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Stabilization for Continuum Models of Large Space Structures in Large Attitude Maneuvers*

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In this work we consider an infinite dimensional model of a large space structure with flexible elements. The control purpose consists of ensuring stability while performing large slew maneuvers. To this aim, a suitably accurate description of the nonlinear coupling between the rigid and the flexible motions of a spacecraft with flexible appendages is necessary. A control law from the state, ensuring global stabilization of the closed-loop solutions for any initial condition, is designed. Furthermore, considering the practical problems arising from the implementation of such a feedback, a control law from the output guaranteeing semiglobal stabilization, i.e., for any initial condition in a chosen region, is proposed. As opposite to the finite dimensional approach, which suffers the spillover effect, the proposed controllers ensure the asymptotic stabilization of all the modes.

Keywords: Large flexible structure; Distributed Parameter model; Stabilization; Boundary control

1. Introduction

In this work, we consider the control of large space structures (LSS), modeled by (nonlinear) distributed parameter systems (or continuum models). The interest in control problems of LSS is shown by the great deal of literature on the subject (see [15,16,21,26–28] for instance). Most of the works are based on finite dimensional models, for which a large amount of control methodologies is available (examples are given in [18,28,29,32]); however, the use of a finite dimensional model presents the well-known drawbacks of mode truncation and possible spillover effects. On the other hand, at the expense of a more complex analysis which naturally limits the viable control strategies, the infinite dimensional setting presents some amenable aspects; among the others, it allows for a system description with a limited number of parameters, and naturally captures the behavior of mechanical systems with distributed flexibility, since all the modes are simultaneously taken into account [4,5,7,8]. It is worth noting that, among the infinite dimensional approaches to LSS control available in the literature, most of them do not consider the state–space formulation [23,30]. On the contrary, the benefit of its use can be appreciated in solving standard control problems based on various performance indices, such as closed-loop stability, tracking and

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regulation accuracy, robustness with respect to model uncertainties and measurement noises [9,10,35].

Another point of interest regards the mathematical models adopted for the LSS in the infinite dimensional setup. In fact, in the literature different type of simplifications can be found, instrumental for applying the available theoretical results; for example, the coupling between the rigid and the flexible motions has been taken into account with different level of approximations. In [4] it has been neglected, and only the flexible dynamics have been stabilized, once the rigid maneuver has been completed, while in [24] a very simplified coupling has been considered leading to a linear model. This nonlinear coupling is of great relevance and has to be considered in all problems of large slew maneuvers with high angular velocity [27,28].

In view of the points recalled above, in this work we intend to set up a benchmark, appropriately significant but at the same time sufficiently simple, containing all the points of interest and all the difficulties arising in large slew maneuvers. This benchmark and the proposed solutions of the control problems constitute, in the authors' aim, the main contribution of the paper, and a starting point for the solution of more sophisticated control problems for more complex structures. Hence, in this work we consider a structure with a suitably accurate description of the (nonlinear) coupling between the rigid and the flexible motions [13]; both the dynamics are taken into account in the control design. The model obtained via the variational approach is then recast in the state–space form, so to exploit the advantages previously recalled; then a nonlinear first order differential equation on an Hilbert space is obtained. Once the well-posedness of such a system is studied, that is the existence and uniqueness of (local) solutions are established, we investigate the state and output feedback control problems. It is shown that the former ensures global stability of the closed-loop solutions for any initial condition. This state feedback, although straightforward, is a contribution instrumental to illustrate the final result of the paper, the output feedback. In fact, the state feedback is distributed, which poses the practical problem of distributed sensing and actuation. This difficulty is got around by considering an output feedback, i.e., a feedback from the measurements of the boundary displacements and rates. In this second case semiglobal stabilization is pursued, i.e., for any initial condition in a given closed and bounded region containing the origin, the system trajectories go asymptotically to zero. As opposite to the finite dimensional approach, which may suffer the spillover effect, the proposed controllers ensure the asymptotic stabilization of all the modes.

The paper is organized as follows. In Section 2, the mathematical model of a LSS is recalled. In Section 3, the state space formulation is introduced and the stabilization problems is addressed. Numerical simulations are presented in Section 4. Some final remarks conclude the paper.

2. Mathematical Model of a Large Flexible Structure

In this section, we recall the mathematical model of a flexible spacecraft developed in [14], referring to the Appendix for details. The spacecraft is composed of a rigid part, representing the main body, and a flexible element, carrying a payload at the extremity. A general rest-to-rest maneuver is equivalent to a single rotation about the so-called Euler axis [37]. Therefore, we consider for each rest-to-rest maneuver a right-hand frame with an axis, say x, coincident with the Euler axis passing through the spacecraft center of mass, so that the maneuver is performed in the plane orthogonal to the Euler axis. The body fixed frame considered has the x-axis superimposed to the Euler one, and the y and z-axes on the plane orthogonal to the Euler axis, with z being coincident with the appendage neutral axis and the y-axis being set such that the triple is a right-hand frame (see Fig. 1).

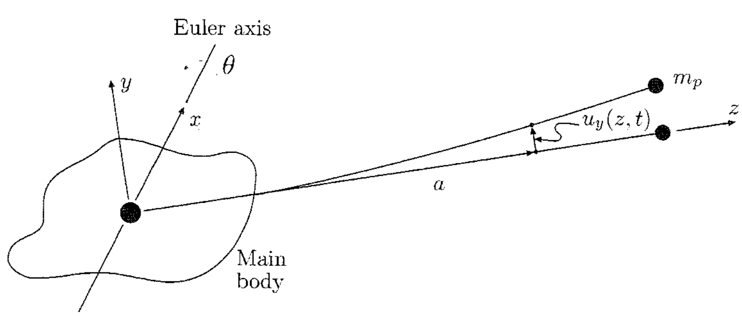

Fig. 1. Flexible structure.

The mathematical model of this system can be obtained by applying the classical variation method. Let us define $-\theta$ the rotation about the x-axis, and $-u_y'(z,t)$ the angular rotation of the mass element dm with respect to the neutral axis due to the bending $u_y(z,t)$. Proceeding as in the Appendix, one finally obtains the dynamic equations of the flexible structure

$$\begin{aligned}
&EI_x u_y''''(z,t) + \rho_l[\ddot{u}_y(z,t) + z\ddot{\theta} - \dot{\theta}^2 u_y(z,t)] = 0,\\
&EI_x u_y'''(l,t) = m_p[\ddot{u}_y(l,t) + l\ddot{\theta} - \dot{\theta}^2 u_y(l,t)] - \nu,\\
&EI_x u_y''(l,t) = -J_{px}[\ddot{\theta} + \ddot{u}_y'(l,t)] + \sigma,\\
&J_{mbx}\ddot{\theta} + \mu = \tau,
\end{aligned} \tag{1}$$

with $u_y(0,t)=0$, $u_y'(0,t)=0$ and

$$\begin{aligned}
\mu = {}& \left(\frac{1}{3}m_b l^2 + J_{px} + m_p l^2\right)\ddot{\theta} + \rho_l\int_b z\ddot{u}_y(z,t)\,dz\\
&+ m_p l\ddot{u}_y(l,t) + J_{px}\ddot{u}_y'(l,t),
\end{aligned}$$

where $m_b = \rho_l l$ is the beam's mass. As usual primes denote the derivatives with respect to the spacial variable, while dots are the time derivatives. Furthermore, J_{mbx} is the moment of inertia with respect to the x-axis of the main body, while J_{px} and m_p are the moment of inertia with respect to the x-axis and the mass of the payload, respectively. Moreover, l is the beam's length, ρ_l is the linear density and EI_x is the flexural rigidity. Finally, τ is the torque acting on the main body, while ν, σ are the force and torque applied by actuators placed on the payload.

Multiplying the first of (1) by z, and integrating on the flexible link, one can re-express μ as

$$\begin{aligned}
\mu = {}& \rho_l\dot{\theta}^2\int_b z u_y(z,t)\,dz - EI_x[l u_y'''(l,t) - u_y''(l,t)\\
&+ u_y''(0,t)] + m_p l(\ddot{u}_y(l,t) + l\ddot{\theta})\\
&+ J_{px}(\ddot{\theta} + \ddot{u}_y'(l,t))\\
= {}& \rho_l\dot{\theta}^2\int_b z u_y(z,t)\,dz - EI_x[l u_y'''(l,t) - u_y''(l,t)\\
&+ u_y''(0,t)] + l(EI_x u_y'''(l,t)\\
&+ m_p\dot{\theta}^2 u_y(l,t) + \nu) + \sigma - EI_x u_y''(l,t),
\end{aligned}$$

where the second and third of (1) have been used. Finally, defining

$$y(z,t) = u_y(z,t) + z\theta(t)$$

as the total (rigid plus flexible) displacement, so that

$$\begin{aligned}
&y'(z,t) = u_y'(z,t) + \theta(t) \quad && u_y(z,t) = y(z,t) - z\theta(t)\\
&y''(z,t) = u_y''(z,t) && \theta(t) = y'(0,t)\\
&\vdots\\
&y''''(z,t) = u_y''''(z,t)
\end{aligned}$$

Eq. (1) become

$$\begin{aligned}
\rho_l\ddot{y}(z,t) = {}& -EI_x y''''(z,t) + \rho_l\dot{y}'^2(0,t)[y(z,t)\\
&- zy'(0,t)]\\
m_p\ddot{y}(l,t) = {}& EI_x y'''(l,t) + m_p\dot{y}'^2(0,t)[y(l,t)\\
&- ly'(0,t)] + \nu\\
J_{px}\ddot{y}'(l,t) = {}& -EI_x y''(l,t) + \sigma\\
J_{mbx}\ddot{y}'(0,t) = {}& EI_x y''(0,t) - \dot{y}'^2(0,t)\\
&\times\left\{\int_b \rho_l z[y(z,t) - zy'(0,t)]\,dz\right.\\
&\left.+ m_p l[y(l,t) - ly'(0,t)]\right\} + \tau - \sigma - \nu l.
\end{aligned}$$

Note that $y(0,t)=0$.

Defining $x = (y(z,t) \quad y(l,t) \quad y'(l,t) \quad y'(0,t))^{\mathrm{T}} = (x_f(t,z) \quad x_b^{\mathrm{T}}(t))^{\mathrm{T}}$, $\eta = \tau - \sigma - \nu l$ and

$$Ax = \begin{pmatrix} EI_x y''''(z,t)\\ -EI_x y'''(l,t)\\ EI_x y''(l,t)\\ -EI_x y''(0,t)\end{pmatrix}, \quad M = \begin{pmatrix}\rho_l & 0 & 0 & 0\\ 0 & m_p & 0 & 0\\ 0 & 0 & J_{px} & 0\\ 0 & 0 & 0 & J_{mbx}\end{pmatrix},$$

$$Bu = \begin{pmatrix}0\\ \nu\\ \sigma\\ \eta\end{pmatrix}, \quad u = \begin{pmatrix}\nu\\ \sigma\\ \eta\end{pmatrix}$$

$$T_n l(x,\dot{x}) = \dot{y}'^2(0,t)\begin{pmatrix} -\rho_l[y(z,t) - zy'(0,t)]\\ -m_p[y(l,t) - ly'(0,t)]\\ \int_b \rho_l z[y(z,t) - zy'(0,t)\,dz + m_p l[y(1,t) - ly'(0,t)]\end{pmatrix}$$

one finally obtains the equations in the form

$$M\ddot{x} + Ax + T_{nl}(x,\dot{x}) = Bu, \tag{2}$$

where $T_{nl}(x,\dot{x})$ is a nonlinear perturbation term arising from the fact that in our modelling we took into account also the variation terms due to the total inertia $J_T(t)$ (see (26) in Appendix). Equation (2) is well posed once we define the space H where the solution x is to be found. Following [4], we see that x is composed of a functional part

$$x_f(z,t) = y(z,t), \quad 0 < z < l,$$

and a boundary component

$$x_b(t) = (\, y(l,t) \quad y'(l,t) \quad y'(0,t)\,)^{\mathrm{T}}, \tag{3}$$

so that H can be chosen as $L_2[0,l] \times \mathbb{R}^3$. Indeed, from the very expression of Ax we see that A can be defined on the following domain

$$\begin{aligned} D(A) = \{ & y(\cdot,t) \in L_2[0,l]: \\ & y'(\cdot,t), y''(\cdot,t), y'''(\cdot,t), y''''(\cdot,t) \\ & \in L_2[0,l], y(0,t) = 0; \\ & (\, y(l,t) \quad y'(l,t) \quad y'(0,t))^{\mathrm{T}} \in \mathbb{R}^3 \} \end{aligned}$$

so that the space H is obtained as the completion of $D(A)$ in the usual inner product on $L_2[0,l] \times \mathbb{R}^3$, defined as

$$\begin{aligned} [x_1, x_2] = & \int_0^l y_1(s,t) y_2(s,t)\, \mathrm{d}s + y_1(l,t) y_2(l,t) \\ & + y_1'(l,t) y_2'(l,t) + y_1'(0,t) y_2'(0,t) \end{aligned}$$

$x_1, x_2 \in H$. Operator A is then densely defined; moreover, it is self-adjoint and

$$[Ax, x] + [x, Ax] \geq 0, \quad x \in D(A) \tag{4}$$

namely A is nonnegative definite. Note that physically the left-hand side of (4) is proportional to the potential energy due to the bending (see Appendix), and this justifies the name of stiffness operator for A, which models the spring-like intrinsic behavior of the flexible beam. Moreover, A admits only a countable set of eigenvalues $\{\omega_k^2\} \cup \{0\}$ and, being self-adjoint, the correspondent set of eigenfunctions $\{\phi_k\}$ form a basis for H [4]. Finally, $M\colon H \to H$ is the generalized inertia operator, which is bounded, self-adjoint and positive definite with bounded inverse, and the input operator $B\colon H_u \to H$ is compact on the control space $H_u = \mathbb{R}^3$.

Remark 1. The main point of this formulation is the inclusion of the boundary variables in the definition of the beam state x [4]; then, semigroup theory is enforced to establish existence and uniqueness of bounded solutions to (2) [6], as shown in the next Section. The related control problems differ substantially from the classical boundary control approach for PDEs, as revisited for instance in [17], where abstract linear equations with unbounded input operators are dealt with.

3. State–Space Formulation

Modern control strategies are based on state–space models, so that we need to reformulate (2) as a first order differential equation on Hilbert space. Moreover, for practical purpose, we need general solutions correspondent to control functions $u \in L_2[0,\infty)^3$, that is controls with finite energy. To this aim, let us consider first the following self-adjoint and positive definite operator

$$A_p x = \begin{pmatrix} EI_x y''''(z,t) \\ -EI_x y'''(l,t) \\ EI_x y''(l,t) \\ -EI_x y''(0,t) + k_p y'(0,t) \end{pmatrix}$$

obtained from operator A by considering a position feedback

$$\tau = -k_p y'(0,t) + \bar{\tau}, \quad k_p > 0. \tag{5}$$

In this way operator A_p shares the properties of A but does not have the zero eigenvalue; therefore, it is also closed since it has a bounded inverse A_p^{-1} [4]. Physically, this feedback specifies the attitude reference, while $\bar{\tau}$ will be used in the following to stabilize the flexible dynamics. Note that $\sqrt{A_p}$ is a well defined positive and self-adjoint operator with a domain $D(\sqrt{A_p}) \supset D(A_p)$ [4]. Now rewrite Eq. (2) as follows

$$\dot{Y} = \mathcal{A}Y + \mathcal{T}(Y) + \mathcal{B}v, \quad Y = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}, \tag{6}$$

where $\bar{\eta} = \bar{\tau} - \sigma - \nu l$, and

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -M^{-1}A_p & 0 \end{pmatrix}, \quad \mathcal{B}v = \begin{pmatrix} 0 \\ M^{-1}Bv \end{pmatrix},$$

$$v = \begin{pmatrix} \nu \\ \sigma \\ \bar{\eta} \end{pmatrix},$$

$$\mathcal{T}(Y) = \begin{pmatrix} 0 \\ -M^{-1}T_{nl}(x, \dot{x}) \end{pmatrix},$$

$$D(\mathcal{A}) = D(A_p) \times D(\sqrt{A_p}).$$

In [4] it is shown that to get a bounded solution for (6) we need to consider the space

$$\mathcal{H} = D(\sqrt{A_p}) \times H$$

with the "energy-like" inner product on $\mathcal{H}$ [3,4]

$$[X, Y] = [\sqrt{A_p}x_1, \sqrt{A_p}y_1] + [Mx_2, y_2],$$

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Then $\mathcal{H}$ is closed under this inner product; moreover, $\mathcal{A}^* = -\mathcal{A}$, $D(\mathcal{A}^*) = D(\mathcal{A})$, and

$$[\mathcal{A}Y, Y] + [Y, \mathcal{A}Y] = 0, \quad Y \in D(\mathcal{A}). \tag{7}$$

Therefore, $\mathcal{A}$ generates a strongly continuous semigroup of contraction $[V(t), t \geq 0]$, $\|V(t)\| \leq 1$, on $\mathcal{H}$ [2]. Finally, note that $\mathcal{T}(Y)$ is a continuous locally Lipschitz function on $\mathcal{H}$.

Remark 2. Note that the explicit (very complicated) expression of $\sqrt{A_p}$ need never be computed since we will use the norm

$$[A_p x, x] + [M\dot{x}, \dot{x}]$$

on $D(\mathcal{A})$, which is dense in $\mathcal{H}$. This is justified by the fact that standard analysis will be always performed on elements in $D(A_p)$ and then extended to elements of H by the usual limiting argument.

The following Lemma works, stating the existence of the so-called local weak solutions (see [31, Theorem 1.4, p. 185]).

Lemma 1. Let $Y(0) \in \mathcal{H}$ and $v \in L_2[0, T]$, $T > 0$. Equation (6) has a unique bounded local weak solution

$$\begin{aligned} Y(t) &= V(t)Y(0) + \int_0^t V(t-s)\mathcal{B}v(s)\,\mathrm{d}s \\ &\quad + \int_0^t V(t-s)\mathcal{T}(Y(s))\,\mathrm{d}s \end{aligned} \tag{8}$$

for $t \in [0, T_{Y(0)})$.

Remark 3. As opposite to strong solutions [23,24,30] obtained for $Y(0) \in D(\mathcal{A})$ and $v \in C^1[0, T]^3$, $T > 0$, considering weak solutions is important from the practical point of view since no smooth initial conditions $Y(0) \in \mathcal{H}$ and more general ($L_2[0, T]$ vs. $C^1[0, T]$) control functions can be used for feedback, such as finite energy piecewise continuous functions. Therefore, in the sequel we will refer only to solutions in the weak sense.

In what follows we consider first a state feedback ensuring *global* stabilization of the closed-loop solutions for any initial condition. Then, considering the practical problems arising from the implementation of such a control law, we will propose an output feedback yielding *semiglobal* stabilization.

Theorem 1. Given Eq. (6), we can find a control

$$v = K(Y(t)) = \begin{pmatrix} -k_1\dot{y}(l,t) \\ -k_2\dot{y}'(l,t) \\ -k_3\dot{y}'(0,t) - \dot{y}'(0,t)\varphi(Y(t)) \end{pmatrix} \tag{9}$$

$k_i > 0$, $i = 1, 2, 3$, where

$$\begin{aligned} \varphi(Y(t)) &= \int_b \rho_l[y(z,t) - zy'(0,t)][\dot{y}(z,t) - z\dot{y}'(0,t)]\,\mathrm{d}z \\ &\quad + m_p[y(l,t) - ly'(0,t)][\dot{y}(l,t) - l\dot{y}'(0,t)] \end{aligned}$$

such that the closed-loop system

$$\dot{Y} = \mathcal{A}Y + \mathcal{T}(Y) + \mathcal{B}K(Y), \quad Y(0) \in \mathcal{H} \tag{10}$$

has a global solution $Y(t)$ for any $t > 0$. Moreover, $Y(t)$ is uniformly bounded and asymptotically stable; in particular, so are the boundary components $x_b(t)$.

Proof. In order to determine the control $v = K(Y)$ ensuring the stability of the system (10), let us consider $W = \frac{1}{2}\|Y\|^2$ as Lyapunov function. According to Lemma 1, we can build a local strong solution of (6) for a given $Y(0) \in D(\mathcal{A})$ and $t \in [0, T_{Y(0)}]$; then the strong time-derivative of W is given by

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Y\|^2 &= [Y, \mathcal{A}Y + \mathcal{T}(Y) + \mathcal{B}v] \\ &= [Y, \mathcal{A}Y] + [Y, \mathcal{T}(Y)] + [Y, \mathcal{B}v] \\ &= -[T_{nl}(x, \dot{x}), \dot{x}] + [\mathcal{B}v, \dot{x}], \end{aligned}$$

where we exploited the property (7) of the operator $\mathcal{A}$. Therefore,

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Y\|^2 \\ &= \dot{y}'^2(0,t)\Big(\int_b \rho_l[y(z,t) - zy'(0,t)][\dot{y}(z,t) - z\dot{y}'(0,t)]\,\mathrm{d}z \\ &\quad + m_p[y(l,t) - ly'(0,t)][\dot{y}(l,t) - l\dot{y}'(0,t)]\Big) \\ &\quad + \dot{y}(l,t)\nu + \dot{y}(l,t)\sigma + \dot{y}'(0,t)(\bar{\tau} - \sigma - \nu l) \\ &= \dot{y}'^2(0,t)\varphi(Y(t)) + \dot{y}(l,t)\nu + \dot{y}'(l,t)\sigma \\ &\quad + \dot{y}'(0,t)(\bar{\tau} - \sigma - \nu l). \end{aligned} \tag{11}$$

Setting $K(Y)$ as in (9), one finally has

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Y\|^2 &= -k_1\dot{y}^2(l,t) - k_2\dot{y}'^2(l,t) - k_3\dot{y}'^2(0,t) \\ &= -[RY, Y] \leq 0, \end{aligned} \tag{12}$$

where

$$R = \begin{pmatrix} 0 & 0 \\ 0 & L \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & k_1 & 0 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & 0 & 0 & k_3 \end{pmatrix}.$$

Note that $K(Y)$ is a locally Lipschitz operator on $\mathcal{H}$. Finally, integrating (12)

$$\begin{aligned} &\|Y(t)\|^2 \\ &= \|Y(0)\|^2 - 2\int_0^t [RY(s), Y(s)]\,\mathrm{d}s, \quad t \leq T_{Y(0)}. \end{aligned} \tag{13}$$

But this actually implies that $Y(t)$ is uniformly bounded by $Y(0)$, so that it can be extended to any $t \geq 0$, thus obtaining a global solution. Moreover,

$$2\int_0^t [RY(s), Y(s)]\,ds \leq \|Y(0)\|^2 - \|Y(t)\|^2 \leq \|Y(0)\|^2,$$

so that

$$\int_0^\infty [RY(s); Y(s)]\,ds \leq \frac{1}{2}\|Y(0)\|^2. \tag{14}$$

While relation (13) implies that $x(t)$, $\dot{x}(t)$ are uniformly bounded, relation (14) implies that the boundary rates $\dot{y}(l,t)$, $\dot{y}'(l,t)$, $\dot{y}'(0,t)$ are in $L_2[0,\infty)$. This is actually crucial in proving that the closed-loop system is asymptotically stable. We first note that the control law (9) can be represented as follows

$$v = -\mathcal{P}\mathcal{B}^*Y + K_0(Y) \tag{15}$$

with

$$K_0(Y) = \begin{pmatrix} 0_{(7\times 1)} \\ -\dot{y}'(0,t)\varphi(Y) \end{pmatrix},$$

$$\mathcal{P} = \mathcal{P}^* = \begin{pmatrix} I_{(4\times 4)} & 0_{(4\times 4)} \\ 0_{(4\times 4)} & P \end{pmatrix} > 0,$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & k_1 & 0 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & 0 & 0 & k_3 \end{pmatrix}.$$

Therefore, the closed-loop system becomes

$$\dot{Y}(t) = (\mathcal{A} - \mathcal{B}\mathcal{P}\mathcal{B}^*)Y(t) + \Gamma(Y(t)) \tag{16}$$

with

$$\Gamma(Y(t)) = \begin{pmatrix} 0_{(4\times 1)} \\ M^{-1}\begin{pmatrix} -\rho_l[y(z,t) - zy'(0,t)]\dot{y}'^2(0,t) \\ -m_p[y(l,t) - ly'(0,t)]\dot{y}'^2(0,t) \\ 0 \\ \left(\int_b \rho_l z[y(z,t) - zy'(0,t)]\,dz + m_p l[y(l,t) - ly'(0,t)]\right)\dot{y}'^2(0,t) - \varphi(Y)\dot{y}'(0,t) \end{pmatrix} \end{pmatrix}.$$

The following lemma states a crucial property of the pair $(\mathcal{A},\mathcal{B})$ [2,4].

Lemma 2. The pair $(\mathcal{A},\mathcal{B})$ is controllable.

Proof. According to [2,4], the pair $(\mathcal{A},\mathcal{B})$ is controllable if

$$\mathcal{B}^* V^*(t) Y = 0, \quad \text{for every } t \geq 0$$

implies $Y = 0$. The proof follows the same arguments of the appendix in [4] and is here omitted. □

On the base of this result, $(\mathcal{A} - \mathcal{B}\mathcal{P}\mathcal{B}^*)$ generates a strongly stable contraction semigroup $[V_p(t), t \geq 0]$ [11], i.e.,

$$\|V_p(t)\| \leq 1 \quad \text{and} \quad \lim_{t\to\infty}\|V_p(t)Z\| = 0, \quad \forall Z \in \mathcal{H}.$$

Furthermore, $\Gamma(Y(t))$ is $L_2[0,\infty)$ since the entries are products of bounded functions with functions in $L_2[0,\infty)$.

From (16) we can write

$$Y(t) = V_p(t)Y(0) + \int_0^t V_p(t-\tau)\Gamma(Y(\tau))\,d\tau = V_p(t)Y(0) + g(t)$$

for any $Y(0) \in D(\mathcal{A})$ and $t > 0$. Now, due to the strong stability of the semigroup $V_p(t)$, the first term on the right side goes to zero as $t \to \infty$. Furthermore, relation (13) implies that function $g(t)$ is finite for any $t > 0$, meaning that function $V_p(t-\tau)\Gamma(\tau)$ is an $L_1[0,\infty)$ function; consequently, given any $\varepsilon > 0$ there exists a $T_\varepsilon > 0$ such that

$$\left\|\int_{T_\varepsilon}^t \Gamma(Y(\tau))\,d\tau\right\| < \varepsilon, \quad t > T_\varepsilon.$$

Therefore, for any $t > T_\varepsilon$ one has

$$\begin{aligned}\|g(t)\| &\leq \left\|\int_0^{T_\varepsilon} V_p(t-\tau)\Gamma(Y(\tau))\,d\tau\right\| + \varepsilon \\ &= \left\|V_p(t-T_\varepsilon)\int_0^{T_\varepsilon} V_p(T_\varepsilon-\tau)\Gamma(Y(\tau))\,d\tau\right\| + \varepsilon \\ &= \|V_p(t-T_\varepsilon)Z(T_\varepsilon)\| + \varepsilon < \bar{\varepsilon},\end{aligned}$$

where the last relation is due to the strong stability of the semigroup $V_p(t)$. Therefore,

$$\lim_{t\to\infty}\|Y(t)\| = 0$$

and in particular $y(l,t)$, $y'(l,t)$, $y'(0,t)$, go asymptotically to zero.

The arguments used from (13) on can be extended to weak solutions, namely for $Y(0) \in \mathcal{H}$, by standard limiting arguments. □

Remark 4. In the proof of Theorem 1, we noted that the control law (9) can be rewritten as

$$v = -\mathcal{P}\mathcal{B}^* Y + K_0(Y), \quad \mathcal{P} = \mathcal{P}^* > 0$$

with $\mathcal{P}\mathcal{B}^* Y$ a linear boundary term and $K_0(Y)$ a distributed nonlinear contribution. This control yields global stability, that is it steers (asymptotically) the system to the origin of the state space regardless the system initial condition $Y(0)$. Nevertheless, the practical implementation of the term $K_0(Y)$ presents technological difficulties, such as distributed sensing and actuation because of the term $\varphi(Y)$, while a control law based only on the boundary components would be amenable.

To avoid the difficulty pointed out in Remark 4, we can rely on semiglobal stabilization [1,10] which requires the prescription of a closed bounded set Ω containing the origin, which can be included in the region of attraction; once the initial condition $Y(0)$ is chosen in Ω, the closed-loop system trajectory $Y(t)$, $t > 0$, stemming from $Y(0)$ remains inside a bounded set containing Ω, and eventually goes to the origin as t goes to infinity. In this case the stabilizing control depends on Ω and works for any $Y(0) \in \Omega$. The natural choice for the output is given by the vector of the boundary quantities, i.e.,

$$Z(t) = \mathcal{C}Y(t) = \begin{pmatrix} x_b(t) \\ \dot{x}_b(t) \end{pmatrix} \tag{17}$$

with $x_b(t)$ as in (3). Next theorem shows that system (6), (17) is actually semiglobally stabilizable by a boundary output feedback control.

Theorem 2. Let us consider the system (6), (17) and a closed bounded set $\Omega \in \mathcal{H}$ containing the origin. Then, for $T(0) \in \Omega$, the control

$$v_\Omega = K_b(Z) = \begin{pmatrix} -k_1 \dot{y}(l,t) \\ -k_2 \dot{y}'(l,t) \\ -(k_3 + k_\Omega + \varphi_b(Y(t)))\dot{y}'(0,t) \end{pmatrix} \tag{18}$$

with $k_i > 0$, $i = 1, 2, 3$, and

$$\varphi_b(Y(t)) = m_p[y(l,t) - ly'(0,t)][\dot{y}(l,t) - l\dot{y}'(0,t)],$$

ensures semiglobal asymptotic stability of the closed-loop system.

Proof. Consider the Lyapunov function $V(Y) = \frac{1}{2}\|Y\|^2$ of Theorem 1. Define the level sets

$$S_c = \left\{Y \in \mathcal{H} : \tfrac{1}{2}\|Y\|^2 \le c\right\}, \quad c > 0.$$

These level sets contain the origin, and for any given closed bounded set $\Omega \in \mathcal{H}$ we can find a $\bar{c}$ such that $\Omega \subset S_{\bar{c}}$. Following the proof of Theorem 1, (11) can be bounded as follows

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Y\|^2 & \\ \le \dot{y}'^2(0,t)&(k_\Omega + m_p[y(l,t) - ly'(0,t)] \\ &\times [\dot{y}(l,t) - l\dot{y}'(0,t)]) \\ &+ \dot{y}(l,t)\nu + \dot{y}'(l,t)\sigma + \dot{y}'(0,t)(\bar{\tau} - \sigma - \nu l) \end{aligned} \tag{19}$$

$t \le T_{Y(0)}$, where k_Ω is such that for any t

$$k_\Omega = \sup_{S_{\bar{c}}} \int_b \rho_l[y(z,t) - zy'(0,t)] \times [\dot{y}(z,t) - z\dot{y}'(0,t)]\,\mathrm{d}z.$$

Now using the control (18) one obtains the same expression (12) of Theorem 1, so ensuring the boundedness of the system trajectories with initial conditions $Y(0) \in \Omega \cap D(\mathcal{A})$. Since control (18) is in the form (15), the same arguments of Theorem 1 apply to show that asymptotic stability for strong solutions of the closed-loop system is obtained, with a region of attraction $S_{\bar{c}}$ containing Ω. The results can be extended to weak solutions as usual. □

The controller (18) depends on the region of attraction Ω through the constant k_Ω, see (19); moreover, it is clearly nonlinear but, as it can be easily seen from the proof, a linear controller can be used once the term $\varphi_b(Y)$ is upper bounded as well, at the expense of a higher gain.

Note that the whole control law resulting from (5), (18) is *finite dimensional* and is a feedback from *measured quantities*, i.e., from the output variables $y(l,t)$, $y'(0,t)$ and $\dot{y}(l,t)$, $\dot{y}'(l,t)$, $\dot{y}'(0,t)$. This control law is just a position and rate feedback, which recovers the usual stabilizing control strategies largely used in finite dimensional settings [21,27,28]. As opposite to the finite dimensional approach, which suffers the spillover effect, the proposed controller ensures the *asymptotic stabilization of all the modes*.

Remark 5. The achieved asymptotic stability in general is not exponential, since the pair $(\mathcal{A}, \mathcal{B})$ is not exponentially stabilizable by bounded controls [19].

4. Simulation Results

In this section, we present some simulations which illustrate the behavior of the controlled system on a simple structure, whose parameters are given in Table 1.

Referring to Eq. (2), a basis for the space H is obtained by adding the mode

$$\Phi_0(z) = \begin{pmatrix} z \\ l \\ 1 \\ 1 \end{pmatrix}$$

corresponding to the rigid body rotation, to the orthogonal system of eigenfunctions $\Phi_k(z)$ of the self-adjoint operator A. The eigenfunctions of the flexible modes

$$\Phi_k(z) = \begin{pmatrix} \phi_k(z) \\ \phi_k(l) \\ \phi'_k(l) \\ \phi'_k(0) \end{pmatrix}$$

and $\Phi_0(z)$ are chosen to be M-orthogonal, i.e.,

$$[M\Phi_i(z), \Phi_k(z)] = \delta_{ik}, \quad i, k \in [0, \infty].$$

Therefore one can write

$$y(z,t) = \sum_{k=0}^{\infty} c_k(t)\phi_k(z).$$

Table 1. Spacecraft parameters.

$J_{mb} = 720$	Kg m^2
$J_p = 25$	Kg m^2
$m_p = 50$	Kg
$l = 10$	m
$\varrho_l = 2$	Kg/m
$E = 1.2 \times 10^{10}$	N/m^2
$I_x = 0.5 \times 10^{-4}$	m^4

A finite dimensional approximation is obtained by considering the span of the first $N+1$ modes

$$H_N = \text{span}\{\Phi_0(z), \ldots, \Phi_N(z)\}$$

so that

$$y(z,t) \simeq y_N(z,t) = \sum_{k=0}^{N} c_k(t)\phi_k(z).$$

Equation (2) can be projected on H_N obtaining

$$\ddot{c}(t) + \Lambda c(t) + \tilde{T}_{nl}(c(t), \dot{c}(t)) = \tilde{U}(c(t), \dot{c}(t))$$

where

$$\Lambda = \text{diag}\{\omega_0, \omega_1, \ldots, \omega_N\},$$

$$\tilde{T}_{nl}(c(t), \dot{c}(t)) = \begin{pmatrix} [T_{nl}(x,\dot{x}), \Phi_0(z)] \\ [T_{nl}(x,\dot{x}), \Phi_1(z)] \\ \vdots \\ [T_{nl}(x,\dot{x}), \Phi_N(z)] \end{pmatrix}$$

$$\tilde{U}(c(t), \dot{c}(t)) = \begin{pmatrix} [Bu, \Phi_0(z)] \\ [Bu, \Phi_1(z)] \\ \vdots \\ [Bu, \Phi_N(z)] \end{pmatrix} \tag{20}$$

and $\omega_0 = 0$ is the rigid motion eigenvalue while ω_k, $k = 1, \ldots, N$, are the eigenfrequencies of A. It is worth recalling that in (20) the control to be considered is a position plus a rate feedback, given by (5) and (9)

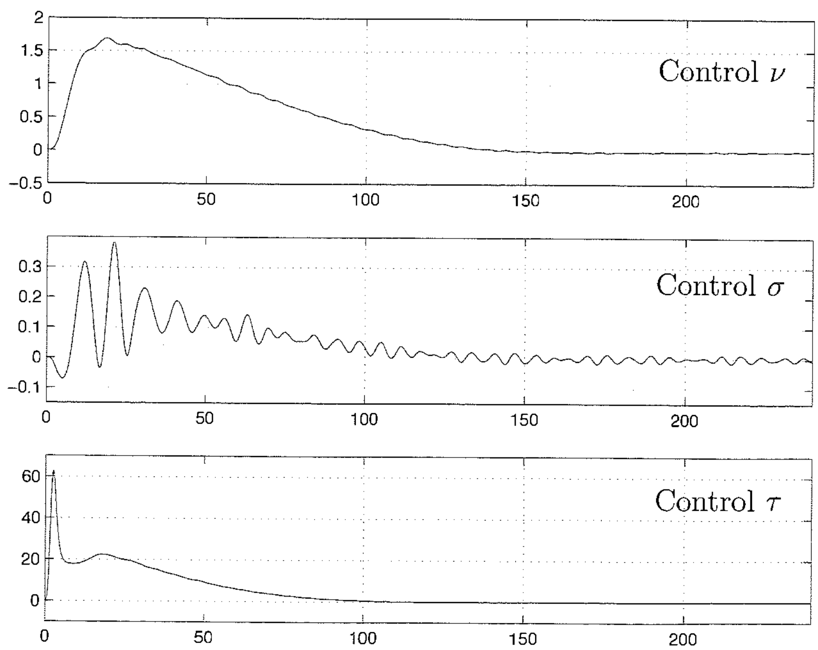

Fig. 2. Controls σ (Nm), ν (N), τ (Nm) (1st simulation).

or (18). Hence, standard integration routines can be applied to determine $c(t)$.

Simulations have been carried out considering $N = 10$ elastic modes and taking into account the semiglobal controller given by Theorem 2. A large slew maneuver of 120° has been considered; the structure is initially idle. Two different sets of controller gains have been considered. Experimentally, one obtains a good trade-off between the control effort and the transient behavior by choosing

$$k_p = 25, \quad k_1 = k_2 = k_3 = 5.$$

The results are summarized in Fig. 2 for the controls σ, ν and τ, and Fig. 3 for the rigid rotation $\theta(t)$ and the tip appendix displacement $u(l, t)$. Further limitation

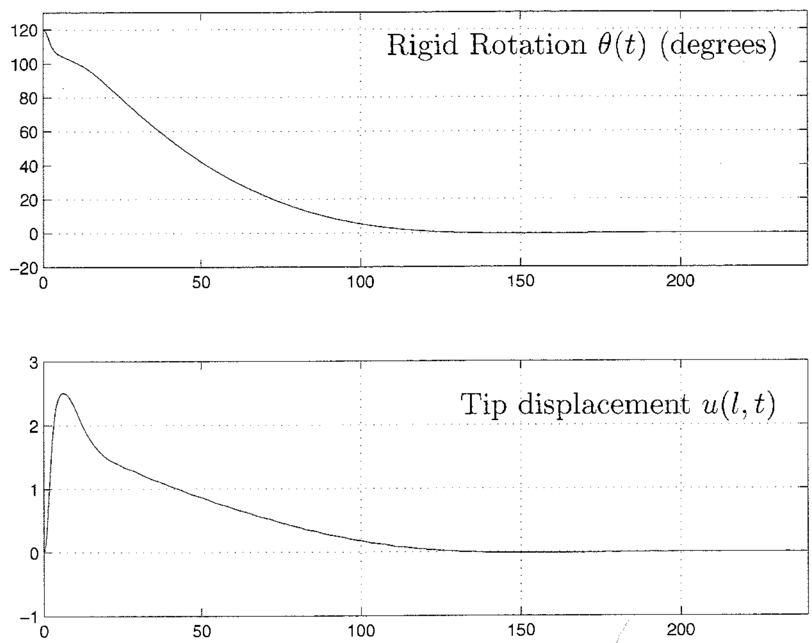

Fig. 3. Rigid rotation $\theta(t)$ and tip appendix displacement $u(l, t)$ (m) (1st simulation).

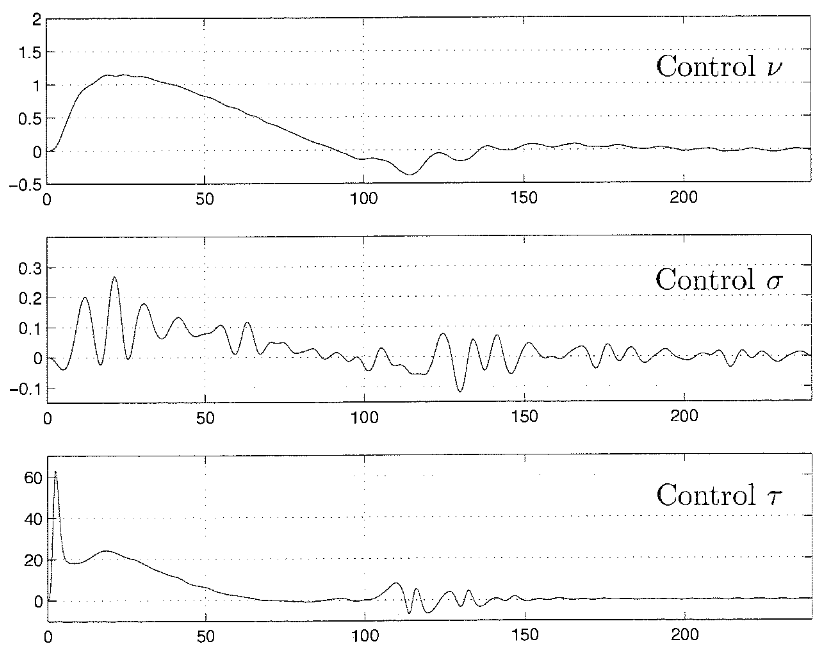

Fig. 4. Controls σ (Nm), ν (N), τ (Nm) (2nd simulation).

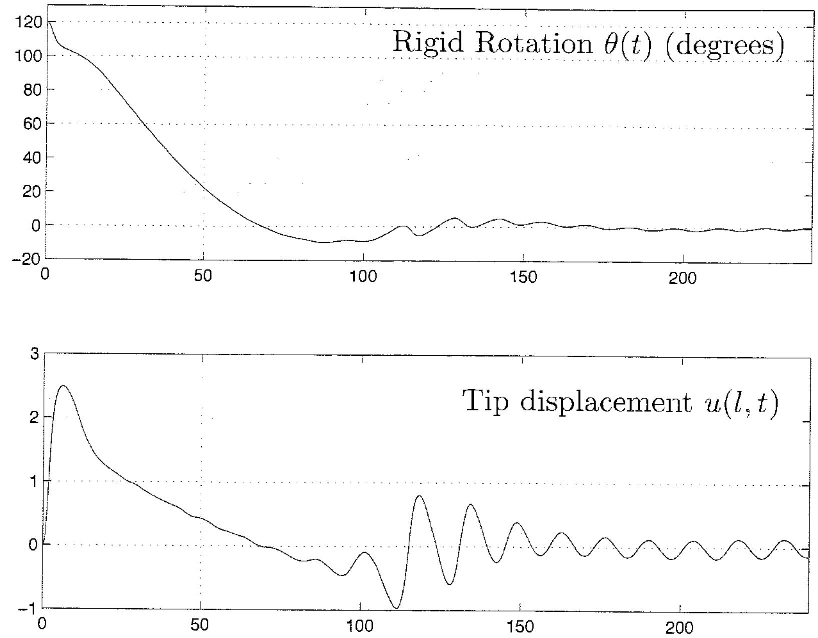

Fig. 5. Rigid rotation $\theta(t)$ and tip appendix displacement $u(l, t)$ (m) (2nd simulation).

on the boundary controls, obtained by choosing

$$k_p = 25, \quad k_1 = k_2 = k_3 = 2.5$$

yields a less satisfying transient behavior. The results are shown in Figs 4 and 5. These results suggest that optimal control design (robust H_∞ control, adaptive control, etc.) should be pursued in order to obtain better closed loop performances.

5. Conclusions

In this work we have considered an infinite dimensional model of a LSS, appropriately describing the coupling between the rigid and the flexible dynamics. These equations have been reformulated as a nonlinear state–space dynamical system, and the existence and uniqueness of local solutions have been shown. Furthermore, two stabilization strategies have been studied: the first is based on a (nonlinear) state feedback and ensures the global stability of the system trajectories; the second is an output feedback which guarantees the semiglobal stabilization of the system trajectories stemming from any initial condition in a chosen closed and bounded set. This second result has an applicative interest since requires only the measurements of boundary quantities, resulting then in a finite dimensional controller. As final remark, we can observe that the proposed controllers ensure the asymptotic stabilization of all the modes, thus avoiding the possible spillover typical of control laws obtained from truncated mode models.

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Appendix

In this Appendix the mathematical model of a LSS, composed of a rigid part (main body) and a flexible element carrying a payload at the extremity, is recalled. The assumed hypotheses are those commonly used in the modeling of elastic structures, namely the Euler–Bernouilli theory for slender beams, the beam cross section symmetry with respect to the neutral axis, the negligibility of the contributions due to torsion, shear and longitudinal deformations [13,14,25,36]. The mathematical model of this system is obtained by applying the classical variation method. The kinetic energy of the rigid main body is given by

$$T_{mb} = \tfrac{1}{2}\omega^{\mathrm{T}} J_{mb}\omega, \quad \omega = \begin{pmatrix} -\dot{\theta} & 0 & 0 \end{pmatrix}^{\mathrm{T}}, \qquad (21)$$

where ω is the angular velocity and J_{mb} is the inertia matrix of the main body, expressed in the body fixed frame having the x-axis superimposed to the Euler one, and the y and z-axes on the plane orthogonal to the Euler axis, with z being coincident with the appendage neutral axis and the y-axis being set such that the triple is a right-hand frame. The kinetic energy of the flexible beam is

$$T_b = \frac{1}{2}\omega^{\mathrm{T}} \int_b J_{bc}(z,t)\,\mathrm{d}z\,\omega + \rho_l \omega^{\mathrm{T}} \int_b \tilde{a}(z)\dot{u}(z,t)\,\mathrm{d}z + \frac{1}{2}\rho_l \int_b \dot{u}^{\mathrm{T}}(z,t)\dot{u}(z,t)\,\mathrm{d}z, \tag{22}$$

where

$$u(z,t) = \begin{pmatrix} 0 \\ u_y(z,t) \\ 0 \end{pmatrix},$$

$$J_{bc}(z,t) \approx \begin{pmatrix} \rho_l z^2 & 0 & 0 \\ 0 & \rho_l z^2 & -\rho_l z u_y(z,t) \\ 0 & -\rho_l z u_y(z,t) & J_z \end{pmatrix}$$

$a(z) = (0 \quad 0 \quad z)^{\mathrm{T}}$ and $\tilde{a}(z)$ is its dyadic representation. Finally, the payload kinetic energy is

$$T_p = \tfrac{1}{2}\omega^{\mathrm{T}} J_p(t)\omega + m_p\omega^{\mathrm{T}}\tilde{a}(l)\dot{u}(l,t) + \tfrac{1}{2}m_p\dot{u}^{\mathrm{T}}(l,t)\dot{u}(l,t) + \omega^{\mathrm{T}} J_{pa}\dot{\vartheta}(l,t) + \tfrac{1}{2}\dot{\vartheta}^{\mathrm{T}}(l,t)J_{pa}\dot{\vartheta}(l,t) \tag{23}$$

with m_p the payload mass, ρ_p its density, and

$$J_{pa} = \begin{pmatrix} J_{px} & 0 & 0 \\ 0 & J_{py} & 0 \\ 0 & 0 & J_{pz} \end{pmatrix}, \quad \vartheta(l,t) = \begin{pmatrix} -u_y'(l,t) \\ 0 \\ 0 \end{pmatrix}$$

$$J_p(t) \simeq \begin{pmatrix} J_{px} + m_p l^2 & 0 & 0 \\ 0 & J_{py} + m_p l^2 & -m_p l u_y(l,t) \\ 0 & -m_p l u_y(l,t) & J_{pz} \end{pmatrix}$$

where J_{px}, J_{py}, J_{pz} are the payload moments of inertia about the x, y and z axes.

Neglecting the torsion, the shear and longitudinal contributions, the potential energy due to the bending can be written as follows [36]

$$U = \frac{1}{2}\int_b EI_x u_y''^2(z,t)\,\mathrm{d}z. \tag{24}$$

The equations of motion are derived by zeroing the variation of the integral

$$I = \int_{t_1}^{t_2} L(t)\,\mathrm{d}t + \int_{t_1}^{t_2} Q(t)\,\mathrm{d}t,$$

where $L = T - U$, and Q are the external (generalized) forces acting on the structure. Hence

$$\delta I = \int_{t_1}^{t_2} [\delta L(t) + \delta Q(t)]\,\mathrm{d}t = 0.$$

Note that $\delta Q = u_g^{\mathrm{T}}\delta\theta + u_m^{\mathrm{T}}\delta u(l,t) + u_p^{\mathrm{T}}\delta\vartheta(l,t)$ in our case. By standard computations, one obtains

$$\delta I = \int_{t_1}^{t_2} (-\dot{A}_1 + u_g)^{\mathrm{T}}\delta\theta\,\mathrm{d}t + \int_{t_1}^{t_2}\int_b (-\dot{A}_2 + A_5)\delta u(z,t)\,\mathrm{d}z\,\mathrm{d}t + \int_{t_1}^{t_2} (-\dot{A}_3 + A_6 + u_m)^{\mathrm{T}}\delta u(l,t)\,\mathrm{d}t - \int_{t_1}^{t_2}\left[\dot{A}_4 + \begin{pmatrix} -EI_x u_y''(l,t) \\ 0 \\ 0 \end{pmatrix} + u_p\right]^{\mathrm{T}} \delta\vartheta(l,t)\,\mathrm{d}t, \tag{25}$$

where

$$J_T(t) = J_{mb} + \int_b J_{bc}(z,t)\,\mathrm{d}z + J_p(t) \tag{26}$$

and

$$A_1 = J_T(t)\omega + \rho_l\int_b \tilde{a}(z)\dot{u}(z,t)\,\mathrm{d}z + m_p\tilde{a}(l)\dot{u}(l,t) + J_{pa}\dot{\vartheta}(l,t),$$
$$A_2 = \rho_l[\dot{u}(z,t) + \tilde{\omega}a(z)],$$
$$A_3 = m_p[\dot{u}(l,t) + \tilde{\omega}a(l)],$$
$$A_4 = J_{pa}[\omega + \dot{\vartheta}(l,t)],$$
$$A_5 = \rho_l\Big[\bar{\omega}^2[a(z) + u(z,t)] - \frac{1}{\rho_l}(0 \quad EI_x u_y''''(z,t) \quad 0)^{\mathrm{T}}\Big],$$
$$A_6 = m_p\Big[\bar{\omega}^2[a(l) + u(l,t)] + \frac{1}{m_p}(0 \quad EI_x u_y'''(l,t) \quad 0)^{\mathrm{T}}\Big],$$
$$\bar{\omega}^2 = \tilde{\omega}^{\mathrm{T}}\tilde{\omega}.$$

Setting to zero the variations, and considering the inputs

$$u_g = (-\tau \quad 0 \quad 0)^{\mathrm{T}}, \quad u_m = (0 \quad \nu \quad 0)^{\mathrm{T}}, \quad u_p = (-\sigma \quad 0 \quad 0)^{\mathrm{T}}$$

one obtains the dynamic equations of the flexible structure (1).