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Nonlinear H^{∞} Tracking Control for Synchronous Motors

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This work deals with the nonlinear H^{∞} control of a permanent magnet synchronous motor subject to parameter variations during its operation. The control aim is to track a desired angular trajectory. The controller design is achieved in two steps: first, a controller based on the nominal parameter values is determined, while the parameter variations and the deterioration of the ideal control law, due to the use of a real actuator, are considered disturbances acting on the nominal plant. Second, a nonlinear H^{∞} controller is designed in order to satisfy the sub-optimal attenuation problem in the case of availability of the whole state vector. This second phase entails the approximated resolution of a (nonlinear) Hamilton–Jacobi–Isaacs equation. The resulting controller is tested in numerical simulations and its performances are compared with those of an LQR controller.

Keywords: Nonlinear H^{∞} control; Nonlinear systems; Parameter uncertainty; Synchronous motor

1. Introduction

The permanent magnet (PM) synchronous motors are interesting actuators which can be usefully utilized in all those applications in which their compact structure, reliability and high performances are important factors. Moreover, PM synchronous motors are easily controllable since the state variables can be considered available for measurement.

The performances of the PM synchronous motors are affected by the nonlinear nature of their dynamics, by the presence of parameters which may vary during operations, and by the deterioration of the ideal input signals by a real actuator. The motor's parameters are the stator winding resistance R and inductance L, the load torque C_l, the inertia J, the viscous friction coefficient f, the torque constant k_m. During the motor's operation they may vary due to heating (the resistance R) or to the geometric characteristics of the motor (the inductance L for instance) or finally to unpredictable operative situations (typically C_l and f). As far as the actual actuators, in a real situation a computational delay is always present. Moreover, the control signals are held during the sampling period by a zero-order holder. Finally, a real inverter can apply only a signal which is a mean, on the sampling period, of the calculated control value. For this facts, starting from the works of [3,4,18,32], which are based on the knowledge of the nominal motor parameters, various works have been done in the direction of taking into account the parameter variation. The improvements in adaptive control led to interesting works ([11,25] and references therein) in which the parameters, supposed unknown but constant or slowly varying, are adapted during the operation. However, these approaches are not applicable when these parameters are rapidly varying. On the other hand, an effort has been produced towards a design

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of a control law taking into account also the deterioration of the continuous control law in a digital control scheme with a real actuator [8,12,13,24,26], even if some work has still to be done in order to cover all the applicative aspects.

In this work a different approach is followed in order to consider at the same time all the aforementioned aspects affecting the performances of a PM synchronous motor. More precisely, regarding the parameter variation and the action of a real actuator as disturbances acting on the system, a control law is obtained so that the best attenuation is obtained for these influences on the motor dynamics. Due to the well-known qualities of the linear H^∞ control, which demonstrates to be robust against load disturbances and motor parameter variations [6,7,9,15–17,20], and considering recent results in nonlinear H^∞ control [19,30], in this paper we design a nonlinear H^∞ control for a PM synchronous motor in the case of tracking of an angular trajectory. The main contribution of this work is that with this approach one can face simultaneously parameter variations and deteriorations due real actuators. Moreover, since in general the resulting H^∞ nonlinear controller cannot be implemented on the available digital signal processors, it is necessary to simplify the control law so designed. Therefore, an approximated controller workable in on-line implementations is derived. This simplification of the exact controller, which is a desired property in real-time applications, represents another contribution of this paper.

In the linear state–space formulation [10] the problem of reducing the H^∞ norm of the closed-loop system can be reformulated as a two person, zero sum, differential game [2]. Therefore, the existence of the controller is related to the solution of the algebraic Riccati equation arising in the linear quadratic differential game theory. In the nonlinear context and in the case of full information, i.e., when all the state variables are available, this Riccati equation is replaced by the so-called Hamilton–Jacobi–Isaacs equation. In our problem, we first design a stabilizing controller on the basis of the nominal model of the motor, and then a nonlinear H^∞ control is determined in order to solve the H^∞ sub-optimal control problem in the case of full information. As anticipated, this last controller leads to a (nonlinear) Hamilton–Jacobi–Isaacs equation; since it cannot be solved analytically, we look for an approximated solution by using a rather standard approach [1,23], consisting of its series expansion and regrouping of the terms of homogeneous powers in the state variables. An infinite number of equations are hence determined; from the practical point of view we compute an approximated nonlinear controller by considering in the controller only the first nonlinear terms involved in the resolution of the Hamilton–Jacobi–Isaacs equation.

The paper is organized as follows. We recall the mathematical model of a PM synchronous motor and we state the control problem in Section 2. In Section 3 some aspects of the nonlinear H^∞ control in the case of full information are briefly recalled and the controller is derived. Comparative simulation results are presented in Section 4. Some observations conclude the paper.

2. Mathematical Model of a Synchronous Motor

The dynamical model of a PM synchronous motor in the (α, β) frame is given by [21,22,25,26]

$$\begin{aligned} \frac{\mathrm{d}i_\alpha}{\mathrm{d}t} &= -\frac{R}{L}i_\alpha + \frac{k_\mathrm{m}}{L}\omega\sin p\vartheta + \frac{1}{L}v_\alpha, \\ \frac{\mathrm{d}i_\beta}{\mathrm{d}t} &= -\frac{R}{L}i_\beta - \frac{k_\mathrm{m}}{L}\omega\cos p\vartheta + \frac{1}{L}v_\beta, \\ \dot\omega &= \frac{k_\mathrm{m}}{J}(-i_\alpha\sin p\vartheta + i_\beta\cos p\vartheta) - \frac{f}{J}\omega - \frac{C_\mathrm{l}}{J}, \\ \dot\vartheta &= \omega, \end{aligned} \tag{1}$$

in which

$$v = \begin{pmatrix} v_\alpha \\ v_\beta \end{pmatrix}, \quad i = \begin{pmatrix} i_\alpha \\ i_\beta \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_\alpha \\ \phi_\beta \end{pmatrix}$$

are stator voltage, current and flux vectors, respectively; $k_\mathrm{m} = p\phi_\mathrm{f}$ is the motor torque constant, with p the pole pair number and ϕ_f the rotor flux generated by the the PMs. The assumptions used here are the linearity of the magnetic materials, the symmetry of the rotor and between the two phases, the nonlinear flux density distribution due to air gap geometry only, the negligibility of the magnetic hysteresis and of the Foucault currents, and the smooth pole hypothesis.

An useful representation of a PM synchronous motor is also that in the fixed rotor frame (d, q), introduced by Park [21,28] and based on the transformation given by the orthogonal matrix

$$T(\vartheta) = \begin{pmatrix} \cos p\vartheta & \sin p\vartheta \\ -\sin p\vartheta & \cos p\vartheta \end{pmatrix}.$$

With $T(\vartheta)$ the vectors i and v, expressed in the (α, β) frame, are transformed into vectors in the (d, q) frame

$$\begin{aligned} i_d &= i_\alpha\cos p\vartheta + i_\beta\sin p\vartheta, \quad v_d = v_\alpha\cos p\vartheta + v_\beta\sin p\vartheta, \\ i_q &= -i_\alpha\sin p\vartheta + i_\beta\cos p\vartheta, \quad v_q = -v_\alpha\sin p\vartheta + v_\beta\cos p\vartheta. \end{aligned} \tag{2}$$

The dynamics (1) expressed in terms of currents and voltages in rotating (d, q) coordinates become

$$\begin{aligned}
\frac{\mathrm{d}i_d}{\mathrm{d}t} &= -\frac{R}{L}i_d + p\omega i_q + \frac{1}{L}v_d \\
&= -\alpha_3 i_d + p\omega i_q + \alpha_5 v_d, \\
\frac{\mathrm{d}i_q}{\mathrm{d}t} &= -\frac{R}{L}i_q - p\omega i_d - \frac{k_\mathrm{m}}{L}\omega + \frac{1}{L}v_q \\
&= -\alpha_3 i_q - p\omega i_d - \alpha_4\omega + \alpha_5 v_q, \\
\dot{\omega} &= \frac{k_\mathrm{m}}{J}i_q - \frac{f}{J}\omega - \frac{C_\mathrm{l}}{J} = \alpha_1 i_q - \alpha_2\omega - c, \\
\dot{\vartheta} &= \omega,
\end{aligned} \tag{3}$$

where

$$c = \frac{C_\mathrm{l}}{J}, \quad \alpha_1 = \frac{k_\mathrm{m}}{J}, \quad \alpha_2 = \frac{f}{J},$$
$$\alpha_3 = \frac{R}{L}, \quad \alpha_4 = \frac{k_\mathrm{m}}{L}, \quad \alpha_5 = \frac{1}{L}.$$

Note that $k_\mathrm{m} i_q$ represents the electromagnetic torque generated by the motor when the q reference axis is aligned with rotor flux vector.

The control aim is the asymptotic tracking of a reference angular trajectory ϑ_r, imposing at the same time a reference i_{dr} for the direct component i_d of the current. To overcome the problem of strong current excitation, consequent to the application of a step reference variation, a polynomial reference trajectory can be used of the form [26]

$$\vartheta_r(t) = \sum_{i=0}^{N} c_i t^i = \begin{cases} \left(35\frac{t^4}{t_\mathrm{f}^4} - 84\frac{t^5}{t_\mathrm{f}^5} + 70\frac{t^6}{t_\mathrm{f}^6} - 20\frac{t^7}{t_\mathrm{f}^7}\right)\vartheta_{r\infty}, & \text{if } t < t_\mathrm{f}, \\ \vartheta_{r\infty}, & \text{if } t \geq t_\mathrm{f}, \end{cases} \tag{4}$$

where $N = 7$ and c_i have been computed for ensuring the constraints $\vartheta_\mathrm{r}(0) = 0$, $\vartheta_\mathrm{r}(t_\mathrm{f}) = \vartheta_{\mathrm{r}\infty}$ and the requirements $\dot{\vartheta}_\mathrm{r}(0) = 0$, $\ddot{\vartheta}_\mathrm{r}(0) = 0$, $\dddot{\vartheta}_\mathrm{r}(0) = 0$, $\dot{\vartheta}_\mathrm{r}(t_\mathrm{f}) = 0$, $\ddot{\vartheta}_\mathrm{r}(t_\mathrm{f}) = 0$, $\dddot{\vartheta}_\mathrm{r}(t_\mathrm{f}) = 0$, t_f being the response time at 99% of the steady-state value. Moreover, the reference i_{dr} is taken as a function of the reference velocity ω_r: when $\omega_\mathrm{r} \leq \omega_\mathrm{n}$ (ω_n is the nominal speed) then $i_{\mathrm{dr}} = 0$, while if $\omega_\mathrm{r} > \omega_\mathrm{n}$ then $i_{\mathrm{dr}} = i_{\mathrm{dr}}(\dot{\vartheta}_\mathrm{r})$. This allows for realizing the so-called field weakening at high speed and leads to the maximization of the produced torque. Note that if $i_{\mathrm{dr}}(\dot{\vartheta})$ is the reference for i_d, $\psi = \alpha_{10} i_{qr}$ is the desired value for the angular acceleration.

These control objectives must be obtained in presence of parameter variations in the real motor. The real parameters in (3) can be rewritten as

$$\begin{aligned} c &= c_0(t) + c_v(t), \\ \alpha_i &= \alpha_{i0} + \alpha_{iv}(t), \quad i = 1, \ldots, 5 \end{aligned} \tag{5}$$

with α_{i0}, $i = 1, \ldots, 5$, the nominal values and $c_0(t)$ the nominal load term C_{l0}/J; here $\alpha_{iv}(t)$, $c_v(t)$ denote their variations. Furthermore, in presence of real actuators the actual controls v_d, v_q differ from the "ideal" ones, v_{di}, v_{qi}, due to the computational delay, the presence of zero-order holders, and the fact that the inverter applies a mean of the ideal control signal. Hence, defining

$$w_q = v_q - v_{qi}, \quad w_d = v_d - v_{di}, \tag{6}$$

the difference between the piecewise constant controls v_q, v_d, actually applied by the actuator to the motor, and the ideal continuous controls v_{qi}, v_{di}, one may regard the degradation of the "ideal" control law due to a real actuator as a disturbance (w_q w_d) acting on the system.

Therefore, the control problem is to find a control v such that, when $\alpha_{iv}(t) = 0$, $c_v(t) = 0$, $w_d(t) = 0$, $w_q(t) = 0$, one has

$$\lim_{t\to\infty} i_d = i_d(\dot{\vartheta}_\mathrm{r}), \quad \lim_{t\to\infty} \vartheta = \vartheta_\mathrm{r},$$

and such that, in presence of parameter variations and disturbances of other source, their influence on a certain penalty variable z is optimally attenuated.

3. Nonlinear H^∞ Control for Synchronous Motors

In this Section a nonlinear H^∞ controller is determined in order to solve the control problem and to obtain the best attenuation of the parameter variation effects on the motor dynamics. We first recall some elements of the state H^∞ control; the reader is referred to [19,30 and references therein] for an exposition of the H^∞ control theory in the case of full information.

3.1. A Review of the State H^∞ Control

The H^∞ sub-optimal control problem in the case of full information, namely when state variables and disturbances are available for feedback, was treated by [30]. This problem is strictly related with the concept of dissipativity [31]. One considers the following nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + k(x)w, \quad u \in \mathbb{R}^m, \quad x \in \mathbb{R}^n, \quad w \in \mathbb{R}^q, \\ y &= h(x), \quad y \in \mathbb{R}^p, \end{aligned} \tag{7}$$

with $f(0)=0$, $f(x)$, $g(x)$, $k(x)$, $h(x)$ matrices of appropriate dimensions, with elements which are C^∞ functions. The control problem is to achieve local asymptotic stability at $x=0$ and disturbance attenuation of the exogenous input w on a certain penalty variable

$$z=\begin{pmatrix} h(x) \\ u \end{pmatrix}.$$

It is well known that this problem can be reformulated as a problem of determination of a feedback $u=\alpha(x)$ which renders the system (7) locally dissipative with respect to a supply rate of the form [31]

$$s(w,z)=\gamma^2\|w\|^2-\|z\|^2.$$

Moreover, this problem can be seen as a two player differential game with zero sum, with u and w respectively the minimizing and the maximizing player controls [2]. The Hamiltonian function associated with this game is

$$H(x,p,w,u) \\ =p^{\mathrm{T}}[f(x)+g(x)u+k(x)w]+\|z\|^2-\gamma^2\|w\|^2,$$

which, in a neighborhood of $(x,p,w,u)=(0,0,0,0)$, has a unique saddle point in (w,u) for each (x,p), given by

$$w_*=\frac{1}{2\gamma^2}k^{\mathrm{T}}(x)p, \quad u_*=-\frac{1}{2}g^{\mathrm{T}}(x)p.$$

Note that

$$H(x,p,w,u_*)\le H_*(x,p)\le H(x,p,w_*,u),$$

where $H_*(x,p)=H(x,p,w_*,u_*)$. This is true, in particular, for $p=V_x^{\mathrm{T}}=\partial V^{\mathrm{T}}/\partial x$, with V a smooth storage function $V:\mathbb{R}^n\to\mathbb{R}$, nonnegative and vanishing at $x=0$. If now $V(x)$ is also positive definite and is such that the strict inequality

$$H_*(x,V_x^{\mathrm{T}})<0$$

holds true for each $x\neq 0$ in a neighborhood of $x=0$, then the system (7) with $u=u_*(x,V_x^{\mathrm{T}})=-\frac{1}{2}g^{\mathrm{T}}(x)V_x^{\mathrm{T}}(x)$ satisfies the following Hamilton–Jacobi–Isaacs (strict) inequality

$$V_x(x)[f(x)+g(x)u_*(x,V_x^{\mathrm{T}})+k(x)w] \\ +\|z\|^2-\gamma^2\|w\|^2<0, \tag{8}$$

namely the system (7) is dissipative in a neighborhood of $(x,w)=(0,0)$, and the state feedback $u=u_*(x,V_x^{\mathrm{T}})$ renders the feedback system locally asymptotically stable. This is the best input of the minimizing player, while $w=w_*(x,V_x^{\mathrm{T}})=(1/2\gamma^2)k^{\mathrm{T}}(x)V_x^{\mathrm{T}}(x)$ is the best strategy of the maximizing player, i.e., the worst possible disturbance affecting the system.

3.2. The Nonlinear H^∞ Controller for Synchronous Motors

The control design is performed in two steps: in the first one calculates the stabilizing control law for the nominal dynamics of the system, thereby obtaining the error dynamics of the system. In the second step the contribution due to parameter variations and the influence of the real actuator are treated as a disturbance acting on the system, and a nonlinear H^∞ controller is derived by solving a Hamilton–Jacobi–Isaacs equation, written for the error system, for sub-optimally counteracting this disturbance.

The first step can be performed making use of a simple (but tedious) backstepping procedure [29]. Since we are interested here mainly in the best attenuation of the effects of the parameter variations and disturbances, this step is summarized in Appendix 1, where a control, which solves the control problem for the nominal plant, is determined and the error dynamics of the PM synchronous motor are derived. With this control the error equations become (see Appendix 1 for further details)

$$\dot{x}_{\mathrm{e}}=A_0x_{\mathrm{e}}+B_0u+k(x_{\mathrm{e}})w \tag{9}$$

which are in the form (7), where $x_{\mathrm{e}}=(\vartheta_{\mathrm{e}}\ \omega_{\mathrm{e}}\ i_{qe}\ i_{de})^{\mathrm{T}}$ is the tracking error vector, and $u=(u_q\ u_d)^{\mathrm{T}}$ is the new input, designed hereinafter for the sub-optimal disturbance attenuation. To this aim u_q, u_d will be functions of the state, i.e., $u_q=\xi_q(x_{\mathrm{e}})$, $u_d=\xi_d(x_{\mathrm{e}})$. Furthermore, w is an extended disturbance vector, which takes into account all the parameter variations and sources of disturbances, $k(x_{\mathrm{e}})$ is a state-dependent matrix, and

$$A_0=\begin{pmatrix} -k_1 & 1 & 0 & 0 \\ -1 & -k_2 & \alpha_{10} & 0 \\ 0 & -\alpha_{10} & -k_3 & 0 \\ 0 & 0 & 0 & -k_4 \end{pmatrix},$$

$$B_0=\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{10}$$

Note that when $w=0$, $x_e=0$ is a globally exponentially stable equilibrium point, as it can be checked by taking the time-derivative of the radially unbounded Lyapunov function $V=\frac{1}{2}x_e^T x_e$ along the system trajectories, obtaining

$$\dot{V} = -x_e^T Q x_e < 0,$$

$$Q = \begin{pmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}, \tag{11}$$

where $k_1, \ldots, k_4 > 0$.

The output is $y = x_e$, while the penalty variable is

$$z = \begin{pmatrix} C_0 x_e \\ u \end{pmatrix}, \quad C_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In z the tracking errors ϑ_e and i_{de} are considered, as well as the control effort u_q, u_d necessary for attenuating the disturbances.

Let us design now the control u so that the disturbance attenuation problem is solved. Referring to Section 3.1, this control is given by

$$u = -\tfrac{1}{2} g^T(x_e) V_{x_e}^T(x_e) = -\tfrac{1}{2} B_0^T V_{x_e}^T(x_e),$$

with $V_{x_e}(x_e)$ the derivative of the Lyapunov function $V(x_e)$, solving the Hamilton–Jacobi–Isaacs inequality (8), which in our case takes the form

$$V_{x_e}(x_e)\left[A_0 x_e - \frac{1}{4} B_0 B_0^T(x) V_{x_e}^T(x_e)\right] + x_e^T C_0^T C_0 x_e + \frac{1}{\gamma^2} V_{x_e} k(x_e) k^T(x_e) V_{x_e}^T(x_e) < 0. \tag{12}$$

The actual solution of this inequality is difficult to find, and the only feasible way is to look for an approximated one. The problem of determining an approximated polynomial solution of a fixed degree, for a Hamilton–Jacobi equation arising in the nonlinear optimal control, has been solved in [1,23]. In the spirit we follow here the same ideas applied in [5,14], where this methodology was applied for the determination of approximated solutions in the discrete-time context, for problems regarding H^∞ control and disturbance rejection in the regulation problem. It is shown that it is possible to determine each term $V^{[i+1]}(x_e)$ from the terms of lower order if one expresses the functions $V(x_e)$, supposed analytic, and $k(x_e)$, $\xi_q(x_e)$, $\xi_d(x_e)$ as a series of homogeneous polynomials

$$\begin{aligned}
V(x_e) &= \sum_{i=1}^{\infty} V^{[i+1]}(x_e), \\
k(x_e) &= K_0 + k_1(x_e) + k_2(x_e) + \mathcal{O}(x_e^{\otimes 3}), \\
\xi_q(x_e) &= \xi_{q1}(x_e) + \xi_{q2}(x_e) + \mathcal{O}(x_e^{\otimes 3}), \\
\xi_{q1}(x_e) &= F_{q1}\vartheta_e + F_{q2}\omega_e + F_{q3} i_{qe} + F_{q4} i_{de}, \\
\xi_d(x_e) &= \xi_{d1}(x_e) + \xi_{d2}(x_e) + \mathcal{O}(x_e^{\otimes 3}), \\
\xi_{d1}(x_e) &= F_{d1}\vartheta_e + F_{d2}\omega_e + F_{d3} i_{qe} + F_{d4} i_{de},
\end{aligned}$$

where "$\otimes$" indicates the tensor product and with $k_i(x_e)$, $\xi_{qi}(x_e)$, $\xi_{di}(x_e)$, $i=1,2$ the terms in x_e, $x_e^{\otimes 2}$ respectively, $\mathcal{O}(x_e^{\otimes 3})$ indicating the terms of higher order, and

$$K_0 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix},$$

$$k_1(x_e) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \chi_1(x_e) & \chi_2(x_e) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{a_2}{\alpha_{10}}\chi_1(x_e) & \frac{a_2}{\alpha_{10}}\chi_2(x_e) & -\chi_1(x_e) & \chi_2(x_e) & \chi_3(x_e) & \frac{p}{\alpha_{50}} i_{de} & 0 & -\frac{p}{\alpha_{50}}\chi_2(x_e) & 0 & 0 & 0 \\ 0 & 0 & -i_{de} & 0 & \chi_4(x_e) & -\frac{p}{\alpha_{50}}\chi_1(x_e) & \frac{p}{\alpha_{10}\alpha_{50}}\chi_2(x_e) & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$k_2(x_e) = \begin{pmatrix} 0&0&0&0& 0 & 0&0&0&0&0&0 \\ 0&0&0&0& 0 & 0&0&0&0&0&0 \\ 0&0&0&0& \xi_{q2}(x_e) - \frac{p}{\alpha_{50}}\chi_2(x_e) i_{de} & 0&0&0&0&0&0 \\ 0&0&0&0& \xi_{d2}(x_e) + \frac{p}{\alpha_{50}}\chi_1(x_e)\chi_2(x_e) & 0&0&0&0&0&0 \end{pmatrix}$$

with $\chi_i(x_e)$, $i=1,\ldots,4$, linear functions of x_e given by (19) and by

$$\begin{aligned}\chi_3(x_e) &= \left(a_3+\frac{1}{\alpha_{50}}F_{q1}\right)\vartheta_e+\left(a_4+\frac{1}{\alpha_{50}}F_{q2}\right)\omega_e\\&\quad+\left(a_5+\frac{1}{\alpha_{50}}F_{q3}\right)i_{qe}+\frac{1}{\alpha_{50}}F_{q4}i_{de}\\\chi_4(x_e) &= \frac{1}{\alpha_{50}}F_{d1}\vartheta_e+\frac{1}{\alpha_{50}}F_{d2}\omega_e+\frac{1}{\alpha_{50}}F_{d3}i_{qe}\\&\quad+\left(\frac{\alpha_{30}-k_4}{\alpha_{50}}+\frac{1}{\alpha_{50}}F_{d4}\right)i_{de}.\end{aligned}$$

Note that the term $V^{[2]}(x_e)$ will appear in a Riccati equation involving the linear approximation of the plant. With this recursive procedure, it is hence possible to determine a polynomial approximation for $V(x_e)$. Therefore, setting

$$\begin{aligned}V_{x_e}(x_e) &= \sum_{i=1}^{\infty}V_{x_e}^{[i]}(x_e)\\&= 2x_e^{\mathrm T}P_1+\left(x_e^{\otimes 2}\right)^{\mathrm T}P_2^{\mathrm T}+\left(x_e^{\otimes 3}\right)^{\mathrm T}P_3^{\mathrm T}\\&\quad+\sum_{i=4}^{\infty}\left(x_e^{\otimes i}\right)^{\mathrm T}P_i^{\mathrm T},\end{aligned}$$

$P_1=P_1^{\mathrm T}$, and substituting in (12), one works out

$$\begin{aligned}&x_e^{\mathrm T}(P_1A_0+A_0^{\mathrm T}P_1+P_1M_0P_1+C_0^{\mathrm T}C_0)x_e\\&\quad+(x_e^{\otimes 2})^{\mathrm T}P_2^{\mathrm T}(A_0+M_0P_1)x_e\\&\quad+(x_e^{\otimes 3})^{\mathrm T}P_3^{\mathrm T}(A_0+M_0P_1)x_e\\&\quad+\frac{1}{4}(x_e^{\otimes 2})^{\mathrm T}P_2^{\mathrm T}M_0P_2(x_e^{\otimes 2})\\&\quad+\frac{4}{\gamma^2}x_e^{\mathrm T}P_1k_1(x_e)k_1^{\mathrm T}(x_e)P_1x_e\\&\quad+(x_e^{\otimes 4})^{\mathrm T}P_4^{\mathrm T}(A_0+M_0P_1)x_e+\cdots+O(x_e^{\otimes 6})<0,\end{aligned}\tag{13}$$

since

$$k_1(x_e)K_0^{\mathrm T}=0,\quad k_2(x_e)K_0^{\mathrm T}=0,$$

and where it has been set

$$M_0=\frac{4}{\gamma^2}K_0K_0^{\mathrm T}-B_0B_0^{\mathrm T}=\begin{pmatrix}0&0&0&0\\0&\frac{4}{\gamma^2}&0&0\\0&0&\frac{4-\gamma^2}{\gamma^2}&0\\0&0&0&\frac{4-\gamma^2}{\gamma^2}\end{pmatrix}.$$

Note that in (13) the terms in $x_e^{\otimes(i+1)}$ can be split into a term $(x_e^{\otimes i})^{\mathrm T}P_i^{\mathrm T}(A_0+M_0P_1)x_e$ and terms which are functions of P_j, $j<i$, which can be properly rewritten in the form

$$(x_e^{\otimes i})^{\mathrm T}R_i^{\mathrm T}x_e$$

with R_i an appropriate matrix. For instance, for $i=3$:

$$\begin{aligned}&\frac{1}{4}(x_e^{\otimes 2})^{\mathrm T}P_2^{\mathrm T}M_0P_2(x_e^{\otimes 2})\\&\quad+\frac{4}{\gamma^2}x_e^{\mathrm T}P_1k_1(x_e)k_1^{\mathrm T}(x_e)P_1x_e\\&\quad=(x_e^{\otimes 3})^{\mathrm T}R_3^{\mathrm T}x_e.\end{aligned}$$

The use of commercial software for symbolic manipulation can greatly help this task. Therefore, (13) can be rewritten as follows:

$$\begin{aligned}&x_e^{\mathrm T}(P_1A_0+A_0^{\mathrm T}P_1+P_1M_0P_1+C_0^{\mathrm T}C_0)x_e\\&\quad+(x_e^{\otimes 2})^{\mathrm T}P_2^{\mathrm T}(A_0+M_0P_1)x_e\\&\quad+(x_e^{\otimes 3})^{\mathrm T}\left[P_3^{\mathrm T}(A_0+M_0P_1)+R_3^{\mathrm T}\right]x_e\\&\quad+\sum_{i=4}^{\infty}(x_e^{\otimes i})^{\mathrm T}\left[P_i^{\mathrm T}(A_0+M_0P_1)+R_i^{\mathrm T}\right]x_e<0,\end{aligned}$$

and we need to solve the following equations:

$$\begin{aligned}P_1A_0+A_0^{\mathrm T}P_1+P_1M_0P_1+C_0^{\mathrm T}C_0&=0,\\P_2^{\mathrm T}(A_0+M_0P_1)&=0,\\P_3^{\mathrm T}(A_0+M_0P_1)+R_3^{\mathrm T}&=0,\\P_i^{\mathrm T}(A_0+M_0P_1)+R_i^{\mathrm T}&=0,\quad i\geq 4.\end{aligned}\tag{14}$$

The first one is a Riccati equation which involves, as anticipated, the linear approximation of the plant. If the matrix $(A_0+M_0P_1)$ is invertible, the other equations can be solved by iteration once the matrices R_i are determined. Therefore, one works out

$$\begin{aligned}P_2=0,\quad P_3&=-(A_0+M_0P_1)^{-\mathrm T}R_3,\\P_i&=-(A_0+M_0P_1)^{-\mathrm T}R_i,\quad i\geq 4.\end{aligned}$$

The control ensuring the sub-optimal disturbance attenuation is hence:

$$\begin{aligned}u=\begin{pmatrix}u_q\\u_d\end{pmatrix}&=-B_0^{\mathrm T}P_1x_e+\frac{1}{2}B_0^{\mathrm T}(A_0+M_0P_1)^{-\mathrm T}R_3x_e^{\otimes 3}\\&\quad+\frac{1}{2}\sum_{i=4}^{\infty}B_0^{\mathrm T}(A_0+M_0P_1)^{-\mathrm T}R_ix_e^{\otimes i}.\end{aligned}\tag{15}$$

Although this control could seem complicated and large in dimension, in Section 4 it is shown how it is possible to reduce its complexity by considering an approximation, suitable for on–line implementations.

4. Simulation Results

Hereinafter an approximation of the controller designed in Section 3 has been applied to a PM synchronous motor, whose nominal characteristics are reported in Appendix 2, Table 1. Considering the control law (15) and (17), we consider only the first nonlinear terms in the control (15). This corresponds to solve only the first three equations (14). Therefore, the implemented nonlinear controller is

$$v = \begin{pmatrix} v_{qi} \\ v_{di} \end{pmatrix} = v_0 + u_L(x_e) + u_{NL}(x_e),$$

$$v_0 = \begin{pmatrix} v_{q0} \\ v_{d0} \end{pmatrix},$$

$$v_{q0} = \frac{1}{\alpha_{50}}\left[\alpha_{30} i_q + p\omega i_d + \alpha_{40}\omega - \frac{a_1 k_1 - a_2}{\alpha_{10}}\vartheta_e + \frac{a_1 + a_2 k_2 - \alpha_{10}^2}{\alpha_{10}}\omega_e - (k_3 + a_2) i_{qe} + \frac{1}{\alpha_{10}}\dot{\phi}_1\right],$$

$$v_{d0} = \frac{1}{\alpha_{50}}\left[\alpha_{30} i_d - p\omega i_q - k_4 i_{de} + \frac{\mathrm{d}i_{dr}}{\mathrm{d}t}\right],$$

$$u_L(x_e) = \frac{1}{\alpha_{50}}\begin{pmatrix} u_{q,L}(x_e) \\ u_{d,L}(x_e) \end{pmatrix} = -\frac{1}{\alpha_{50}} B_0^{\mathrm{T}} P_1 x_e,$$

$$u_{NL}(x_e) = \frac{1}{\alpha_{50}}\begin{pmatrix} u_{q,NL}(x_e) \\ u_{d,NL}(x_e) \end{pmatrix} = \frac{1}{2\alpha_{50}} B_0^{\mathrm{T}} (A_0 + M_0 P_1)^{-\mathrm{T}} R_3 x_e^{\otimes 3}.$$

In particular, the implementation of $u_{NL}(x_e)$ requires the determination of the nonzero elements of the matrix R_3. In Appendix 2, Table 2, are reported those terms of R_3 which are significantly different from zero, say greater than 10^{-8}. Taking into account all these entries, the (4×1) vector $R_3 x_e^{\otimes 3}$ is given by the elements

$$\begin{aligned} E_1 &= R_{3,1}\vartheta_e^3 + R_{3,2}\vartheta_e^2\omega_e + R_{3,3}\vartheta_e^2 i_{qe} + R_{3,4}\vartheta_e\omega_e^2 \\ &\quad + R_{3,5}\vartheta_e\omega_e i_{qe} + R_{3,6}\vartheta_e i_{qe}^2 + R_{3,7}\vartheta_e i_{de}^2 \\ &\quad + R_{3,8} i_{qe} i_{de}^2 + R_{3,9}\omega_e i_{de}^2 \\ E_2 &= R_{3,10} i_{qe} i_{de}^2 + R_{3,11}\omega_e i_{de}^2 \\ E_3 &= R_{3,12} i_{qe}^3 + R_{3,13} i_{qe} i_{de}^2 \\ E_4 &= R_{3,14} i_{de}^3. \end{aligned}$$

It is clear that, from a practical point of view, this (approximated) controller may suffer from high calculation times. This problem can be partially mitigated by considering, in each particular application, the terms which give the main contributions in the control action. Therefore, a further approximation, consisting of considering only certain terms of the vector $u_{NL}(x_e)$ and based on simulations, may vary from case to case (different size, different parameter incertitude, etc.) and has to be evaluated for the particular application. Once this further approximation is done, the controller can be implemented on a microprocessor. For the PM synchronous motor considered in this section, such a preliminary investigation has shown that the contribution of $u_{q,NL}(x_e)$ to v_{qi} is negligible, while $u_{d,NL}(x_e)$ furnishes an important contribution, as shown in simulations later on. Therefore, it is possible to implement the only term E_4, which is hence the only nonlinear term in the approximated controller. This is an important advantage since such a simplification reduces the complexity of the control law and allows for real time implementations. In other words, this approximated controller keeps the improvements due to the contribution of the nonlinear terms, but at the same time represents a compromise between performance improvements and computational complexity.

In the simulations presented here we have considered a reference trajectory ϑ_r(rad) given by

$$\vartheta_r = \begin{cases} \sigma_1 & \text{if } t \in [0, t_f = 0.3)\,\text{s} \\ \vartheta_{f1} & \text{if } t \in [t_f, t_{s2} = 0.6)\,\text{s} \\ \sigma_2 & \text{if } t \in [t_{s2}, t_{s2} + t_f = 0.9)\,\text{s} \\ \vartheta_{f2} & \text{if } t \in [0.9, t_{s3} = 1.2)\,\text{s} \\ \sigma_3 & \text{if } t \in [1.2, t_{s3} + t_f = 1.5)\,\text{s} \\ (t - t_{s4})^2 + \vartheta_{f3} & \text{if } t \geq 1.5\,\text{s} \end{cases}$$

with t_f the response time at 99% of the steady-state value; here σ_i have the form (4), i.e.,

$$\sigma_1 = \vartheta_{f1}\left(35\frac{t^4}{t_f^4} - 84\frac{t^5}{t_f^5} + 70\frac{t^6}{t_f^6} - 20\frac{t^7}{t_f^7}\right),$$

$$\sigma_2 = \vartheta_{f1} + (\vartheta_{f2} - \vartheta_{f1})\left(35\frac{(t - t_{s2})^4}{t_f^4} - 84\frac{(t - t_{s2})^5}{t_f^5} + 70\frac{(t - t_{s2})^6}{t_f^6} - 20\frac{(t - t_{s2})^7}{t_f^7}\right),$$

$$\sigma_3 = \vartheta_{f2} + (\vartheta_{f3} - \vartheta_{f2})\left(35\frac{(t - t_{s3})^4}{t_f^4} - 84\frac{(t - t_{s3})^5}{t_f^5} + 70\frac{(t - t_{s3})^6}{t_f^6} - 20\frac{(t - t_{s3})^7}{t_f^7}\right),$$

where $\vartheta_{f1} = 45$, $\vartheta_{f2} = -35$, $\vartheta_{f3} = 50$ rad. Moreover, a reference value for i_{dr} (A) identically zero has been considered. Note that a nonzero reference direct current would not have sensibly influenced the performances of the controller.

Three different situations have been considered in what follows. The first simulation (denoted by "a") corresponds to the application of the approximated controller with $E_4 = 0$, namely with regards to the application of the linear control $v = v_0 + u_L(x_e)$. The gains used in this case are

$$k_1 = 250, \quad k_2 = 250, \quad k_3 = 300, \quad k_4 = 300,$$

and have been chosen in order to obtain, with the nominal control $v_0 = \begin{pmatrix} v_{q0} \\ v_{d0} \end{pmatrix}$ and in the case of absence of parameter uncertainties, a satisfactory closed-loop behavior. We recall that the closed-loop in this case is given by (9) with $u = 0$ and $w = 0$, and that the behavior is exponentially stable, see (11). In fact, simple passages show that

$$\|x_e(t)\| \leq \|x_e(0)\| e^{-k_{\min} t}, \quad k_{\min} = \min_{i=1,2,3,4} \{k_i\}.$$

In the second simulation (denoted by "b") the nonlinear contribution due to E_4 has been considered. In both cases a and b the matrix P_1, solution of the Riccati equation, is

$$P_1 = \begin{pmatrix} 2.00 \times 10^{-3} & 2.36 \times 10^{-6} & 1.87 \times 10^{-6} & 0 \\ 2.36 \times 10^{-6} & 4.07 \times 10^{-9} & 3.08 \times 10^{-9} & 0 \\ 1.87 \times 10^{-6} & 3.08 \times 10^{-9} & 4.48 \times 10^{-9} & 0 \\ 0 & 0 & 0 & 3.33 \times 10^{-3} \end{pmatrix}$$

which ensures an L_2 gain less or equal to $\gamma = 4.714 \times 10^{-3}$. A third case (denoted by "c") has been discussed; in this last case the controller is an LQR controller determined on the basis of the gains

$$k_1 = 375, \quad k_2 = 375, \quad k_3 = 450, \quad k_4 = 450,$$

which are 50% higher than the ones used for the cases a and b. In this way we can confront the nonlinear controller with a linear (and hence simpler) controller with higher gains. In fact, the previous gains have been chosen in order to obtain performances comparable with the nonlinear controller (case b), establishing a fair comparison between the cases b and c.

In these simulations we have supposed that the motor parameters and the load torque are subject to variations. More precisely we have considered that the nominal load torque is piecewise constant as follows:

$$C_{10} = \begin{cases} 5\,\text{N m} & t < 0.7\,\text{s} \\ 15\,\text{N m} & t \in [0.7, 1.6)\ \text{s} \\ 10\,\text{N m} & t \in [1.6, 2.1)\ \text{s} \\ 11.06 + 0.32\dot{\vartheta}_r^2\,\text{N m} & t \geq 2.1\,\text{s} \end{cases}$$

while the real value is subject to a superimposed sinusoidal variation. In particular note that for $t \geq 2.1$ s the load torque is supposed to depend on the (desired) speed; this is interesting in some industrial applications. Moreover, variations of 60% for J, 50% for f, 100% for R, 35% for L have been considered as benchmark for testing the performances of the three controllers

$$\begin{aligned} C_1 &= C_{10} + 0.05 d_c \sin \frac{2\pi}{T} t \\ k_m &= k_{m0} \\ J &= J_0\left(1 + 0.6 \sin \frac{\omega}{2\pi} t\right) \\ f &= f_0(1 + 0.5 \sin 50\omega t) \\ R &= R_0(1 + e^{-t/0.1}) \\ L &= L_0\left(1 + 0.35 \sin \frac{\omega}{p\pi} t\right) \end{aligned}$$

with $d_c = 15$ N m, $T = 0.14$ s. Finally, an inverter with switching time of 100 μs and nominal voltage of 300 V has been considered.

Figures 1–6 summarize these simulations. In particular, Fig. 1 shows the error angular position ϑ_e in the three cases; in order to use the same scale Fig. 1c has been truncated. Actually, it presents a negative peak of nearly -2.28 rad at about 1.38 s. It is clear that the nonlinear controller (curve b) ensures better performances. Better results have been obtained also for the error current i_{de} (Fig. 2) and in terms of instantaneous power (Fig. 3), denoted by $\mathcal{P}(t)$ and given by [21]

$$\mathcal{P}(t) = R i_d^2 + R i_q^2 + L\left(i_d \frac{\mathrm{d}i_d}{\mathrm{d}t} + i_q \frac{\mathrm{d}i_q}{\mathrm{d}t}\right) + k_m \omega i_q.$$

These better performances are explained by the fact that the nonlinear term due to E_4 determines a control action $u_{d,\mathrm{NL}}$ (Fig. 4) very effective in preventing control saturation of the inputs v_q, v_d (Figs 5 and 6).

5. Conclusions

In this work we have determined a nonlinear H^∞ controller for a PM synchronous motor in the case of tracking of an angular trajectory. The motor is subject to parameter variations and the presence of a real actuator has been considered. The controller design implies the resolution of a Hamilton–Jacobi–Isaacs equation. In general, this kind of equation has no analytic solution and this would have represented the main drawback in the design of the controller. However, we have avoided this problem using an iterative procedure, quite standard in the literature, in order to

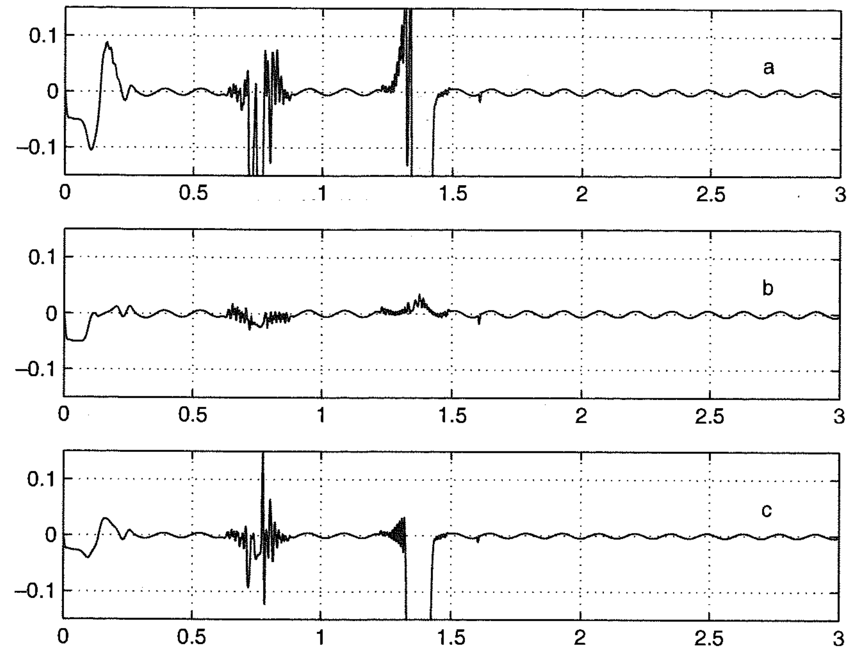

Fig. 1. Error angular position ϑ_e (rad).

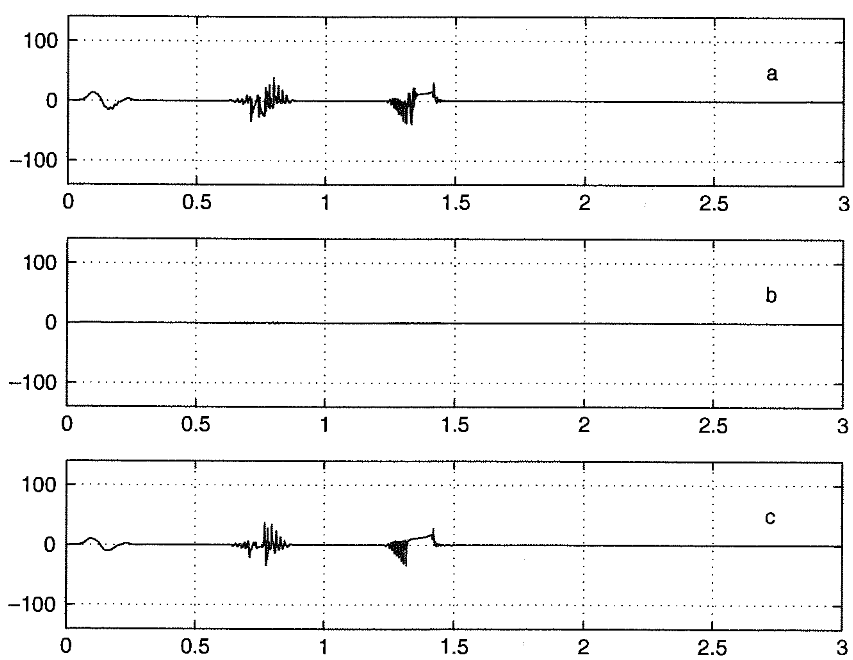

Fig. 2. Error current i_{de} (A).

derive a solution approximated at the third power of the state variables. Moreover, we have stressed that in practical on-line implementations of the controller, even this approximated solution is not workable, since the computational time becomes unacceptably high. Hence, we proceeded to a further approximation, taking into account only the main contributions of the nonlinear terms to the control law. The resulting controller has shown a better performance with respect to simpler controllers, derived on the basis of the linear model of the system. The main advantage of the proposed controller is that it can simultaneously

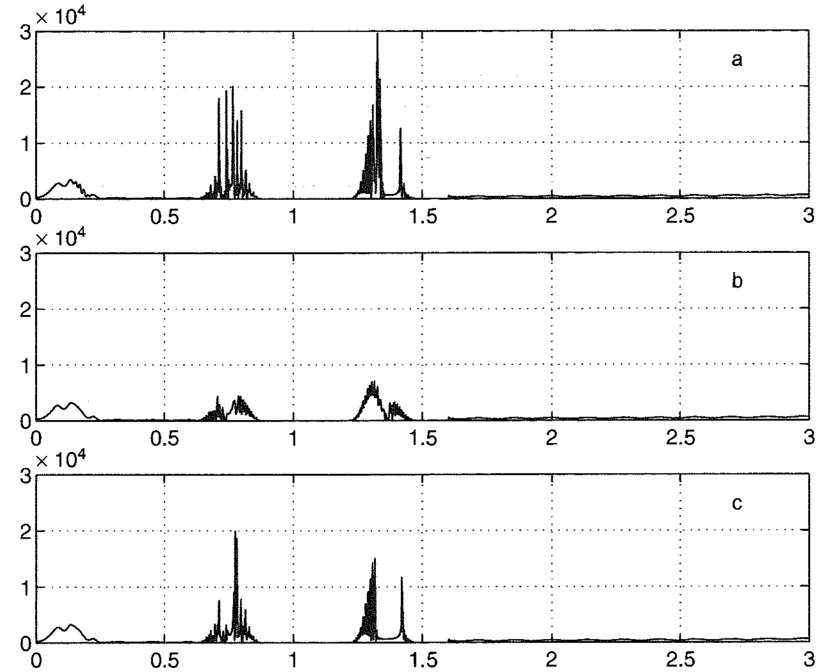

Fig. 3. Instantaneous power $\mathcal{P}(t)$ (N m s^{-1}).

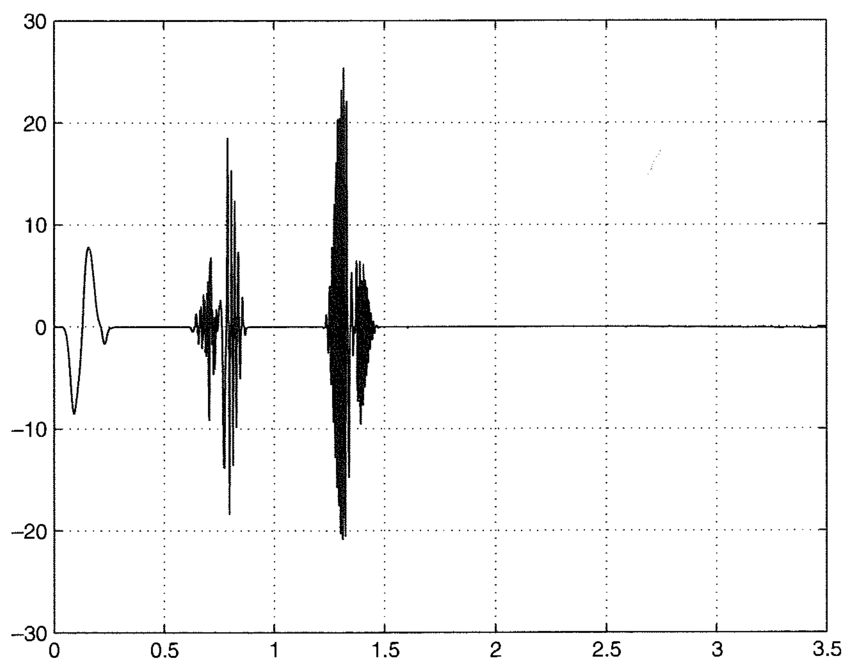

Fig. 4. Nonlinear contribution $u_{d,\mathrm{NL}}$ to v_d (V) (controller b).

take into account perturbations in the motor parameters and in the input acting on the system, due to the presence of a real actuator. Finally, the present methodology can be extended without technical difficulties to different types of synchronous motors or in order to consider also disturbances on the measured variables, while the extention to the discrete-time context would require a more attentive analysis, either for the backstepping procedure [27] or the resolution of the Hamilton–Jacobi–Isaacs

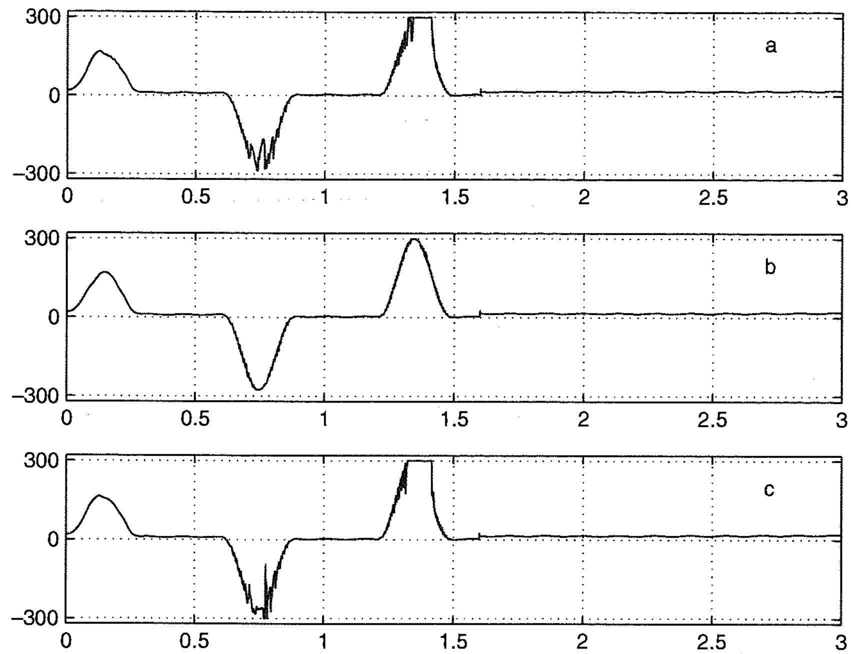

Fig. 5. Voltage v_q (V).

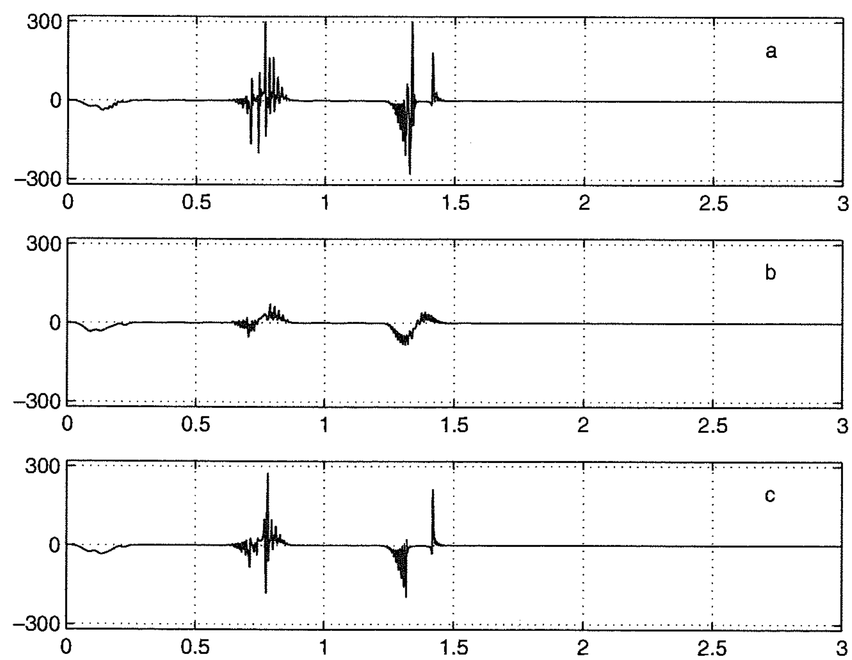

Fig. 6. Voltage v_d (V).

equation [14]. A potential drawback of this method reflects the difficulty of solving analytically the Hamilton–Jacobi–Isaacs equation, especially in the case of highly nonlinear motors. In fact, in these cases one needs to determine a solution of appropriate (possibly high) degree and this implies the determination of a large number of coefficients appearing in the approximated solution. The fact that this calculus is performed off-line would mitigate this inconvenience.

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Appendix 1: Derivation of Eq. (9)

In what follows we calculate the stabilizing control law for the nominal dynamics of the system by using a backstepping procedure [29]. Let

$$\vartheta_e = \vartheta - \vartheta_r$$

be the tracking error and define

$$\omega_r = \dot\vartheta_r - k_1\vartheta_e = -k_1\vartheta + (\dot\vartheta_r + k_1\vartheta_r),$$

the angular reference, with $k_1 > 0$ the gain used in the backstepping procedure. The tracking error for ω is

$$\omega_e = \omega - \omega_r.$$

Hence

$$\begin{aligned}
\dot\vartheta_e &= -k_1\vartheta_e + \omega_e,\\
\dot\omega_e &= \alpha_{10} i_q - \alpha_{20}\omega - c_0 - \dot\omega_r + \alpha_{1v} i_q - \alpha_{2v}\omega - c_v\\
&= -\vartheta_e - k_2\omega_e + \alpha_{10} i_{qe} + \alpha_{1v}\left(\frac{a_1}{\alpha_{10}}\vartheta_e - \frac{a_2}{\alpha_{10}}\omega_e + i_{qe}\right)\\
&\quad + \alpha_{2v}(k_1\vartheta_e - \omega_e) + \alpha_{1v}\phi_1\frac{1}{\alpha_{10}} - \alpha_{2v}\dot\vartheta_r - c_v,
\end{aligned} \tag{16}$$

with

$$\phi_1 = c_0 + \ddot\vartheta_r + \alpha_{20}\dot\vartheta_r,$$

$i_{qe} = i_q - i_{qr}$, and $i_{qr} = \psi/\alpha_{10}$ the reference for the i_q current. The desired value ψ for the angular acceleration is given by

$$\begin{aligned}
\psi &= \alpha_{10} i_{qr} = -\vartheta_e - k_2\omega_e + \alpha_{20}\omega + c_0 + \dot\omega_r\\
&= a_1\vartheta_e - a_2\omega_e + \phi_1,
\end{aligned}$$

where $k_2 > 0$ and

$$a_1 = k_1^2 - k_1\alpha_{20} - 1, \quad a_2 = k_1 + k_2 - \alpha_{20}.$$

As far as the errors i_{qe}, $i_{de} = i_d - i_{dr}$ are concerned, one uses the relations

$$\begin{aligned}
\omega &= \omega_e + \omega_r = -k_1\vartheta_e + \omega_e + \dot\vartheta_r\\
i_q &= i_{qe} + i_{qr} = \frac{a_1}{\alpha_{10}}\vartheta_e - \frac{a_2}{\alpha_{10}}\omega_e + i_{qe} + \frac{1}{\alpha_{10}}\phi_1\\
i_d &= i_{de} + i_{dr},
\end{aligned}$$

and (6), and computes the derivatives of the current references as

$$\begin{aligned}
\frac{di_{qr}}{dt} &= \frac{1}{\alpha_{10}}\dot\psi = \frac{1}{\alpha_{10}}(a_1\dot\vartheta_e - a_2\dot\omega_e + \dot\phi_1)\\
&= -\frac{a_1k_1 - a_2}{\alpha_{10}}\vartheta_e + \frac{a_1 + a_2k_2}{\alpha_{10}}\omega_e - a_2 i_{qe} + \frac{1}{\alpha_{10}}\dot\phi_1\\
&\quad - \alpha_{1v}\frac{a_2}{\alpha_{10}}\left(\frac{a_1}{\alpha_{10}}\vartheta_e - \frac{a_2}{\alpha_{10}}\omega_e + i_{qe}\right)\\
&\quad - \alpha_{2v}\frac{a_2}{\alpha_{10}}(k_1\vartheta_e - \omega_e) - \lambda_1,\\
\frac{di_{dr}}{dt} &= \frac{di_{dr}(\dot\vartheta_r)}{d\dot\vartheta_r}\ddot\vartheta_r,\\
\lambda_1 &= \alpha_{1v}\frac{a_2}{\alpha_{10}^2}\phi_1 - \alpha_{2v}\frac{a_2}{\alpha_{10}}\dot\vartheta_r - c_v\frac{a_2}{\alpha_{10}}.
\end{aligned}$$

Therefore, one finally works out the following expressions for the error current derivatives

$$\begin{aligned}
\frac{di_{qe}}{dt} &= -\alpha_{30} i_q - p\omega i_d - \alpha_{40}\omega + \alpha_{50} v_{qi} + \frac{a_1k_1 - a_2}{\alpha_{10}}\vartheta_e\\
&\quad - \frac{a_1 + a_2k_2}{\alpha_{10}}\omega_e + a_2 i_{qe} - \frac{1}{\alpha_{10}}\dot\phi_1\\
&\quad + \alpha_{1v}\frac{a_2}{\alpha_{10}}\left(\frac{a_1}{\alpha_{10}}\vartheta_e - \frac{a_2}{\alpha_{10}}\omega_e + i_{qe}\right)\\
&\quad + \alpha_{2v}\frac{a_2}{\alpha_{10}}(k_1\vartheta_e - \omega_e) - \alpha_{3v}\\
&\quad \times\left(\frac{a_1}{\alpha_{10}}\vartheta_e - \frac{a_2}{\alpha_{10}}\omega_e + i_{qe}\right)\\
&\quad - \alpha_{4v}(-k_1\vartheta_e + \omega_e) + \alpha_{5v} v_{qi} + \lambda_1\\
&\quad - \alpha_{3v}\phi_1\frac{1}{\alpha_{10}} - \alpha_{4v}\dot\vartheta_r + (\alpha_{50} + \alpha_{5v})w_q,\\
\frac{di_{de}}{dt} &= -\alpha_{30} i_d + p\omega i_q + \alpha_{50} v_{di} - \frac{di_{dr}}{dt} - \alpha_{3v} i_{de}\\
&\quad + \alpha_{5v} v_{di} - \alpha_{3v} i_{dr} + (\alpha_{50} + \alpha_{5v})w_d.
\end{aligned}$$

The "ideal" controls v_{qi}, v_{di} can be designed so that the nominal part of these dynamics are canceled, while the remaining parts, due to the parameter uncertainties, are sub-optimally compensated. Hence, setting

$$\begin{aligned}
v_{qi} &= \frac{1}{\alpha_{50}}\Bigg[\alpha_{30} i_q + p\omega i_d + \alpha_{40}\omega - \frac{a_1k_1 - a_2}{\alpha_{10}}\vartheta_e\\
&\quad + \frac{a_1 + a_2k_2 - \alpha_{10}^2}{\alpha_{10}}\omega_e - (k_3 + a_2)i_{qe}\\
&\quad + \frac{1}{\alpha_{10}}\dot\phi_1 + u_q\Bigg]\\
v_{di} &= \frac{1}{\alpha_{50}}\left[\alpha_{30} i_d - p\omega i_q - k_4 i_{de} + \frac{di_{dr}}{dt} + u_d\right]
\end{aligned} \tag{17}$$

with $k_3, k_4 > 0$, after some tedious algebra one works out

$$\frac{di_{qe}}{dt} = -\alpha_{10}\omega_e - k_3 i_{qe} + u_q + \alpha_{1v}\frac{a_2}{\alpha_{10}}\left(\frac{a_1}{\alpha_{10}}\vartheta_e - \frac{a_2}{\alpha_{10}}\omega_e + i_{qe}\right) + \alpha_{2v}\frac{a_2}{\alpha_{10}}(k_1\vartheta_e - \omega_e) - \alpha_{3v}\left(\frac{a_1}{\alpha_{10}}\vartheta_e - \frac{a_2}{\alpha_{10}}\omega_e + i_{qe}\right) + \alpha_{4v}(k_1\vartheta_e - \omega_e) + \alpha_{5v}\left(a_3\vartheta_e + a_4\omega_e + a_5 i_{qe} + \frac{1}{\alpha_{50}}\xi_q(x_e) - \frac{p}{\alpha_{50}}(k_1\vartheta_e - \omega_e)i_{de}\right) + \alpha_{5v}\dot{\vartheta}_r\frac{p}{\alpha_{50}}i_{de} - \alpha_{5v}i_{dr}\frac{p}{\alpha_{50}}(k_1\vartheta_e - \omega_e) + \left(\lambda_1 - \alpha_{3v}\phi_1\frac{1}{\alpha_{10}} - \alpha_{4v}\dot{\vartheta}_r + \alpha_{5v}\lambda_2 + (\alpha_{50} + \alpha_{5v})w_q\right),$$

$$\frac{di_{de}}{dt} = -k_4 i_{de} + u_d - \alpha_{3v}i_{de} + \alpha_{5v}\left[\frac{\alpha_{30} - k_4}{\alpha_{50}}i_{de} + \frac{1}{\alpha_{50}}\xi_d(x_e) + \frac{p}{\alpha_{50}}(k_1\vartheta_e - \omega_e)\left(\frac{a_1}{\alpha_{10}}\vartheta_e - \frac{a_2}{\alpha_{10}}\omega_e + i_{qe}\right)\right] - \alpha_{5v}\dot{\vartheta}_r\frac{p}{\alpha_{50}}\left(\frac{a_1}{\alpha_{10}}\vartheta_e - \frac{a_2}{\alpha_{10}}\omega_e + i_{qe}\right) + \alpha_{5v}\phi_1\frac{p}{\alpha_{10}\alpha_{50}}(k_1\vartheta_e - \omega_e) + \alpha_{5v}\lambda_3 - \alpha_{3v}i_{dr} + (\alpha_{50} + \alpha_{5v})w_d, \tag{18}$$

where

$$a_3 = \frac{1}{\alpha_{50}}\left(-\frac{1}{\alpha_{10}}(a_1k_1 - \alpha_{30}a_1 - a_2) - \alpha_{40}k_1\right),$$

$$a_4 = \frac{1}{\alpha_{50}}\left(\frac{1}{\alpha_{10}}(a_2k_2 + a_1 - \alpha_{30}a_2) + \alpha_{40} - \alpha_{10}\right),$$

$$a_5 = \frac{1}{\alpha_{50}}(-k_3 - a_2 + \alpha_{30}),$$

$$\lambda_2 = \frac{1}{\alpha_{50}}\left(\frac{1}{\alpha_{10}}\dot{\phi}_1 + \alpha_{30}\frac{1}{\alpha_{10}}\phi_1 + p\dot{\vartheta}_r i_{dr} + \alpha_{40}\dot{\vartheta}_r\right),$$

$$\lambda_3 = \frac{1}{\alpha_{50}}\left(\alpha_{30}i_{dr} - p\dot{\vartheta}_r\frac{1}{\alpha_{10}}\phi_1 + \frac{di_{dr}}{dt}\right),$$

and where it has been considered that the new controls u_q, u_d, designed hereinafter for the sub-optimal disturbance attenuation, are functions of the state, i.e., $u_q = \xi_q(x_e)$, $u_d = \xi_d(x_e)$.

Equations (16) and (18) can be rewritten as equation (9), with A_0, B_0 as in (10) and

$$x_e = \begin{pmatrix} \vartheta_e \\ \omega_e \\ i_{qe} \\ i_{de} \end{pmatrix}, \quad u = \begin{pmatrix} u_q \\ u_d \end{pmatrix},$$

$$w = \begin{pmatrix} \alpha_{1v} \\ \alpha_{2v} \\ \alpha_{3v} \\ \alpha_{4v} \\ \alpha_{5v} \\ \alpha_{5v}\dot{\vartheta}_r \\ \alpha_{5v}\phi_1 \\ \alpha_{5v}i_{dr} \\ \alpha_{1v}\phi_1\frac{1}{\alpha_{10}} - \alpha_{2v}\dot{\vartheta}_r - c_v \\ \lambda_1 - \alpha_{3v}\phi_1\frac{1}{\alpha_{10}} - \alpha_{4v}\dot{\vartheta}_r + \alpha_{5v}\lambda_2 + (\alpha_{50} + \alpha_{5v})w_q \\ \alpha_{5v}\lambda_3 - \alpha_{3v}i_{dr} + (\alpha_{50} + \alpha_{5v})w_d \end{pmatrix},$$

the tracking error, the new input and the extended disturbance, respectively. Finally,

$$k(x_e) = \left(\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ \chi_1(x_e) & \chi_2(x_e) & 0 & 0 & 0 \\ \frac{a_2}{\alpha_{10}}\chi_1(x_e) & \frac{a_2}{\alpha_{10}}\chi_2(x_e) & -\chi_1(x_e) & \chi_2(x_e) & a_3\vartheta_e + a_4\omega_e + a_5 i_{qe} + \frac{1}{\alpha_{50}}\xi_q(x_e) - \frac{p}{\alpha_{50}}\chi_2(x_e)i_{de} \\ 0 & 0 & -i_{de} & 0 & \frac{\alpha_{30}-k_4}{\alpha_{50}}i_{de} + \frac{1}{\alpha_{50}}\xi_d(x_e) + \frac{p}{\alpha_{50}}\chi_1(x_e)\chi_2(x_e) \end{array}\right.$$
$$\left.\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{p}{\alpha_{50}}i_{de} & 0 & -\frac{p}{\alpha_{50}}\chi_2(x_e) & 0 & 1 & 0 \\ -\frac{p}{\alpha_{50}}\chi_1(x_e) & \frac{p}{\alpha_{10}\alpha_{50}}\chi_2(x_e) & 0 & 0 & 0 & 1 \end{array}\right)$$

with

$$\chi_1(x_e) = \frac{a_1}{\alpha_{10}}\vartheta_e - \frac{a_2}{\alpha_{10}}\omega_e + i_{qe}, \quad \chi_2(x_e) = k_1\vartheta_e - \omega_e. \tag{19}$$

Appendix 2: Motor and Nonlinear H^∞ Controller Parameters

Table 1. Nominal parameters of the PM synchronous motor.

$R_0 = 0.6\,\Omega$	$J_0 = 0.0011\,\mathrm{kg\,m^2}$	$L_0 = 0.0014\,\mathrm{H}$
$f_0 = 0.0014\,\mathrm{N\,m\,s}$	$p = 4$	$k_{m0} = 0.48\,\mathrm{Wb}$

Table 2. Nonzero elements of matrix R_3.

$R_{3,1} = 37.5787$	$R_{3,2} = 0.119496$	$R_{3,3} = 0.141531$
$R_{3,4} = 1.02660 \times 10^{-4}$	$R_{3,5} = 1.98039 \times 10^{-4}$	$R_{3,6} = 1.36400 \times 10^{-4}$
$R_{3,7} = 3.11889$	$R_{3,8} = 4.11839 \times 10^{-2}$	$R_{3,9} = -4.52183 \times 10^{-2}$
$R_{3,10} = -2.92623 \times 10^{-4}$	$R_{3,11} = 1.55044 \times 10^{-4}$	$R_{3,12} = 0.334377$
$R_{3,13} = 1.35142 \times 10^{-4}$	$R_{3,14} = 4.12782$	