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On the Nonlinear Ripple-Free Sampled-data Robust Regulator

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In this paper, we present a control scheme for robust regulation of a discretized nonlinear system ensuring a ripple-free behavior in the intersampling time. We show that, as in the case of linear systems, the controller must contain an internal model to ensure that the continuous output tracking error decays asymptotically to zero. We also show that under certain conditions, a linear controller obtained from the discretization of the linear approximation of the nonlinear system, guarantees the robust regulation.

Keywords: Exponential holder; Nonlinear digital control; Robust regulator theory.

1. Introduction

The extensive use of digital computers for implementing control laws gives rise to the problem, of great relevance in control theory, of describing the dynamic behavior of the coupling of continuous time systems with digital devices via A/D and D/A converters. In fact, when a control law is implemented via digital devices, two solutions are possible. The first is to design a continuous control law and to use sampling periods sufficiently small with respect to the plant dynamics. The second is to discretize the plant dynamics and to design a digital control law on the basis of the of sampled measures [22]; this law is then converted into an analog signal generally using zero-order holders. This second solution is in general more adequate since the stability property is ensured [20,21], but only at the sampling instants; therefore, such a digital control law is in open-loop in the intersampling. In particular, the problem caused by a digital control law applied via zero order holders to a continuous time system is the presence of ripple in the output tracking error signal. In fact, it is well known that if the steady-state of the references to be tracked or the disturbances acting on the plant is not constant (sinusoidal, polynomial, etc.) then the sampled-data controller implemented by a zero-order holder gives rise to a ripple error. This means that the asymptotic tracking is guaranteed at the sampling instants, and the steady-state is zero, but in the intersampling an error exists and does not decay. This is due to the fact that in the classical control schemes using zero-order holders the condition under which this ripple vanishes is violated. In fact, it is well known that a necessary and sufficient condition for guaranteeing a ripple-free tracking is the continuous internal model principle [11,26]; in fact, the ripple can be eliminated if and only if an internal model of the reference and/or disturbance is present in the controller structure. This means that when using simple zero-order holders it is not possible to reconstruct the internal model, except for the constant case. Therefore, this problem should be solved employing a different approach, such as that

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which makes use of an hybrid controller constituted by a digital compensator followed by an analog internal model.

Another desirable characteristic in a control scheme is that of robustness; this is, the capability of maintaining the output tracking error within certain predefined bounds while ensuring the stability of the closed-loop system. Among the different approaches, the regulator theory provides an interesting scheme to accomplish, under certain assumptions, a robust asymptotic reference tracking. In fact, in recent works [8,16] a robust regulator scheme for nonlinear systems was presented. This is an error feedback controller which relies on the existence of an internal model, obtained by finding, if possible, an immersion of the exosystem dynamics into an observable one, which allows the generation of all the possible steady-state inputs for the admissible values of the system parameters. In addition, a remarkable feature is that the controller is constructed on the basis of the linear approximation of the nonlinear system and in the case when the immersion is linear the controller becomes fully linear.

For discretized linear systems, among others, in [11] an hybrid controller was presented. There, it was pointed out that a continuous internal model was necessary and sufficient to provide ripple-free response. In [25], an hybrid robust controller was presented. This controller was formed by a discrete linear controller plus an analog linear immersion which guarantees a ripple free behavior. With a different approach, in [12,13] and references therein, a multirate controller was used in order to robustly eliminate the ripples in the intersampling in the case of linear systems whose outputs can be sampled at different rates. Also, in [27], an hybrid multirate robust controller was presented for linear systems with delays; in this case the compensator amounts to a continuous internal model followed by a multirate digital compensator. Other works have treated in some fashion the same problem [14,17,24]. In particular, in [3], the problem of attenuating the ripple in the tracking error has been faced for linear systems by introducing a performance index given by the maximum peak of the tracking error, and by solving a ℓ_1 optimization problem. Finally, for the simple case of a unit step reference signal and also for linear systems, in [23] a procedure is proposed for determining a controller satisfying design constraints, expressed in terms of transient specifications or in terms of system norm quantities, with emphasis on the computational efficiency.

In this work, along the lines of [22,25], we present a ripple-free robust regulator scheme for nonlinear systems. We show that it is possible to obtain a robust controller on the basis of the discretization of the linear approximation of the continuous nonlinear system, that is, we do not need to discretize the entire continuous nonlinear system. Moreover, we show that the discretized solution coincides with the continuous solution at the sampled instants.

The paper is organized as follows: in Section 2 we give some preliminaries on the robust regulator results, while in Section 3 we introduce the main result of the paper. Section 4 is devoted to an illustrative example and finally, some conclusions are drawn.

2. Preliminaries

Let us consider the nonlinear time-invariant system described by

$$\dot{x} = f(x, u, w, \mu), \tag{1}$$

$$\dot{w} = s(w), \tag{2}$$

$$e = h(x, w, \mu), \tag{3}$$

where $x \in R^n$, $u \in R^m$ are the state and input variable of the plant, whose dynamics are described by Eq. (1), and $\mu \in R^p$ is a parameter vector which may take values in a neighborhood $\mathcal{P} \subset R^p$ of the nominal ones. Moreover, $w \in R^r$ is the state of an external signal generator, which provides the reference signals and the perturbation signal as well, and is described by (2). It is worth noting that we suppose that the exosystem (2) does not depend on μ; this is a hypothesis which is often verified in practical problems, since the exosystem models the reference and disturbance signals affecting the plant. Finally, Eq. (3) describes the tracking error $e \in R^p$, namely the difference between the plant output y and the reference $y_r = r(w)$.

The *Robust Regulation Problem* (RRP) for this system is defined as the problem of rejecting the disturbance signals, tracking the references and maintaining the closed-loop stability property, when the parameters vary in a neighborhood of the nominal values. It is well known that this problem can be solved by determining a certain submanifold of the state space (x, w), where the tracking error is zero, and by rendering it attractive and invariant by feedback. More formally, the RRP consists of finding a dynamic controller of the form

$$\begin{aligned} \dot{z} &= \varphi(z, e), \\ u &= \vartheta(z), \end{aligned} \tag{4}$$

such that, for all admissible parameter values μ in a neighborhood $\mathcal{P} \subset R^p$ of the nominal values, the following conditions hold:

(S) *Stability*. The equilibrium point $(x,z)=(0,0)$ of the closed-loop system without disturbance,

$$\dot{x}=f(x,\vartheta(z),0,\mu),$$
$$\dot{z}=\varphi(z,h(x,0,\mu)),$$

is asymptotically stable.

(R) *Regulation*. For each initial condition $(x(0),z(0),w(0))$ in a neighborhood of the origin, the solution of the closed-loop system,

$$\dot{x}=f(x,\vartheta(z),w,\mu),$$
$$\dot{z}=\varphi(z,h(x,w,\mu)),$$
$$\dot{w}=s(w),$$

satisfies the condition $\lim_{t\to\infty} e(t)=0$.

A local solution to this problem can be found in [1,16]. This solution is stated in terms of the existence of nonlinear mappings $x_{ss}=\pi(w,\mu)$, and $z_{ss}=\sigma(w,\mu)$ satisfying the regulator equations:

$$\begin{aligned}\frac{\partial\pi(w,\mu)}{\partial w}s(w)&=f(\pi(w,\mu),\vartheta(\sigma(w,\mu)),w,\mu),\\ \frac{\partial\sigma(w,\mu)}{\partial w}s(w)&=\varphi(\sigma(w,\mu),0)\\ 0&=h(\pi(w,\mu),w,\mu),\end{aligned}\tag{5}$$

for all admissible values of $\mu\in\mathcal{P}$. More precisely, it is known that one of the conditions under which the RRP can be solved is the existence of mappings

$$x_{ss}=\pi(w,\mu),\quad u_{ss}=\gamma(w,\mu)=\begin{pmatrix}\gamma_1(w,\mu)\\ \vdots\\ \gamma_m(w,\mu)\end{pmatrix}$$

with $\pi(0,\mu)=0$ and $\gamma(0,\mu)=0$, for all $\mu\in\mathcal{P}$, both defined in a neighborhood of the origin of $(w,\mu)=(0,0)$, solving the equations

$$\begin{aligned}\frac{\partial\pi(w,\mu)}{\partial w}s(w)&=f(\pi(w,\mu),\gamma(w,\mu),w,\mu)\\ 0&=h(\pi(w,\mu),w,\mu)\end{aligned}\tag{6}$$

for all the admissible values of $\mu\in\mathcal{P}$, and such that, for each $i=1,\dots,m$ there exists a set of real numbers $a_{0,i},a_{1,i},\dots,a_{r_i-1,i}$ satisfying

$$\begin{aligned}L_s^{r_i}\gamma_i(w,\mu)&=-a_{0,i}\gamma_i(w,\mu)-a_{1,i}L_s\gamma_i(w,\mu)\\ &\quad+\cdots-a_{r_i-1,i}L_s^{r_i-1}\gamma_i(w,\mu).\end{aligned}\tag{7}$$

where $L_s^k\gamma_i(w,\mu)$ stands for the Lie derivative defined as $L_s^k\gamma_i(w,\mu)=[\partial L_s^{k-1}\gamma_i(w,\mu)/\partial w]s(w)$; $k\geq 1$ with $L_s^0\gamma_i(w,\mu)=\gamma_i(w,\mu)$.

The mapping $x_{ss}=\pi(w,\mu)$ represents the steady-state zero output submanifold and $u_{ss}=\gamma(w,\mu)$ is the steady-state input which makes invariant this steady-state zero output submanifold. Conditions (7) express the fact that this steady-state input can be generated, independently of the values of the parameter vector μ, by the linear dynamical system

$$\begin{aligned}\dot{\xi}&=\Phi\xi,\\ u_{ss}&=H\xi\end{aligned}\tag{8}$$

where

$$\xi=\begin{pmatrix}\xi_1\\ \vdots\\ \xi_m\end{pmatrix},\quad \Phi=\mathrm{diag}\{\Phi_1,\dots,\Phi_m\}$$

$$H=\begin{pmatrix}H_1&0&\cdots&0\\ 0&H_2&\cdots&0\\ \vdots&0&\ddots&0\\ 0&0&\cdots&H_m\end{pmatrix}\tag{9}$$

$$\xi_i=\begin{pmatrix}\xi_{i1}\\ \vdots\\ \xi_{ir_i}\end{pmatrix},$$

$$\Phi_i=\begin{pmatrix}0&1&0&\cdots&0\\ 0&0&1&\cdots&0\\ \vdots&\vdots&\vdots&\ddots&\vdots\\ 0&0&0&\cdots&1\\ -a_{0,i}&-a_{1,i}&-a_{2,i}&\cdots&-a_{r_i-1,i}\end{pmatrix},$$

$$H_i=(1\quad 0\quad\cdots\quad 0)_{1\times r_i}$$

and $\xi_{ij}=L_s^{j-1}\gamma_i(w,\mu)$, $i=1,\dots,m$, $j=1,\dots,r_i$. This system can be viewed as an immersion of the exosystem (2) into a linear observable system; it is also worth noting that necessary and sufficient conditions for the solution of the RRP could be given considering, instead of (7), a nonlinear function [1,16]. This would determine a controller in the more generic nonlinear form (4).

The controller which solves the RRP is given by

$$\begin{aligned}\dot{\zeta}_1&=(A_0+B_0K-G_1C_0)\zeta_1+G_1e\\ \dot{\zeta}_2&=-G_2C_0\zeta_1+\Phi\zeta_2+G_2e\\ u&=K\zeta_1+H\zeta_2,\end{aligned}\tag{10}$$

where

$$A_0 = \left.\frac{\partial f(x,w,u,0)}{\partial x}\right|_{\substack{x=0\\w=0\\u=0}}, \quad B_0 = \left.\frac{\partial f(x,w,u,0)}{\partial u}\right|_{\substack{x=0\\w=0\\u=0}}, \quad C_0 = \left.\frac{\partial h(x,w,0)}{\partial x}\right|_{\substack{x=0\\w=0}} \tag{11}$$

are the nominal matrices of the linear approximation of system (1), (3), and K and G_1, G_2 make stable the matrices $(A_0 + B_0K)$ and

$$\begin{pmatrix} A_0 & -B_0H \\ 0 & \Phi \end{pmatrix} - \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} \begin{pmatrix} C_0 & 0 \end{pmatrix},$$

respectively. Obviously the stabilizability and detectability of the pairs

$$(A_0, B_0) \quad \text{and} \quad \left[\begin{pmatrix} A_0 & -B_0H \\ 0 & \Phi \end{pmatrix}, \begin{pmatrix} C_0 & 0 \end{pmatrix}\right], \tag{12}$$

respectively, are necessary conditions for solvability of the RRP.

To show that the controller (10) solves the RRP for the system (1)–(3), it is sufficient to verify that Eq. (5) are satisfied for all $\mu \in \mathcal{P}$ [1,16]. The linear approximation of the closed-loop system at the nominal parameter value ($\mu = 0$) with $w = 0$ is given by

$$\begin{pmatrix} \dot{x} \\ \dot{\zeta}_1 \\ \dot{\zeta}_2 \end{pmatrix} = \begin{pmatrix} A_0 & B_0K & B_0H \\ G_1C_0 & A_0 + B_0K - G_1C_0 & 0 \\ G_2C_0 & -G_2C_0 & \Phi \end{pmatrix} \begin{pmatrix} x \\ \zeta_1 \\ \zeta_2 \end{pmatrix},$$

whose dynamic matrix is similar to

$$\begin{pmatrix} A_0 + B_0K & B_0K & B_0H \\ 0 & A_0 - G_1C_0 & -B_0H \\ 0 & -G_2C_0 & \Phi \end{pmatrix}, \tag{13}$$

which can be made stable thanks to the assumptions on the pairs (12). Moreover, since the mappings $\pi(w,\mu)$, $\gamma(w,\mu)$ satisfy Eq. (6), one can choose

$$z_{ss} = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = \sigma(w,\mu) = \begin{pmatrix} 0 \\ \sigma_2(w,\mu) \end{pmatrix},$$

with

$$\zeta_2 = \sigma_2(w,\mu) = \begin{pmatrix} \gamma_1(w,\mu) \\ L_s\gamma_1(w,\mu) \\ \vdots \\ L_s^{r_1-1}\gamma_1(w,\mu) \\ \vdots \\ \gamma_m(w,\mu) \\ L_s\gamma_m(w,\mu) \\ \vdots \\ L_s^{r_m-1}\gamma_m(w,\mu) \end{pmatrix} \tag{14}$$

so that

$$\dot{\sigma}(w,\mu) = \frac{\partial \sigma(w,\mu)}{\partial w} s(w) = L_s\sigma(w,\mu) = \begin{pmatrix} 0 \\ \Phi \end{pmatrix} \zeta_2 = \varphi(\sigma(w,\mu), 0) \tag{15}$$

that is, the second equation of (5) is satisfied. Furthermore, setting

$$u_{ss} = \gamma(w,\mu) = H\zeta_2 = Hz_{ss} = \vartheta(z_{ss}) = \vartheta(\sigma(w,\mu)),$$

the first equation of (5) is satisfied too. Now, the dynamic matrix of the linear approximation of the closed-loop system (1), (10), (2) is given by

$$\begin{pmatrix} A & BK & BH \\ G_1C_0 & A_0 + B_0K - G_1C_0 & 0 \\ G_2C_0 & -G_2C_0 & \Phi \end{pmatrix}$$

which is stable for all the parameter variation in a suitable neighborhood $\mathcal{P}$ of the nominal values, since the matrix (13) is stable, and then the closed-loop system (1), (10), (2) has a center manifold at $(0,0,0)$ given by

$$x_{ss} = \pi(w,\mu), \quad z_{ss} = \sigma(w,\mu) \quad \forall \mu \in \mathcal{P}$$

which, by hypothesis, satisfies the third equation of (5).

When dealing with controllers implemented by digital devices and zero-order holders, it is well-known that the sampled version of the continuous time controller (10) might determine an unstable closed-loop system [21]. In this case, one has to design the controller directly on the basis of the sampled version of the nonlinear system (1)–(3), namely

$$x_d(k+1) = f_d(\delta, x_d, u_d, w_d, \mu), \quad (16)$$

$$w_d(k+1) = s_d(\delta, w_d), \quad (17)$$

$$e_d = h(x_d, w_d, \mu), \quad (18)$$

where $f_d(\delta, x_d, u_d, w_d, \mu)$, $s_d(\delta, w_d)$ are functions of the sampling period δ (see [6,22] for details) and $x_d = x_d(k) = x(k\delta)$, $u_d = u_d(k) = u(k\delta)$, $w_d = w_d(k) = w(k\delta)$.

It is possible to formulate the discrete time counterpart of the RRP, whose solvability can be recast in terms of existence of the nonlinear mappings $x_{d,ss} = \pi_d(w_d, \mu)$, $z_{d,ss} = \sigma_d(w_d, \mu)$, $u_{d,ss} = \gamma_d(w_d, \mu)$ solving the regulator equations

$$\pi_d(s_d(w_d), \mu) = f_d(\pi_d(w_d, \mu), \vartheta_d(\sigma_d(w_d, \mu)), w_d, \mu), \quad (19)$$

$$\sigma_d(s_d(w_d), \mu) = \varphi_d(\sigma_d(w_d, \mu), 0), \quad (20)$$

$$0 = h(\pi_d(w_d, \mu), w_d, \mu). \quad (21)$$

Clearly, for the solution of the RRP in the discrete time setting, one could proceed in the same way and construct a controller which guarantees a zero output tracking error only at the sampling instants, but in the intersampling time the output tracking error would generally be not zeroed, due to the fact that the internal model cannot be reproduced when using zero order holders, except in the particular case of constant reference signals.

In order to achieve zero output also in the intersampling, we propose in the next section a controller designed on the basis of the linear approximation of the discretized system and on the internal model dynamics obtained for the continuous time problem; it is known that this last requirement is necessary and sufficient to ensure a ripple-free output tracking.

3. Ripple-Free Robust Regulation for Sampled-data Systems

From the discussion of the previous section it is clear that, in order to eliminate the ripple in the output tracking error during the intersampling, we need to reproduce the continuous internal model (8) from its discrete time realization

$$\xi_d(k+1) = e^{\Phi\delta}\xi_d(k).$$

This can be done implementing the continuous steady-state input as

$$u_{ss}(k\delta + \theta) = He^{\Phi\theta}\xi_d(k), \quad \theta \in [0, \delta). \quad (22)$$

Note that the term $e^{\Phi\theta}$, defined as the *exponential holder*, is realized by an analog device. The solution of Eq. (8) is given by

$$\xi(t) = e^{\Phi t}\xi(0),$$

and setting $t = k\delta + \theta$, $t_0 = k\delta$, we get

$$\xi(k\delta + \theta) = e^{\Phi(k\delta+\theta)}\xi(0) = e^{\Phi\theta}e^{k\delta\Phi}\xi(0) = e^{\Phi\theta}\xi(k\delta),$$
$$u_{ss}(k\delta + \theta) = H\xi(k\delta + \theta) = He^{\Phi\theta}\xi(k\delta)$$

$\theta \in [0, \delta)$ which describe *exactly* not only the behavior at the sampling instants but also the intersampling behavior, since

$$\xi(k\delta + \theta) = \xi(t) \qquad u_{ss}(k\delta + \theta)$$
$$t \in [k\delta, (k+1)\delta), \quad \theta \in [0, \delta), \quad k = 0, 1, 2, \ldots$$

In what follows, we will consider, therefore, an input which is piecewise continuous and dependent on the parameter $\theta \in [0, \delta)$. Denoting such a control by $u(t) = \vartheta_d(z_d, \theta)$, the sampling of the system (1) will be denoted by [20,22]

$$x_d(k+1) = F_d(\delta, x_d, \vartheta_d(z_d, \theta), w_d, \mu). \quad (23)$$

Note that this description is not the same as in (16)–(18), because in those equations an input which is constant in the intersampling is used, while in (23) the input is not constant between successive sampling instants.

We can now formulate the *Ripple-Free Robust Output Regulator Problem* (RFRRP) as the problem of finding a dynamic controller of the form

$$z_d(k+1) = \varphi_d(z_d, e_d) \quad (24)$$

$$u(t) = \vartheta_d(z_d, \theta) \qquad \theta \in [0, \delta) \quad (25)$$

such that for each $\mu \in \mathcal{P}$ the following conditions are satisfied

(S_{rf}) *Stability*. The solution of the system

$$x_d(k+1) = F_d(\delta, x_d, \vartheta_d(z_d, \theta), 0, \mu),$$
$$z_d(k+1) = \varphi_d(z_d, h(x_d, 0, \mu))$$

at the sampling instants goes asymptotically to zero;

($\mathbf{R}_{rf}$) *Regulation.* For each initial condition ($x(0), z_d(0), w(0)$) the solution of the closed-loop system

$$\begin{aligned}\dot{x} &= f(x, \vartheta_d(z_d, \theta), w, \mu) \quad \theta \in [0, \delta),\\ z_d(k+1) &= \varphi_d(z_d, h(x_d, w_d, \mu)),\\ \dot{w} &= s(w),\end{aligned}$$

guarantees that

$$\lim_{t\to\infty} e(t) = 0.$$

As for the RRP problem, the solution of the RFRRP can be expressed considering the matrices

$$A_{d0} = e^{A_0\delta}, \quad B_{d0} = \int_0^\delta e^{A_0\tau} d\tau B_0, \quad C_{d0} = C_0$$

obtained from sampling the linear approximation of the system (1), (3), at the nominal parameter values ($\mu = 0$).

We now state the main result of the paper.

Theorem 1. Let us assume the following hypotheses hold.

(H_{d1}) The pair (A_0, B_0) is stabilizable;

(H_{d2}) There exist mappings

$$x_{ss} = \pi(w, \mu), \quad u_{ss} = \gamma_d(w, \mu) = \begin{pmatrix} \gamma_1(w, \mu) \\ \vdots \\ \gamma_m(w, \mu) \end{pmatrix}$$

with $\pi(0, \mu) = 0$ and $\gamma(0, \mu) = 0$, for all $\mu \in \mathcal{P}$, both defined in a neighborhood of the origin of $(w, \mu) = (0, 0)$, solving the equations (6), for all the admissible values of $\mu \in \mathcal{P}$, and such that, for each $i = 1, \ldots, m$ there exists a set of real numbers $a_{0,i}, a_{1,i}, \ldots, a_{r_i-1,i}$ satisfying (7);

(H_{d3}) The pair

$$\begin{pmatrix} A_{d0} & -M_{d0} \\ 0 & \Phi_d \end{pmatrix}, \quad \begin{pmatrix} C_{d0} & 0 \end{pmatrix}$$

is detectable, with

$$M_{d0} = \int_0^\delta e^{A_0\tau} B_0 \tilde{H} e^{\Phi(\delta-\tau)} d\tau, \quad \Phi_d = e^{\Phi\delta},$$

and Φ given by (9).

Then the RFRRP is solvable by the following controller

$$\begin{aligned}\zeta_{d1}(k+1) &= (A_{d0} + B_{d0}K_d - G_{d1}C_{d0})\zeta_{d1} + G_{d1}e_d,\\ \zeta_{d2}(k+1) &= -G_{d2}C_{d0}\zeta_{d1} + \Phi_d\zeta_{d2} + G_{d2}e_d,\\ u(k\delta + \theta) &= K_d\zeta_{d1} + He^{\Phi\theta}\zeta_{d2} \qquad \theta \in [0, \delta),\end{aligned} \tag{26}$$

where K_d, and $G_d = (G_{d1}\ G_{d2})^T$ render stable the matrices $A_{d0} + B_{d0}K_d$, and

$$\begin{pmatrix} A_{d0} & -M_{d0} \\ 0 & \Phi_d \end{pmatrix} - \begin{pmatrix} G_{d1} \\ G_{d2} \end{pmatrix} \begin{pmatrix} C_{d0} & 0 \end{pmatrix},$$

respectively.

Proof. First, let us verify that condition ($\mathbf{S}_{rf}$) is fulfilled. To this aim, let us write the system (1), with $w = 0$ and the control law given by (26), as

$$\begin{aligned}\dot{x}(t) &= A(\mu)x(t) + B(\mu)(K_d\zeta_{d1}(k)\\ &\quad + He^{\Phi\theta}\zeta_{d2}(k)) + f_1(x(t), u(k\delta + \theta), 0, \mu),\\ \zeta_{d1}(k+1) &= (A_{d0} + B_{d0}K_d - G_{d1}C_{d0})\zeta_{d1} + G_{d1}e_d(k),\\ \zeta_{d2}(k+1) &= -G_{d2}C_{d0}\zeta_{d1} + \Phi_d\zeta_{d2} + G_{d2}e_d(k);\\ & t \in [k\delta, (k+1)\delta), \quad k = 0, 1, 2, \ldots, \quad \theta \in [0, \delta),\end{aligned}$$

where it has been considered that $u(t) = u(k\delta + \theta)$ and that the function f_1 takes into account the nonlinear terms. Since we need to ensure that the solution of the previous system goes to zero at the time instants $k\delta$ and for the nominal parameter values, let us consider the sampling of the first equation, with $\mu = 0$

$$\begin{aligned}x_d(k+1) &= A_{d0}x_d + B_{d0}K_d\zeta_{d1} + M_{d0}\zeta_{d2}\\ &\quad + f_{d1}(\delta, x_d, u(k\delta + \theta), 0, 0)|_{\theta=0},\\ \zeta_{d1}(k+1) &= (A_{d0} + B_{d0}K_d - G_{d1}C_{d0})\zeta_{d1} + G_{d1}C_{d0}x_d,\\ \zeta_{d2}(k+1) &= -G_{d2}C_{d0}\zeta_{d1} + \Phi_d\zeta_{d2} + G_{d2}C_{d0}x_d,\end{aligned} \tag{27}$$

where, since $\theta = t - k\delta$,

$$\int_{k\delta}^{(k+1)\delta} e^{A_0((k+1)\delta-\tau)} B_0 He^{\Phi(\tau-k\delta)} d\tau = \int_0^\delta e^{A_0\tau} B_0 He^{\Phi(\delta-\tau)} d\tau = M_{d0}.$$

Considering the linear approximation of system (27), one has

$$\begin{pmatrix} x_d(k+1) \\ \zeta_{d1}(k+1) \\ \zeta_{d2}(k+1) \end{pmatrix} = \begin{pmatrix} A_{d0} & B_{d0}K_d & M_{d0} \\ G_{d1}C_{d0} & (A_{d0} + B_{d0}K_d - G_{d1}C_{d0}) & 0 \\ G_{d2}C_{d0} & -G_{d2}C_{d0} & \Phi_d \end{pmatrix} \begin{pmatrix} x_d(k) \\ \zeta_{d1}(k) \\ \zeta_{d2}(k) \end{pmatrix},$$

whose dynamic matrix, as in the continuous case, is similar to the matrix

$$\begin{pmatrix} A_{d0}+B_{d0}K_d & B_{d0}K_d & M_{d0} \\ 0 & (A_{d0}-G_{d1}C_{d0}) & -M_{d0} \\ 0 & -G_{d2}C_{d0} & \Phi_d \end{pmatrix},$$

which can be made stable, thanks to (H_{d1}) and (H_{d3}). This shows that $(\mathbf{S}_{\rm rf})$ is verified.

As for $(\mathbf{R}_{\rm rf})$, we first show that the error $e_d(k)$ is zeroed at the sampling instants, and then that the tracking error is zero also in the intersampling. For, it is possible to show [6] that if there exists a solution $x_{\rm ss}=\pi(w,\mu)$, $u_{\rm ss}=\gamma(w,\mu)$ to Eq. (6), then there also exists a solution

$$x_{d,\rm ss}=\pi_d(w_d,\mu),$$

$$u_{d,\rm ss}=\gamma_d(w_d,\mu)=\begin{pmatrix}\gamma_{d1}(w_d,\mu)\\ \vdots \\ \gamma_{dm}(w_d,\mu),\end{pmatrix}$$

to the equations

$$\begin{aligned}\pi_d(s_d(w_d),\mu)&=f_d(\pi_d(w_d,\mu),\gamma_d(w_d,\mu),w_d,\mu),\\ 0&=h(\pi_d(w_d,\mu),w_d,\mu),\end{aligned} \tag{28}$$

where

$$\begin{aligned} s_d(w_d)&=\mathrm{e}^{\delta L_s}(w)|_{w_d}=\sum_{i=0}^{\infty}\frac{\delta^i}{i!}L_s^i(w)|_{w_d}\\ &=w|_{w_d}+\delta L_s(w)|_{w_d}+\frac{\delta^2}{2!}L_s^2(w)|_{w_d}\\ &\quad+\mathcal{O}(\delta^3). \end{aligned} \tag{29}$$

Moreover, it may be shown that

$$\sigma_d(s_d(w_d),\mu)=\varphi_d(\sigma_d(w_d,\mu),0), \tag{30}$$

admits a solution as well and that this solution coincides with the function $\sigma(w,\mu)$ at the sampling instants, namely at w_d. More formally, we may observe that the structure of a solution to Eq. (30) can be given, as in the continuous case, by

$$\sigma_d(w_d,\mu)=\begin{pmatrix}0\\ \sigma_{d2}(w_d,\mu)\end{pmatrix}.$$

Now, as in [6], we assume that the function $\sigma_{d2}(w_d,\mu)$ can be expressed as a power expansion in δ around the continuous-time solution $\sigma_2(w,\mu)$, namely

$$\begin{aligned}\sigma_{d2}(w_d,\mu)&=\sum_{j=0}^{\infty}\frac{\delta^j}{j!}\sigma_{d2,j}(w_d,\mu),\\ \sigma_{d2,0}(w_d,\mu)&=\sigma_2(w,\mu)|_{w_d},\end{aligned} \tag{31}$$

which is convergent for δ small enough. First, making use of the exchange theorem of Lie series [22], the second component of the left-hand side (30) can be written as follows

$$\begin{aligned}\sigma_{d2}(s_d(w_d),\mu)&=\sum_{j=0}^{\infty}\frac{\delta^j}{j!}\sigma_{d2,j}(\mathrm{e}^{\delta L_s}w|_{w_d},\mu)\\ &=\sum_{j=0}^{\infty}\frac{\delta^j}{j!}\mathrm{e}^{\delta L_s}(\sigma_{d2,j}(w,\mu))|_{w_d} \qquad (32)\\ &=\sum_{j=0}^{\infty}\sum_{h=0}^{j}\binom{j}{h}\frac{\delta^j}{j!}L_s^h(\sigma_{d2,j-h}(w,\mu))|_{w_d}.\end{aligned} \tag{33}$$

The second component of the right-hand side of (30) yields

$$\mathrm{e}^{\delta\Phi}\sigma_{d2}(w_d,\mu)=\sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\frac{\delta^{j+h}}{j!h!}\Phi^j\sigma_{d2,h}(w,\mu)|_{w_d},$$

so that from (30) one obtains

$$\begin{aligned}&\sum_{j=0}^{\infty}\sum_{h=0}^{j}\binom{j}{h}\frac{\delta^j}{j!}L_s^h(\sigma_{d2,j-h}(w,\mu))|_{w_d}\\ &\quad=\sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\frac{\delta^{j+h}}{j!h!}\Phi^j\sigma_{d2,h}(w,\mu)|_{w_d}.\end{aligned}$$

Equating the terms in the same power of δ one works out

$$\begin{aligned}\sigma_{d2,0}(w_d,\mu)&=\sigma_{d2,0}(w_d,\mu),\\ \sigma_{d2,1}(w_d,\mu)+L_s\sigma_{d2,0}(w,\mu)|_{w_d}&=\sigma_{d2,1}(w_d,\mu)+\Phi\sigma_{d2,0}(w_d,\mu),\ \sigma_{d2,2}(w_d,\mu)+2L_s\sigma_{d2,1}(w,\mu)|_{w_d}\\ +L_s^2\sigma_{d2,0}(w,\mu)|_{w_d}&=\sigma_{d2,2}(w_d,\mu)+2\Phi\sigma_{d2,1}(w_d,\mu)+\Phi^2\sigma_{d2,0}(w_d,\mu),\\ &\ \ \vdots\end{aligned}$$

which thanks to result (15) can be identically satisfied by setting

$$\sigma_{d2,j}(w,\mu)=\sigma_{d2,0}(w,\mu),\quad j=1,2,\ldots.$$

Hence, by (32), Eq. (31) becomes

$$\begin{aligned}\sigma_{d2}(w_d,\mu)&=\sum_{j=0}^{\infty}\frac{\delta^j}{j!}\sigma_{d2,0}(w_d,\mu)\\ &=\mathrm{e}^{\delta L_s}\sigma_{d2,0}(w,\mu)|_{w_d},\end{aligned}$$

and therefore $\sigma_{d2,0}(w,\mu)$ can be taken as $\sigma_{d2,0}(w,\mu)=\sigma_2(w,\mu)$, namely the continuous solution. From this, and due to the structure of the controller (26), it follows immediately that Eq. (30) is satisfied and thus $e_d(k)$ goes asymptotically to zero.

Finally, to show that the continuous time tracking error $e(t)$ goes to zero asymptotically, note that the controller (26), when $e_d(k)=0$, generates exactly the continuous steady-state input (22) needed for zeroing the continuous tracking error. □

Remark 2. It is worth noting that for constructing the controller (26) it is not necessary to know either the continuous steady-state $x_{ss}=\pi(w,\mu)$ or the discrete one $x_{d,ss}=\pi_d(w_d,\mu)$, but only the continuous steady-state input $u_{ss}(t)=\gamma(w,\mu)$. However, in the special case when the function $\gamma(w,\mu)$ is polynomial, as for example in the case of triangular systems with polynomial terms, the matrix Φ can be determined without knowing $\gamma(w,\mu)$ exactly, but only the maximal degree of the polynomial. Another remarkable feature of the controller is that it is based on the discretized linear part of the system description. Note also that the input to the process is a combination of a discrete and a continuous part. The complete continuous controller is equivalent to the discrete one only in the sense that both provide the exact steady-state input needed to maintain the dynamics of the system within the zero output submanifold. Clearly the transient behavior provided by both the controllers is different.

4. An Illustrative Example

To illustrate the performance of the proposed scheme, we consider the dynamical model of the controller Duffing oscillator given by [18]

$$\begin{aligned}\dot{x}_1&=x_2,\\ \dot{x}_2&=x_1-x_1^3-\mu_1x_2+\mu_2\cos\beta t+u,\\ e&=x_1-r\sin\alpha t.\end{aligned}$$

Choosing the exosystem as

$$\begin{aligned}\dot{w}_1&=\alpha w_2,\\ \dot{w}_2&=-\alpha w_1,\\ \dot{w}_3&=\beta w_4,\\ \dot{w}_4&=-\beta w_3,\end{aligned}$$

with $w_1(0)=r$, $w_2(0)=0$, $w_3(0)=0$, $w_4(0)=1$, the system may be expressed in the state (x,w) as

$$\begin{aligned}\dot{x}_1&=x_2,\\ \dot{x}_2&=x_1-x_1^3-\mu_1x_2+\mu_2w_4+u,\\ e&=x_1-w_1.\end{aligned}$$

The solution of Eqs (6) is then found by taking $e=0=x_1-w_1$ from which

$$\pi_1(w,\mu)=w_1;$$

and, by forward substitutions,

$$\begin{aligned}\pi_2(w,\mu)&=\alpha w_2,\\ \gamma(w,\mu)&=-(1+\alpha^2)w_1+\mu_1\alpha w_2-\mu_2w_4+w_1^3=:z_1,\end{aligned}$$

from which we obtain

$$\begin{aligned}L_s\gamma(w,\mu)&=-\alpha(1+\alpha^2)w_2-\mu_1\alpha^2w_1+\mu_2\beta w_3+3\alpha w_1^2w_2=:z_2,\\
L_s^2\gamma(w,\mu)&=\alpha^2(1+\alpha^2)w_1-\mu_1\alpha^2w_2+\mu_2\beta^2w_4+3\alpha^2(2w_1w_2^2-w_1^3)\\
&=-\alpha^2(z_1-\mu_2w_4-w_1^3)+\beta^2\mu_2w_4+3\alpha^2(2w_1w_2^2-w_1^3)\\
&=-\alpha^2z_1+(\beta^2-\alpha^2)\mu_2w_4+6\alpha^2w_1w_2^2-2\alpha^2w_1^3=:z_3,\\
L_s^3\gamma(w,\mu)&=-\alpha^2z_2-\beta(\beta^2-\alpha^2)\mu_2w_3+6\alpha^3(w_2^3-2w_1^2w_2)=:z_4,\\
L_s^4\gamma(w,\mu)&=-\alpha^2z_3-\beta^2(\alpha^2-\beta^2)\mu_2w_4-9\alpha^2(\alpha^2w_1w_2^2-2\alpha^2w_1^3)\\
&=-\alpha^2z_3-\beta^2(\alpha^2-\beta^2)\mu_2w_4-9\alpha^2(z_3+\alpha^2z_1-(\beta^2-\alpha^2)\mu_2w_4)\\
&=-9\alpha^4z_1-10\alpha^2z_3-(\beta^2-9\alpha^2)(\beta^2-\alpha^2)\mu_2w_4=:z_5,\\
L_s^5\gamma(w,\mu)&=-9\alpha^4z_2-10\alpha^2z_4+\beta(\beta^2-9\alpha^2)(\beta^2-\alpha^2)\mu_2w_3=:z_6,\\
L_s^6\gamma(w,\mu)&=-9\alpha^4z_3-10\alpha^2z_5+\beta^2(\beta^2-9\alpha^2)(\beta^2-\alpha^2)\mu_2w_4\\
&=-9\alpha^4z_3-10\alpha^2z_5-\beta^2(z_5+9\alpha^4z_1+10\alpha^2z_3)\\
&=-9\alpha^4\beta^2z_1-(10\alpha^2\beta^2+9\alpha^4)z_3-(\beta^2+10\alpha^2)z_5=\dot{z}_6,\end{aligned}$$

and thus the immersion (9) is written as

$$\Phi = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -9\alpha^4\beta^2 & 0 & -10\alpha^2\beta^2 - 9\alpha^4 & 0 & -\beta^2 - 10\alpha^2 & 0 \end{pmatrix}.$$

It is interesting to note that if we apply the procedure given in [16] for polynomial steady-state inputs, we would obtain an eight-dimensional immersion. Choosing $\alpha = 1$, $\beta = 2$, $\delta = 0.3$ s, $\mu_1 = 0.15$, $\mu_2 = 0.15$ and the closed-loop poles at $[0.22, 0.26]$ and $[0.25, 0.3, 0.35, 0.4, 0.45, 0.5, 0.55, 0.6]$, the parameter values of the controller

$$\begin{aligned} \xi(k+1) &= F_d\xi(k) + G_d e_d(k), \\ u &= \left[K_d \; He^{\Phi\theta} \right] \xi(k); \quad \theta \in [0, \delta), \end{aligned}$$

are

$$F_d = \begin{pmatrix} -2.7399 & 0.1134 & 0 & 0 & 0 & 0 & 0 & 0 \\ -12.6501 & -0.2292 & 0 & 0 & 0 & 0 & 0 & 0 \\ 14.0577 & 0 & 1 & 0.3 & 0.0450 & 0.0045 & 0.0003 & 0 \\ -26.3178 & 0 & -0.0007 & 1 & 0.2990 & 0.045 & 0.0042 & 0.0003 \\ -112.7510 & 0 & -0.0116 & -0.0007 & 0.9841 & 0.299 & 0.0404 & 0.0042 \\ 106.9334 & 0 & -0.1520 & -0.0116 & -0.2076 & 0.9841 & 0.2399 & 0.0404 \\ 946.0962 & 0 & -1.4552 & -0.1520 & -1.9923 & -0.2076 & 0.4182 & 0.2399 \\ -532.9393 & 0 & -8.6370 & -1.4552 & -11.9079 & -1.9923 & -3.5665 & 0.4182 \end{pmatrix};$$

$$G_d = \begin{pmatrix} 3.4492 \\ 10.7117 \\ -14.0577 \\ 26.3178 \\ 112.7510 \\ -106.9334 \\ -946.0963 \\ 532.9394 \end{pmatrix}; \quad He^{\Phi\theta} = \begin{pmatrix} 3/2\cos\theta - 3/5\cos 2\theta + 1/10\cos 3\theta \\ 3/2\sin\theta - 3/10\sin 2\theta + 1/30\sin 3\theta \\ 13/24\cos\theta - 2/3\cos 2\theta + 1/8\cos 3\theta \\ 13/24\sin\theta - 1/3\sin 2\theta + 1/24\sin 3\theta \\ 1/24\cos\theta - 1/15\cos 2\theta + 1/40\cos 3\theta \\ 1/24\sin\theta - 1/30\sin 2\theta + 1/120\sin 3\theta \end{pmatrix}$$

$$K_d = (-7.5097 \quad -4.1282).$$

The simulation results are shown in Figs 1–3 for several values of μ_1. In all the figures, a stands for $\mu_1 = 0.3$, b for $\mu_1 = 0.15$, and c for $\mu_1 = -0.15$.

As may be observed, the output tracking error converges to zero asymptotically, even for values of μ_1 different from the nominal value.

5. Conclusions

In this work we have presented a robust regulator scheme for discretized nonlinear systems which ensures ripple-free behavior. This feature is accomplished by the insertion, in the discrete controller structure, of a continuous subsystem, namely, an internal model, which allows reproduction of the steady-state inputs necessary for zeroing the continuous output tracking error for all the admissible values of the system parameters. We show that if the immersion is linear, then, as in the continuous case, the controller is also linear. We show also that, as we may expect, the solution of the continuous case coincides at the sampling instants, with the solution of the robust discretized problem. An illustrative example demonstrates the performance of the presented scheme.

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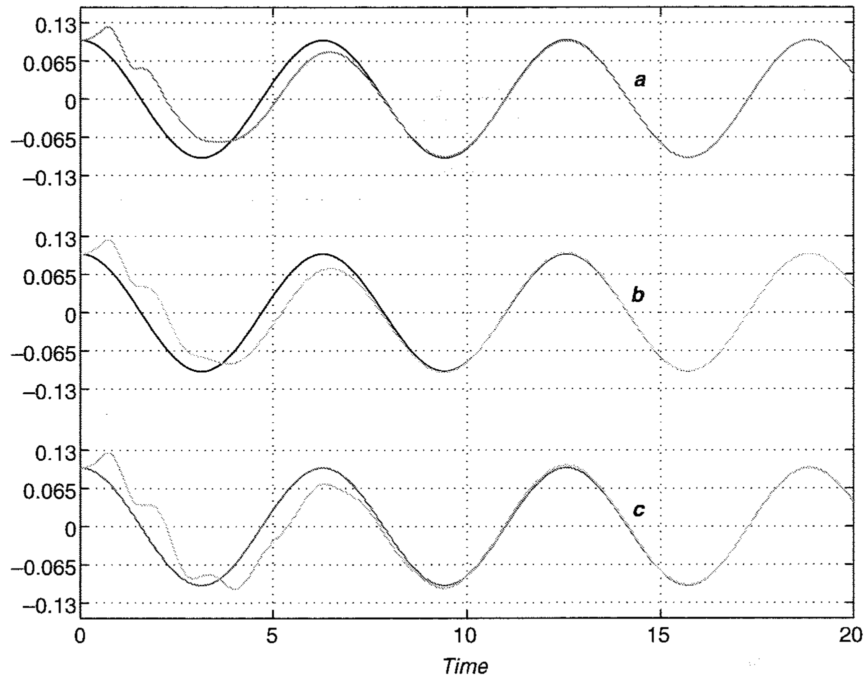

Fig. 1. Output signal vs reference signal.

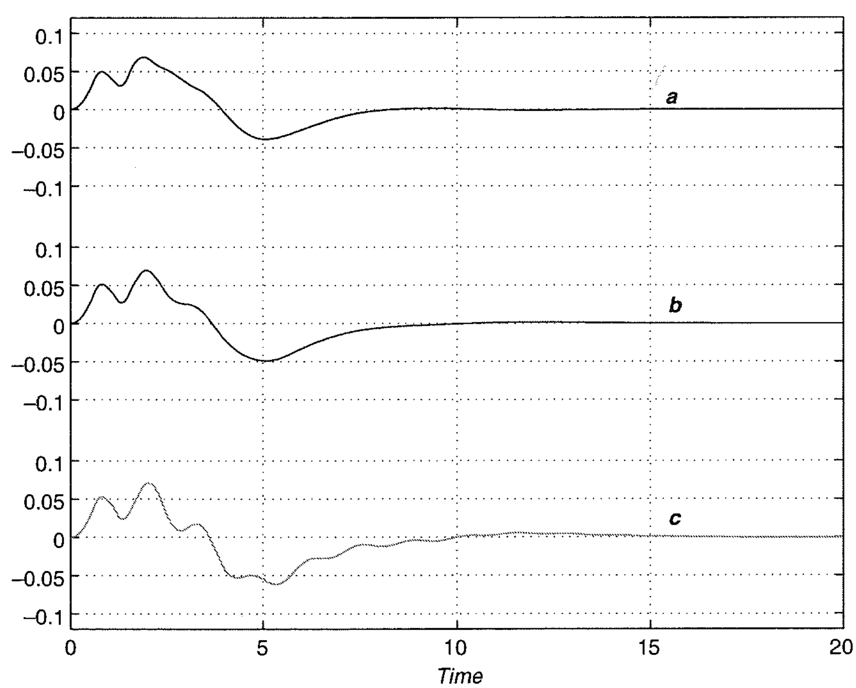

Fig. 2. Output tracking error.

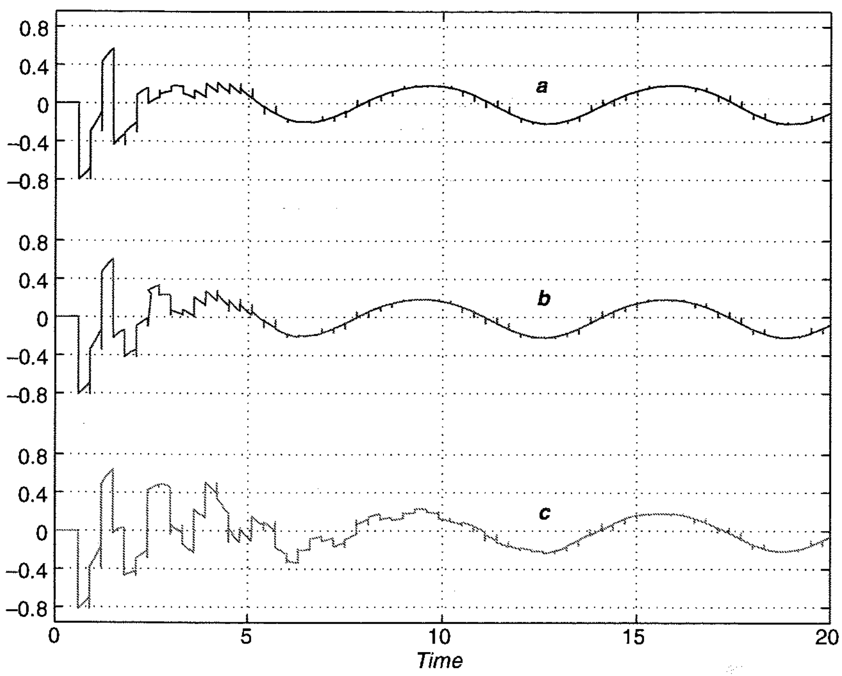

Fig. 3. Input signal.

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