v_α, v_β given in (17) and setting $\alpha_i\omega - \hat{\alpha}_i\tilde{\omega} = \alpha_i\tilde{\omega} + \tilde{\alpha}_i\hat{\omega}$, $i = 4, 6$, one obtains

$$\frac{\mathrm{d}i_{\beta e}}{\mathrm{d}t} = -\alpha_6(\omega_e + \tilde{\omega}) q_0 - (k_3 + \delta) i_{\beta e} - \tilde{\alpha}_3 i_\beta - \tilde{\alpha}_4 q_0 \hat{\omega} + \frac{\tilde{\alpha}_5}{\alpha_5} \varrho_\beta - \tilde{\alpha}_6(\omega_r - \hat{\omega}) q_0 + (W_\beta - q_0(\alpha_4 - 2\alpha_6))\tilde{\omega}$$

$$\frac{\mathrm{d}i_{\alpha e}}{\mathrm{d}t} = \alpha_6(\omega_e + \tilde{\omega}) q_1 - (k_4 + \delta) i_{\alpha e} - \tilde{\alpha}_3 i_\alpha - \tilde{\alpha}_4 q_1 \hat{\omega} + \frac{\tilde{\alpha}_5}{\alpha_5} \varrho_\alpha + \tilde{\alpha}_6(\omega_r - \hat{\omega}) q_1 + (W_\alpha + q_1(\alpha_4 - 2\alpha_6))\tilde{\omega} \quad (31)$$

where W_α, W_β are as in (18). Now, choosing

$$\mu_1(t, y, \hat{\omega}) = -(\psi + W_\alpha i_{\alpha e} + W_\beta i_{\beta e})$$

and since

$$|\alpha_4 - 2\alpha_6| = k_m \left| \frac{1}{L} - \frac{2}{J} \right| \leq 2\delta$$

and $|q_0| \leq 1$, $|q_1| \leq 1$, so that

$$(\alpha_4 - \alpha_6)(q_1 i_{\alpha e} - q_0 i_{\beta e})\tilde{\omega} \leq \delta(i_{\alpha e}^2 + \tilde{\omega}^2) + \delta(i_{\beta e}^2 + \tilde{\omega}^2) = \delta(i_{\alpha e}^2 + i_{\beta e}^2 + 2\tilde{\omega}^2)$$

(25) becomes

$$\dot{V} \leq -k_1 \vartheta_e^2 - (\omega_e \quad \omega) Q \begin{pmatrix} \omega_e \\ \tilde{\omega} \end{pmatrix} + 2\delta\tilde{\omega}^2 - k_3 i_{\beta e}^2 - k_4 i_{\alpha e}^2 - \tilde{\alpha}_3 \left[i_{\alpha e} i_\alpha + i_{\beta e} i_\beta + \frac{\dot{\tilde{\alpha}}_3}{\lambda_3} \right] - \tilde{\alpha}_4 \left[(-i_{\alpha e} q_1 + i_{\beta e} q_0)\hat{\omega} + \frac{\dot{\tilde{\alpha}}_4}{\lambda_4} \right] - \frac{\tilde{\alpha}_5}{\alpha_5} \left[-i_{\alpha e}\varrho_\alpha - i_{\beta e}\varrho_\beta + \frac{\dot{\tilde{\alpha}}}{\lambda_5} \right] - \tilde{\alpha}_6 \left[(\omega_r - \hat{\omega})(-i_{\alpha e} q_1 + i_{\beta e} q_0) + \frac{\dot{\tilde{\alpha}}_6}{\lambda_6} \right] \quad (32)$$

Note that with the expression chosen for $\mu_1(t, y, \hat{\omega})$ one has

$$\dot{\hat{\omega}} = -(W_\beta - \alpha_6 q_0) i_{\beta e} - (W_\alpha + \alpha_6 q_1) i_{\alpha e} - (\bar{k}_1 + \alpha_2)\tilde{\omega} - \tilde{c} - \frac{\tilde{\alpha}_1}{\alpha_1}\psi - \tilde{\alpha}_2\hat{\omega} \quad (33)$$

With the expression so far computed, setting

$$\left. \begin{aligned} \dot{\tilde{\alpha}}_3 &= \lambda_3(-i_{\alpha e} i_\alpha - i_{\beta e} i_\beta) \\ \dot{\tilde{\alpha}}_4 &= \lambda_4(i_{\alpha e} q_1 - i_{\beta_e} q_0)\hat{\omega} \\ \dot{\tilde{\alpha}}_5 &= \lambda_5(i_{\alpha e}\varrho_\alpha + i_{\beta e}\varrho_\beta) \\ \dot{\tilde{\alpha}}_6 &= \lambda_6(i_{\alpha e} q_1 - i_{\beta e} q_0)(\omega_r - \hat{\omega}) \end{aligned} \right\} \quad (34)$$

(32) yields (19). Finally, note that equations (6), (15), (31), (33), (26), (30), (28), (34) are in the form (20).

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