



## Discrete time sliding mode control with application to induction motors<sup>☆</sup>

B. Castillo-Toledo<sup>a</sup>, S. Di Gennaro<sup>b,\*</sup>, A.G. Loukianov<sup>a</sup>, J. Rivera<sup>c</sup>

<sup>a</sup> CINESTAV del IPN – Unidad Guadalajara, Av. Científica, Col. El Bajío, Zapopan, 45010, Jalisco, Mexico

<sup>b</sup> Department of Electrical and Information Engineering, and Center of Excellence DEWS, University of L'Aquila, Poggio Di Roio, 67040 L'Aquila, Italy

<sup>c</sup> Centro Universitario de Ciencias Exactas e Ingenierías de la Universidad de Guadalajara, Av. Revolución, Col. Olímpica, Guadalajara, 44430, Jalisco, Mexico

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### ABSTRACT

This work deals with a sliding mode control scheme for discrete time nonlinear systems. The control law synthesis problem is subdivided into a finite number of subproblems of lower complexity, which can be solved independently. The sliding mode controller is designed to force the system to track a desired reference and to eliminate unwanted disturbances, compensating at the same time matched and unmatched parameter variations. Then, an observer is designed to eliminate the need of the state in the controller implementation. This design technique is illustrated determining a dynamic discrete time controller for induction motors.

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### 1. Introduction

The advent of digital technology and its widespread use have revolutionized the computer-based implementation of advanced control schemes. In fact, recent advances in digital microprocessor technology have given considerable credit to digital control systems, exhibiting relatively low operational cost, flexibility in implementation, simple and functional interactive communication among several control loops. At the same time, there has been a growing interest in the design of controllers based on the digital model of the system. When the system is continuous, the first step is to obtain an accurate sampled model. This has motivated an interesting research activity in the area of discrete time control and has determined the development of digital control methods. Starting from the first studies on the sampling of continuous time nonlinear systems (Monaco & Normand-Cyrot, 1985, 1988), many tools have been developed in the last two decades to control

digital and sampled nonlinear systems, see for instance Monaco and Di Giamberardino (1996), Monaco and Normand-Cyrot (1997, 2001, 2007) and references therein. The aim of this work is to give a further contribution in this field. The main objective is to design a discrete time feedback controller which ensures stability and achieves a specified transient response for discrete time nonlinear systems. Moreover, to provide a certain robust stability margin against bounded uncertainties, a simple approach is used here, based on the sliding mode approach (Utkin, 1993). Sliding mode control is a particular type of variable structure control, designed to drive and constrain the system state to lie within a neighborhood of a switching function. The advantages of the sliding mode technique are well known. First, this method enables the decomposition of the design problem into two independent subproblems: (a) selection of discontinuity surfaces with the desired sliding motion, and (b) determination of a control law to force the sliding mode along this manifold. This allows the suppression of the effects of matched parameter uncertainties and disturbances, and total invariance is obtained when the motion of the system is in sliding mode.

In this paper, an iterative procedure is proposed to design a discrete time sliding mode control law for a class of nonlinear systems. This controller complies with the bounds on the control resources, and is such that the system state is driven toward a certain sliding manifold and stays there for all sampled time instants, avoiding chattering. Furthermore, a discrete time observer is designed to estimate the non-measurable states and perturbation. A separation principle is then applied to verify the

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\* Corresponding author. Tel.: +39 862434461; fax: +39 6233233142.

E-mail addresses: [toledo@gdl.cinvestav.mx](mailto:toledo@gdl.cinvestav.mx) (B. Castillo-Toledo), [digennar@ing.univaq.it](mailto:digennar@ing.univaq.it) (S. Di Gennaro), [louk@gdl.cinvestav.mx](mailto:louk@gdl.cinvestav.mx) (A.G. Loukianov), [jorge.rivera@cucei.udg.mx](mailto:jorge.rivera@cucei.udg.mx) (J. Rivera).

stability of the closed-loop system with this resulting dynamic state feedback. Finally, the proposed control strategy is applied to control voltage-fed induction motors.

Induction motors represent challenging nonlinear systems and constitute an important area of application of recent control methodologies. From a practical point of view, they present many appealing characteristics which render them suitable for the industrial environment (robustness, low maintenance, high performances, etc.). A classical technique for induction motor is the field oriented control, due to Blaschke (1972). More recently, various nonlinear control design approaches have been applied to induction motors to improve their performance, e.g. passivity (Ortega, Nicklasson, & Espinoza-Pérez, 1996), sliding mode (Dodds, Vittek, & Utkin, 1998; Utkin, 1993), adaptive input-output linearization (Marino, Peresada, & Valigi, 1993), backstepping (Tan & Chang, 1999), adaptive sliding mode (Loukianov, 2002). All these approaches are designed using the continuous time model of the plant, which are then approximated (discretized) for real time implementation. This discretization yields relatively low performance. To get better results from real time implementations, the controller must be designed on the basis of an accurate discrete time model of the plant (Castillo-Toledo, Di Gennaro, Loukianov & Rivera, 2008; Monaco & Di Giamberardino, 1996; Monaco & Normand-Cyrot, 1985). Hence, as a further contribution of this paper, an approximated model for current-fed induction motors is derived following Monaco and Di Giamberardino (1996) and Ortega and Taoutaou (1996), and a dynamic controller is derived on the basis of this model to achieve rotor speed and flux amplitude tracking.

The paper is organized as follows. The main results are presented in Section 2, where a discrete time sliding mode control is developed. A nonlinear discrete time observer is designed, and the stability of the closed-loop system is studied. The application of the proposed control method to induction motors is developed in Section 3. In Section 4, simulation results are presented for validation of the proposed methodology. Finally, some remarks conclude the paper.

## 2. Design of digital control by block backstepping

In this section, we will determine a digital control for a particular class of systems, which includes many systems of interest in applications.

### 2.1. The class of systems under study

Consider a nonlinear system

$$\begin{aligned} x_{k+1} &= f(x_k, u_k) + d(w_k) \\ y_k &= h(x_k) \\ \eta_k &= \gamma(x_k) \end{aligned} \quad (1)$$

where  $k \in Z$  denotes the discrete time, with  $Z$  the set of the nonnegative integers. The state vector  $x_k$  is defined on a neighborhood  $X$  of the origin of  $R^n$ ,  $u_k \in R^m$  is the input vector,  $y_k \in R^p$  is the vector of the variables to be controlled, and  $\eta_k \in R^q$  is the measured output of the system. Here  $f(\cdot, \cdot)$ ,  $h(\cdot)$ ,  $\gamma(\cdot)$ ,  $d(\cdot)$  are smooth vector fields of class  $C_{[t, \infty)}^\infty$ , with  $f(0, 0) = 0$ ,  $h(0) = 0$ ,  $\gamma(0) = 0$ . Moreover,  $d(w_k)$  is a perturbation term representing modeling errors, aging, disturbances, etc. Initially  $d(w_k)$  will be supposed known; this assumption will be removed later on.

The control objective is to force the output  $y_k$  to track asymptotically a reference signal  $y_r(w_k)$ , rejecting the effects of an external disturbance  $d(w_k)$ . The tracking error is defined as the

difference between  $y_k$  and the reference signal  $y_r(w_k)$  to be tracked i.e.

$$e_k = y_k - y_r(w_k). \quad (2)$$

The reference signal  $y_r(w_k)$  and the disturbance  $d(w_k)$  are assumed to be bounded, with bounded increments, and generated by a given external system described by

$$\begin{aligned} w_{k+1} &= s(w_k), \quad w_k \in R^s \\ y_{r,k} &= y_r(w_k) \\ d_k &= d(w_k). \end{aligned} \quad (3)$$

While  $y_r(w_k)$  is known,  $d(w_k)$  has to be estimated. In this section we first derive a controller for system (1) assuming known the state  $x_k$  and the perturbation  $d(w_k)$ . Then, an observer for the unmeasured state variable and for the perturbation is designed.

In this work, we consider systems (1) that, under an appropriate nonsingular transformation, can be described by

$$\begin{aligned} x_{0,k+1} &= f_0(x_{0,k}, x_{1,k}, \dots, x_{q,k}, w_k) \\ x_{i,k+1} &= f_i(x_{0,k}, x_{1,k}, \dots, x_{i,k}, x_{i+1,k}) + d_i(w_k) \\ x_{q,k+1} &= f_q(x_{0,k}, x_{1,k}, \dots, x_{q,k}) \\ &\quad + B_q(x_{0,k}, x_{1,k}, \dots, x_{q,k})u_k + d_q(w_k) \end{aligned} \quad (4)$$

$$y_k = x_{1,k} \quad (5)$$

$$\eta_k = \gamma(x_k)$$

$i = 1, \dots, q - 1$ , where  $x_k = (x_{0,k}^T \ x_{1,k}^T \ \dots \ x_{q,k}^T)^T \in X \subset R^n$ ,  $x_{j,k} \in R^{n_j}$ ,  $j = 0, 1, \dots, q$ ,  $n = \sum_{j=0}^q n_j$ . Finally,  $x_{0,k}$  represents the state of the dynamics, possibly unstable, which enters in the remaining dynamics through bounded functions. This is the case of the induction motor, for instance, where the dynamics of the angular position  $\theta$  are unstable, with  $\theta$  appearing in the other dynamics through bounded trigonometric functions.

The following assumption is considered hereinafter.

(A.1.a). *The equations*

$$\begin{aligned} F_i(x_{0,k}, x_{1,k}, \dots, x_{i,k}, x_{i+1,k}, w_k, x_{i+1,k}) \\ = f_i(x_{0,k}, x_{1,k}, \dots, x_{i,k}, x_{i+1,k}) + d_i(w_k) - x_{r,i,k+1} \\ - K_i(x_{i,k} - x_{r,i,k}) = 0 \end{aligned}$$

$i = 1, \dots, q - 1$ , with  $K_i$  given matrices and  $x_{r,i,k} = y_{r,k}$ , admit unique solutions in  $x_{i+1,k}$  given by

$$x_{r,i+1,k} = \kappa_{i+1}(x_{0,k}, x_{1,k}, \dots, x_{i,k}, w_k)$$

and for all  $x \in X$  the matrix  $B_q$  is bounded in norm with bounded inverse  $B_q^{-1}$ , namely

$$\|B_q(x_{0,k}, x_{1,k}, \dots, x_{q,k})\| \leq \beta_q$$

and

$$\|B_q^{-1}(x_{0,k}, x_{1,k}, \dots, x_{q,k})\| \leq \beta_q^-$$

for certain constants  $\beta_q, \beta_q^- > 0$ .  $\triangleleft$

As a particular case, verified in certain applications, some of the maps  $f_i$ ,  $i = 0, \dots, q - 1$ , can take the form

$$\begin{aligned} f_i(x_{0,k}, x_{1,k}, \dots, x_{i,k}, x_{i+1,k}) &= \bar{f}_i(x_{0,k}, x_{1,k}, \dots, x_{i,k}) \\ &\quad + B_i(x_{0,k}, x_{1,k}, \dots, x_{i,k})x_{i+1,k} \end{aligned} \quad (6)$$

as in Loukianov (1998, 2002) or in the block backstepping (Khalil, 1996). In this case Assumption (A.1.a) can be substituted by the following.

(A.1.b). *For all  $x \in X$  the matrices  $B_i$ ,  $i = 1, \dots, q$ , are bounded in norm*

$$\|B_i(x_{0,k}, x_{1,k}, \dots, x_{i,k})\| \leq \beta_i$$

and there exist the inverses  $B_i^{-1}$  with

$$\|B_i^{-1}(x_{0,k}, x_{1,k}, \dots, x_{i,k})\| \leq \beta_i^-$$

for certain constants  $\beta_i, \beta_i^- > 0$ .  $\triangleleft$

**Remark 1.** When the maps  $f_i$  have the structure (6), pseudo-inverse matrices  $B_i^\ominus = B_i^T(B_i B_i^T)^{-1}$  could be used in assumption (A.1.b).  $\triangleleft$

A final assumption regards the control amplitude. As quite obvious in applications, we require the control to be capable of ensuring tracking of the desired reference and, at the same time, rejection of the disturbance acting on the system. In order to formalize this requirement, consider a given reference  $x_{r,q,k}$  and define

$$\begin{aligned}\Delta_{q,k}^\circ &= x_{q,k+1}^\circ - x_{q,k} \\ x_{q,k+1}^\circ &= f_q(x_{0,k}, x_{1,k}, \dots, x_{q,k}) + d_q(w_k) \\ \Delta_{r,k} &= x_{r,q,k} - x_{r,q,k+1}\end{aligned}$$

and suppose that

$$\|\Delta_{q,k}^\circ + \Delta_{r,k}\| \leq \delta_q, \quad \delta_q > 0. \quad (7)$$

Condition (7) means that, for the  $q$ th subsystem (4), the control is capable of imposing at step  $k+1$  that  $x_{q,k+1} = x_{r,q,k+1}$ , when  $x_{q,k} = x_{r,q,k}$ , namely when the  $q$ th reference signal is tracked at step  $k$ . In fact, the  $q$ th subsystem (4) can be rewritten as

$$\begin{aligned}x_{q,k+1} - x_{r,q,k+1} &= B_q(x_{0,k}, x_{1,k}, \dots, x_{q,k}) \\ &\quad \times [B_q^{-1}(x_{0,k}, x_{1,k}, \dots, x_{q,k}) (\Delta_{r,k} + \Delta_{q,k}^\circ) + u_k]\end{aligned}$$

with  $\|B_q^{-1}(\cdot)\| \leq \beta_q^-$ . Hence, in the sequel we assume that the following holds.

(A.2). The maximal value  $u_{\max}$  of the control is such that

$$u_{\max} > \delta_q \beta_q^-. \quad \triangleleft$$

## 2.2. The backstepping procedure

The control design procedure consists of a step-by-step construction of a new system with states  $z_{i,k} = x_{i,k} - x_{r,i,k}$ ,  $i = 1, \dots, q$ , where  $x_{r,i,k}$  is the desired value for  $x_{i,k}$ , which will be defined by such a construction.

We start by defining as new variable the tracking error (2)

$$z_{1,k} = e_k = x_{1,k} - x_{r,1,k}$$

with  $x_{r,1,k} = y_r(w_k)$  the reference value for  $x_{1,k}$ , having dynamics

$$z_{1,k+1} = f_1(x_{0,k}, z_{1,k} + x_{r,1,k}, x_{2,k}) + \bar{d}_1(w_k) \quad (8)$$

where

$$\bar{d}_1(w_k) = d_1(w_k) - x_{r,1,k+1}(w_k) = d_1(w_k) - y_r(s(w_k)).$$

In Eq. (8),  $x_{2,k}$  is viewed as a control input used to impose the following desired dynamics

$$z_{1,k+1} = K_1 z_{1,k}. \quad (9)$$

The design matrix  $K_1$  is Schur, namely it has the eigenvalues inside the unit circle, to ensure the asymptotic stability of (9). Therefore, on the basis of Assumption (A.1.a), one determines the solution in  $x_{2,k}$  for the equation

$$\begin{aligned}F_1(z_{1,k}, x_{2,k}, w_k) &= f_1(x_{0,k}, z_{1,k} + x_{r,1,k}, x_{2,k}) \\ &\quad - K_1 z_{1,k} + \bar{d}_1(w_k) = 0.\end{aligned} \quad (10)$$

This solution is given by  $x_{r,2,k} = \kappa_2(x_{0,k}, z_{1,k}, w_k)$  which represents the reference behavior for  $x_{2,k}$ . Proceeding in the same way, one introduces  $z_{2,k} = x_{2,k} - x_{r,2,k}$ , having dynamics

$$z_{2,k+1} = f_2(x_{0,k}, z_{1,k} + x_{r,1,k}, z_{2,k} + x_{r,2,k}, x_{3,k}) + \bar{d}_2(w_k)$$

where  $\bar{d}_2(w_k) = d_2(w_k) - x_{r,2,k+1}$ . One imposes the desired dynamics

$$z_{2,k+1} = K_2 z_{2,k} \quad (11)$$

where  $K_2$  is Schur. By assumption (A.1.a), the equation

$$\begin{aligned}F_2(z_{1,k}, z_{2,k}, x_{3,k}, w_k) &= f_2(x_{0,k}, z_{1,k} + x_{r,1,k}, z_{2,k} + x_{r,2,k}, x_{3,k}) \\ &\quad - K_2 z_{2,k} + \bar{d}_2(w_k) = 0\end{aligned}$$

has solution in  $x_{3,k}$  given by  $x_{r,3,k} = \kappa_3(x_{0,k}, z_{1,k}, z_{2,k}, w_k)$  which is the reference value for  $x_{3,k}$ . Iterating these steps, one finally introduces the variable  $z_{q,k} = x_{q,k} - x_{r,q,k}$ ,  $x_{r,q,k} = \kappa_q(x_{0,k}, z_{1,k}, \dots, z_{q-1,k}, w_k)$ , with dynamics

$$\begin{aligned}z_{q,k+1} &= f_q(x_{0,k}, z_{1,k} + x_{r,1,k}, \dots, z_{q,k} + x_{r,q,k}) \\ &\quad + B_q(x_{0,k}, z_{1,k} + x_{r,1,k}, \dots, z_{q,k} + x_{r,q,k}) u_k + \bar{d}_q(w_k)\end{aligned}$$

where  $\bar{d}_q(w_k) = d_q(w_k) - x_{r,q,k+1}$ . It is worth mentioning that the new variables  $z_{i,k}$ ,  $i = 0, 1, \dots, q$ , determine a nonlinear transformation

$$\begin{aligned}z_{0,k} &= x_{0,k} = \varphi_0(x_{0,k}) \\ z_{1,k} &= x_{1,k} - x_{r,1,k} = \varphi_1(x_{0,k}, x_{1,k}, w_k) \\ z_{2,k} &= x_{2,k} - \kappa_2(x_{0,k}, x_{1,k} - x_{r,1,k}, w_k) \\ &= \varphi_2(x_{0,k}, x_{1,k}, x_{2,k}, w_k) \\ z_{3,k} &= x_{3,k} - \kappa_3(x_{0,k}, x_{1,k} - x_{r,1,k}, x_{2,k} - x_{r,2,k}, w_k) \\ &= \varphi_3(x_{0,k}, x_{1,k}, x_{2,k}, x_{3,k}, w_k)\end{aligned} \quad (12)$$

$\vdots$

$$\begin{aligned}z_{q,k} &= x_{q,k} - \kappa_q(x_{0,k}, \dots, x_{q-1,k} - x_{r,q-1,k}, w_k) \\ &= \varphi_q(x_{0,k}, x_{1,k}, x_{2,k}, \dots, x_{q,k}, w_k).\end{aligned}$$

It is easy to check that, by means of this transformation  $z_k = \varphi(x_k, w_k) = (\varphi_0^T \quad \varphi_1^T \quad \dots \quad \varphi_q^T)^T$ , system (4) is diffeomorphic to

$$\begin{aligned}z_{0,k+1} &= f_0(z_{0,k}, z_{1,k} + x_{r,1,k}, \dots, z_{q,k} + x_{r,q,k}, w_k) \\ z_{1,k+1} &= K_1 z_{1,k} + \Delta f_1 \\ z_{2,k+1} &= K_2 z_{2,k} + \Delta f_2 \\ &\vdots\end{aligned} \quad (13)$$

$$\begin{aligned}z_{q-1,k+1} &= K_{q-1} z_{q-1,k} + \Delta f_{q-1} \\ z_{q,k+1} &= f_q(z_{0,k}, z_{1,k} + x_{r,1,k}, \dots, z_{q,k} + x_{r,q,k}) \\ &\quad + B_q(z_{0,k}, z_{1,k} + x_{r,1,k}, \dots, z_{q,k} + x_{r,q,k}) u_k + \bar{d}_q(w_k)\end{aligned}$$

with

$$\begin{aligned}\Delta f_1 &= f_1(z_{0,k}, z_{1,k} + x_{r,1,k}, z_{2,k} + x_{r,2,k}) \\ &\quad - f_1(z_{0,k}, z_{1,k} + x_{r,1,k}, x_{r,2,k}) \\ \Delta f_2 &= f_2(z_{0,k}, z_{1,k} + x_{r,1,k}, z_{2,k} + x_{r,2,k}, z_{3,k} + x_{r,3,k}) \\ &\quad - f_2(z_{0,k}, z_{1,k} + x_{r,1,k}, z_{2,k} + x_{r,2,k}, x_{r,3,k})\end{aligned}$$

$\vdots$

$$\begin{aligned}\Delta f_{q-1} &= f_{q-1}(z_{0,k}, z_{1,k} + x_{r,1,k}, \dots, z_{q,k} + x_{r,q,k}) \\ &\quad - f_{q-1}(z_{0,k}, z_{1,k} + x_{r,1,k}, \dots, x_{r,q,k}).\end{aligned}$$

It is worth noting that when  $f_i$  are in the form (6), under (A.1.b) one gets

$$\begin{aligned}x_{r,2,k} &= (B_1(z_{0,k}, z_{1,k} + x_{r,1,k}))^{-1} (K_1 z_{1,k} \\ &\quad - f_1(z_{0,k}, z_{1,k} + x_{r,1,k}) - \bar{d}_1(w_k)) \\ &\vdots \\ x_{r,q,k} &= (B_{q-1}(\cdot))^{-1} (K_{q-1} z_{q-1,k} - f_{q-1}(\cdot) - \bar{d}_{q-1}(w_k))\end{aligned}$$

and system (4) is diffeomorphic to

$$z_{0,k+1} = f_0(z_{0,k}, z_{1,k} + x_{r,1,k}, \dots, z_{q,k} + x_{r,q,k}, w_k)$$

$$z_{1,k+1} = K_1 z_{1,k} + B_1 z_{2,k}$$

$$z_{2,k+1} = K_2 z_{2,k} + B_2 z_{3,k}$$

⋮

$$z_{q-1,k+1} = K_{q-1} z_{q-1,k} + B_{q-1} z_{q,k}$$

$$z_{q,k+1} = f_q(z_{0,k}, z_{1,k} + x_{r,1,k}, \dots, z_{q,k} + x_{r,q,k}) + B_q(z_{0,k}, z_{1,k} + x_{r,1,k}, \dots, z_{q,k} + x_{r,q,k})u_k + \bar{d}_q(w_k).$$

When not specified, in the following it will be clear from the context if hypothesis (A.1.a) or (A.1.b) applies.

### 2.3. Digital sliding mode control

In this section, we will design a sliding mode controller for system (13), such that the reference signal  $y_r(w_k)$  is tracked and the disturbance  $d(w_k)$  is rejected. This will be achieved in presence of constraints on the input

$$\|u_k\| \leq u_{\max}. \quad (14)$$

The maximal value  $u_{\max}$  is assumed to fulfill Assumption (A.2).

As usual in the sliding mode technique (Utkin, 1993), the control forces the system evolution on a certain surface which guarantees the achievement of the control requirements. A natural choice is the sliding surface  $S_k = z_{q,k} = 0$ . With this choice, the last of (13) can be also rewritten as follows

$$S_{k+1} = f_q(z_{0,k}, z_{1,k} + x_{r,1,k}, \dots, S_q + x_{r,q,k}) + B_q(z_{0,k}, z_{1,k} + x_{r,1,k}, \dots, S_q + x_{r,q,k})u_k + \bar{d}_q(w_k). \quad (15)$$

The next step is to find a control law which fulfills the bound (14) and forces the system evolution on  $S_k = 0$ . Such a control is

$$u_k = \begin{cases} u_{k,eq} & \text{for } \|u_{k,eq}\| \leq u_{\max} \\ \frac{u_{k,eq}}{\|u_{k,eq}\|} u_{\max} & \text{for } \|u_{k,eq}\| > u_{\max} \end{cases} \quad (16)$$

where the equivalent control  $u_{k,eq}$  is calculated imposing  $S_k \equiv 0$  in (15)

$$0 = f_q(z_{0,k}, z_{1,k} + x_{r,1,k}, \dots, x_{r,q,k}) + B_q(z_{0,k}, z_{1,k} + x_{r,1,k}, \dots, x_{r,q,k})u_k + \bar{d}_q(w_k).$$

Using (A.1.b)

$$u_{k,eq} = -\bar{B}_q^{-1} [\bar{f}_q + \bar{d}_q(w_k)] \quad (17)$$

$$\bar{f}_q = f_q(z_{0,k}, z_{1,k} + x_{r,1,k}, \dots, z_{q-1,k} + x_{r,q-1,k}, x_{r,q,k})$$

$$\bar{B}_q = B_q(z_{0,k}, z_{1,k} + x_{r,1,k}, \dots, z_{q-1,k} + x_{r,q-1,k}, x_{r,q,k})$$

or, in the original coordinates

$$u_{k,eq} = -[B_q(x_{0,k}, x_{1,k}, \dots, x_{q-1,k}, x_{r,q,k})]^{-1} \times [f_q(x_{0,k}, x_{1,k}, \dots, x_{q-1,k}, x_{r,q,k}) + \bar{d}_q(w_k)].$$

A simple analysis will prove that the dynamics of the closed-loop system over the surface  $S_k \equiv 0$ , i.e. the sliding mode dynamics, are stable. For, let us rewrite (15) and (17) as

$$S_{k+1} = S_k + \tilde{f}_s + \tilde{B}_q u_k \quad (18)$$

$$u_{k,eq} = -\tilde{B}_q^{-1} (S_k + \tilde{f}_s) \quad (19)$$

where

$$\tilde{f}_s = -z_{q,k} + \bar{f}_q + \bar{d}_q(w_k) = (\Delta_{q,k}^\circ + \Delta_{r,k})|_{x_k = \varphi^{-1}(z_k, w_k)}$$

$$\tilde{B}_q = B_q(z_{0,k}, \dots, z_{q-1,k} + x_{r,q-1,k}, S_k + x_{r,q,k})$$

and

$$\tilde{f}_q = f_q(z_{0,k}, \dots, z_{q-1,k} + x_{r,q-1,k}, S_k + x_{r,q,k}).$$

It follows that

$$\|u_{k,eq}\| \leq \|\tilde{B}_q^{-1}\| \|S_k + \tilde{f}_s\|. \quad (20)$$

Let us now consider the two cases given in (16). When  $\|u_{k,eq}\| \leq u_{\max}$ , the equivalent control  $u_{k,eq}$  is applied, bringing the system trajectory on the sliding manifold  $S_k = 0$  in one step. When  $\|u_{k,eq}\| > u_{\max}$ ,

$$S_{k+1} = S_k + \tilde{f}_s + \tilde{B}_q u_{\max} \frac{u_{k,eq}}{\|u_{k,eq}\|} = (S_k + \tilde{f}_s) \left( 1 - \frac{u_{\max}}{\|u_{k,eq}\|} \right). \quad (21)$$

Along any solution of (21), the increment of the Lyapunov function  $V_k = \|S_k\|$  is given by

$$\begin{aligned} \Delta V &= \|S_{k+1}\| - \|S_k\| \\ &= \|S_k + \tilde{f}_s\| \left( 1 - \frac{u_{\max}}{\|u_{k,eq}\|} \right) - \|S_k\| \\ &\leq \left( \|S_k + \tilde{f}_s\| - \frac{u_{\max}}{\|\tilde{B}_q^{-1}\|} \right) - \|S_k\|. \end{aligned}$$

It is now sufficient to note that, from Assumptions (A.1.a) and (A.2)

$$\|S_k + \tilde{f}_s\| - \frac{u_{\max}}{\|\tilde{B}_q^{-1}\|} \leq \|S_k\| + \delta_q - \frac{u_{\max}}{\|\tilde{B}_q^{-1}\|} < \|S_k\|$$

to deduce that  $\|S_k\|$  decreases monotonically. Therefore, also  $\|u_{k,eq}\|$  decreases monotonically, since from (20)

$$\|u_{k,eq}\| \leq \beta_q^- (\|S_k\| + \delta_q)$$

and there will be a certain time instant  $\bar{k}$  such that  $\|u_{k,eq}\| \leq u_{\max}$ , for  $k \geq \bar{k}$ . At this time the the equivalent control  $u_{k,eq}$  is applied, bringing the system trajectory on the sliding manifold  $S_k = 0$  at time  $\bar{k} + 1$ .

The motion on the sliding manifold  $S_k = z_{q,k} = 0$ , i.e. the sliding mode dynamics, is described by a reduced  $(n - n_q)$ th order system

$$\begin{aligned} z_{0,k+1} &= f_0(z_{0,k}, z_{1,k} + x_{r,1,k}, \dots, x_{r,q,k}, w_k) \\ z_{1,k+1} &= K_1 z_{1,k} + \Delta f_1 \\ z_{2,k+1} &= K_2 z_{2,k} + \Delta f_2 \\ &\vdots \\ z_{q-2,k+1} &= K_{q-2} z_{q-2,k} + \Delta f_{q-2} \\ z_{q-1,k+1} &= K_{q-1} z_{q-1,k} \end{aligned} \quad (22)$$

since  $\Delta f_{q-1} = 0$ . Moreover, since the matrices  $K_i$ ,  $i = 1, \dots, q - 1$  are Schur,  $z_{q-1,k}$  asymptotically decays to zero, as well as  $\Delta f_{q-2}$ . Iteratively, one checks that also  $z_{q-2,k}, \dots, z_{1,k}$  and  $\Delta f_{q-3}, \dots, \Delta f_1$  asymptotically decay to zero, fulfilling the control objective. Hence, the residual dynamics on the sliding manifold are

$$z_{0,k+1} = f_0(z_{0,k}, x_{r,1,k}, \dots, x_{r,q,k}, w_k).$$

#### 2.4. A discrete time observer and stabilization via dynamic feedback

The measurability of the whole state and the knowledge of the disturbance are strong hypotheses often not verified, as for example, in the case of induction motors. This problem may be overcome by using observers and estimators. In what follows we consider system (1) which, after a coordinate transformation  $(\eta_k^T \ \zeta_k^T)^T = T(x_k)$ , can be put in the following form

$$\begin{aligned} \eta_{k+1} &= \phi_1(\eta_k, \zeta_k, d_\eta, u_k) \\ \zeta_{k+1} &= \phi_2(\eta_k, \zeta_k, u_k) \end{aligned} \quad (23)$$

where  $\zeta_k$  is the vector of the non-measurable state variables and  $d_\eta(w_k)$  is the disturbance, depending on  $w_k$  and having dynamics

$$d_{\eta,k+1} = Sd_{\eta,k}$$

supposed linear. In the case of the induction motor it will be shown that it is possible to exploit the particular structure of the equations in order to obtain an observer which ensures exponential convergence. For, let us consider the following particular structure for Eq. (23)

$$\begin{aligned} \eta_{k+1} &= A_{11}\eta_k + A_{13}d_{\eta,k} + A_{14}u_k + f_\eta(\eta_k)\zeta_k \\ \zeta_{k+1} &= f_\zeta(\eta_k, \zeta_k) \end{aligned}$$

with  $f_\eta(\eta_k)$  a globally bounded function, as in the case of the induction motor. The proposed observer is the following

$$\begin{aligned} \hat{\eta}_{k+1} &= A_{11}\eta_k + A_{13}\hat{d}_{\eta,k} + A_{14}u_k + f_\eta(\eta_k)\hat{\zeta}_k + G_1(\eta_k - \hat{\eta}_k) \\ \hat{\zeta}_{k+1} &= f_\zeta(\eta_k, \hat{\zeta}_k) \\ \hat{d}_{\eta,k+1} &= S\hat{d}_{\eta,k} + G_2(\eta_k - \hat{\eta}_k). \end{aligned} \quad (24)$$

We consider the following assumptions.

(A.3). The pair  $\begin{pmatrix} 0 & A_{13} \\ 0 & S \end{pmatrix}$ ,  $(\mathbb{I}_d \ 0)$  is detectable, where  $\mathbb{I}_d$  is the identity matrix.  $\triangleleft$

(A.4). The dynamics of  $\zeta_k - \hat{\zeta}_k$  are exponentially stable.  $\triangleleft$   
Introducing the errors

$$e_{1,k} = \eta_k - \hat{\eta}_k, \quad e_{2,k} = \zeta_k - \hat{\zeta}_k, \quad e_{3,k} = d_k - \hat{d}_k$$

under (A.3) and (A.4) it is easy to show that the error dynamics

$$\begin{pmatrix} e_{1,k+1} \\ e_{3,k+1} \end{pmatrix} = \begin{pmatrix} -G_1 & A_{13} \\ -G_2 & S \end{pmatrix} \begin{pmatrix} e_{1,k} \\ e_{3,k} \end{pmatrix} + \begin{pmatrix} f_\eta(\eta_k) \\ 0 \end{pmatrix} e_{2,k}$$

$$e_{2,k+1} = f_\zeta(\eta_k, \zeta_k) - f_\zeta(\eta_k, \hat{\zeta}_k)$$

are globally exponentially stable for an appropriate choice of the gain matrix  $G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$ . In fact, by (A.4),  $e_{2,k}$  goes to zero exponentially. Moreover, the dynamics of  $e_{1,k}$ ,  $e_{3,k}$  are globally input-to-state stable (Khalil, 1996), with exponential rate thanks to (A.3).

In the remainder of this section we will show that the feedback

$$\hat{u}_k = \begin{cases} \hat{u}_{k,eq} & \text{for } \|\hat{u}_{k,eq}\| \leq u_{\max} \\ \frac{\hat{u}_{k,eq}}{\|\hat{u}_{k,eq}\|} u_{\max} & \text{for } \|\hat{u}_{k,eq}\| > u_{\max} \end{cases} \quad (25)$$

$$\hat{u}_{k,eq} = -B_q^{-1}(\hat{x}_k) [f_q(\hat{x}_k) + \bar{d}_q(\hat{w}_k)]$$

with  $\hat{x}_k = T^{-1}(\eta_k, \hat{\zeta}_k)$  given from (24), asymptotically stabilizes system (1). This result can be easily proved making use the following theorem.

**Theorem 1** (Lin and Byrnes (1994), Th. 4.3 – Separation Principle). *The asymptotic stabilization problem of system (1) is solvable via estimated state feedback (24) and (25) if, and only if, system (1) is asymptotically stabilizable and exponentially detectable.  $\triangleleft$*

We summarize our conclusions in the next result.

**Theorem 2.** *If the nonlinear system (4) can be put in the form (23), with measured output vector  $\eta_k = \gamma(x_k)$ , under conditions (A.1)–(A.4) the control law (24) and (25) asymptotically stabilizes the system (1).  $\triangleleft$*

### 3. Digital control of induction motors

In this section, a discrete time sliding mode controller is designed on the basis of the sampled dynamics of an induction motor, given in Appendix. The variables to be controlled are the rotor velocity and the rotor flux squared modulus, while the disturbance to be rejected is the load torque. The measured variables are the rotor speed and stator currents. For the rotor fluxes and for the load torque an observer is designed.

#### 3.1. Control design

The sampled dynamics for the induction motor in the stator fixed reference frame  $(\alpha, \beta)$  can be approximated as follows (see Appendix)

$$\theta_{k+1} = \theta_k + \delta\omega_k + a_1 (i_{\beta,k}\phi_{\alpha,k} - i_{\alpha,k}\phi_{\beta,k}) - \frac{\delta^2}{2J} C_{L,k}$$

$$\omega_{k+1} = \omega_k + a_2 (i_{\beta,k}\phi_{\alpha,k} - i_{\alpha,k}\phi_{\beta,k}) - \delta C_{L,k}/J$$

$$\phi_{\alpha,k+1} = \chi_{1,k} = \rho_{1,k} \cos \Delta\theta_k - \rho_{2,k} \sin \Delta\theta_k$$

$$\phi_{\beta,k+1} = \chi_{2,k} = \rho_{1,k} \sin \Delta\theta_k + \rho_{2,k} \cos \Delta\theta_k$$

$$i_{\alpha,k+1} = i_{\alpha,k} + \delta (\alpha\beta\phi_{\alpha,k} + p\beta\omega_k\phi_{\beta,k} - \gamma i_{\alpha,k}) + \delta u_{\alpha,k}/\sigma$$

$$i_{\beta,k+1} = i_{\beta,k} + \delta (\alpha\beta\phi_{\beta,k} - p\beta\omega_k\phi_{\alpha,k} - \gamma i_{\beta,k}) + \delta u_{\beta,k}/\sigma$$

where

$$\chi_k(\omega_k, \phi_k, I_k, w_k) = \begin{pmatrix} \chi_{1,k} \\ \chi_{2,k} \end{pmatrix} = e^{\Delta\theta_k \mathfrak{S}} \rho_k$$

$$\rho_k(\phi_k, I_k) = \begin{pmatrix} \rho_{1,k} \\ \rho_{2,k} \end{pmatrix} = a_0\phi_k + a_3 I_k$$

$$\Delta\theta_k(\omega_k, \phi_k, I_k, w_k) = p(\theta_{k+1} - \theta_k)$$

$$= p\delta\omega_k + pa_1 (i_{\beta,k}\phi_{\alpha,k} - i_{\alpha,k}\phi_{\beta,k}) - \frac{p\delta^2}{2J} C_{L,k}$$

$$e^{\Delta\theta_k \mathfrak{S}} = \begin{pmatrix} \cos \Delta\theta_k & -\sin \Delta\theta_k \\ \sin \Delta\theta_k & \cos \Delta\theta_k \end{pmatrix}, \quad \mathfrak{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and  $a_0, \dots, a_3, J, \alpha, \beta, \gamma, \sigma$  are constants,  $\delta$  is the sampling period,  $C_{L,k}$  is the load torque,  $\phi_k = (\phi_{\alpha,k} \ \phi_{\beta,k})^T$ ,  $I_k = (i_{\alpha,k} \ i_{\beta,k})^T$ ,  $u_k = (u_{\alpha,k} \ u_{\beta,k})^T$  are the rotor flux, stator current, voltage vectors. Note that  $e^{\Delta\theta_k \mathfrak{S}}$  is a counterclockwise rotation matrix.

This system is in the form (1), with the vectors of the variables to be controlled and measurable given by

$$y_k = (\omega_k \ \Phi_k)^T, \quad \eta_k = (\theta_k \ \omega_k \ i_{\alpha,k} \ i_{\beta,k})^T.$$

Here,  $\Phi_k = \|\phi_k\|^2 = \phi_{\alpha,k}^2 + \phi_{\beta,k}^2$  is the square rotor flux magnitude. It is worth noting that

$$\Phi_{k+1} = \phi_{k+1}^T \phi_{k+1} = (a_0\phi_k + a_3 I_k)^T (a_0\phi_k + a_3 I_k).$$

Moreover,  $y_{r,k} = (\omega_{r,k} \ \Phi_{r,k})^T$  is the reference signal vector to be tracked, where  $\omega_{r,k}$  and  $\Phi_{r,k}$  are appropriate bounded signals with bounded increments, generated by the external system (3) along with the disturbance  $C_{L,k} = C_{L,k}(w_k)$  to be rejected, supposed bounded.

It is well known that the control objective can be met ensuring an appropriate electromechanical torque, proportional to the term

$$I_k^T \mathfrak{S} \phi_k = i_{\beta,k} \phi_{\alpha,k} - i_{\alpha,k} \phi_{\beta,k}$$

and an appropriate reactive power

$$I_k^T \phi_k = i_{\alpha,k} \phi_{\alpha,k} + i_{\beta,k} \phi_{\beta,k}.$$

In fact, while the angular velocity requirement determines a certain acceleration to be imposed to the mechanical system, and hence a certain electromechanical torque  $C_{m,k} = \mu I_k^T \mathfrak{S} \phi_k$  ( $\mu$  is a constant, see Appendix), the flux requirement imposes a flux vector with a certain module, which can be obtained thanks to a certain reactive power  $P_k = I_k^T \phi_k$ . This can be done imposing

$$\begin{pmatrix} \phi_k^T \\ -\phi_k^T \mathfrak{S} \end{pmatrix} I_k = \begin{pmatrix} \psi_{1,k} \\ \psi_{2,k} \end{pmatrix} = \Psi_k$$

with  $\psi_{1,k} = P_k$ ,  $\psi_{2,k} = C_{m,k}/\mu$  appropriate functions, which gives

$$I_k = \frac{1}{\Phi_k} (\phi_k \quad \mathfrak{S} \phi_k) \Psi_k = e^{\theta_{\phi,k} \mathfrak{S}} \Psi_k \quad (26)$$

for  $\Phi_k \neq 0$ . Note that  $I_k$  results to be  $\Psi_k$  rotated of  $\theta_{\phi,k}$  counterclockwise. Eq. (26) gives the expression of the reference current  $I_{r,k}$ . This is the classical field oriented control (FOC, see for instance Fekih and Chowdhury (2004)). In this way the current vector imposes a reference module and a reference angle to the flux vector. From this discussion it is clear that it is convenient to consider the following change of coordinates

$$\begin{aligned} x_{0,k} &= \begin{pmatrix} x_{01,k} \\ x_{02,k} \end{pmatrix} = \begin{pmatrix} \theta_{\phi,k} \\ \theta_k \end{pmatrix}, \\ x_{1,k} &= \begin{pmatrix} x_{11,k} \\ x_{12,k} \end{pmatrix} = y_k, \quad x_{2,k} = \begin{pmatrix} x_{21,k} \\ x_{22,k} \end{pmatrix} = I_k \end{aligned} \quad (27)$$

where  $\theta_{\phi,k} = \arctan(\phi_{\beta,k}/\phi_{\alpha,k})$  is the flux vector angle, and  $x_k = (x_{0,k}^T \quad x_{1,k}^T \quad x_{2,k}^T)^T$ . Clearly, the inverse transformation is

$$\begin{aligned} \theta_k &= x_{02,k} & \phi_{\alpha,k} &= x_{12,k} \cos x_{01,k} & i_{\alpha,k} &= x_{21,k} \\ \omega_k &= x_{11,k}, & \phi_{\beta,k} &= x_{12,k} \sin x_{01,k}, & i_{\beta,k} &= x_{22,k}. \end{aligned}$$

In the new coordinates the induction motor sampled dynamics are

$$\begin{aligned} x_{0,k+1} &= f_0(x_{0,k}, x_{1,k}, x_{2,k}, w_k) \\ x_{1,k+1} &= f_1(x_{0,k}, x_{1,k}, x_{2,k}) + d_1(w_k) \\ x_{2,k+1} &= f_2(x_{0,k}, x_{1,k}, x_{2,k}) + B_2 u_k \end{aligned} \quad (28)$$

i.e. are in the form (4) with  $q = 2$ , and

$$f_0(x_k, w_k) = \begin{pmatrix} \arctan \frac{\chi_{2,k}(\omega_k, \phi_k, I_k, w_k)}{\chi_{1,k}(\omega_k, \phi_k, I_k, w_k)} \\ \theta_k + \delta \omega_k + a_1 I_k^T \mathfrak{S} \phi_k - \frac{\delta^2}{J} C_{L,k} \end{pmatrix}$$

$$f_1(x_k) = \begin{pmatrix} \omega_k + a_2 I_k^T \mathfrak{S} \phi_k \\ a_0^2 \Phi_k + 2a_0 a_3 \phi_k^T I_k + a_3^2 I_k^T I_k \end{pmatrix}$$

$$f_2(x_k) = (1 - \delta \gamma) I_k + \delta \beta (\alpha \mathbb{I}_d - p \omega_k \mathfrak{S}) \phi_k$$

$$B_2 = \frac{\delta}{\sigma} \mathbb{I}_d, \quad d_1(w_k) = \begin{pmatrix} -\frac{\delta}{J} C_{L,k} \\ 0 \end{pmatrix}.$$

Notice that the first equation refers to unstable internal dynamics, since they describe the rotation of the rotor flux vector and the rotor position. Hence, their instability is not a physical problem. Moreover, from a mathematical point of view, these variables appear in the dynamics of  $x_{1,k}, x_{2,k}$  through bounded functions.

In what follows we derive a sliding mode controller, using the control procedure of Section 2. To this aim, one considers first the nonlinear transformation (12)

$$z_{0,k} = x_{0,k}, \quad z_{1,k} = x_{1,k} - y_{r,k}, \quad z_{2,k} = x_{2,k} - I_{r,k}$$

with  $I_{r,k}$  given by (26). In order to determine explicitly  $I_{r,k}$ , consider that, from (26)

$$I_k^T \phi_k = \psi_{1,k}, \quad I_k^T \mathfrak{S} \phi_k = \psi_{2,k}, \quad I_k^T I_k = \frac{1}{\Phi_k} (\psi_{1,k}^2 + \psi_{2,k}^2)$$

$\Phi_k \neq 0$ . Hence, setting

$$\begin{aligned} z_{1,k+1} &= x_{1,k+1} - y_{r,k+1} \\ &= \begin{pmatrix} \omega_k + a_2 \psi_{2,k} - \delta C_{L,k}/J - \omega_{r,k+1} \\ a_0^2 \Phi_k + 2a_0 a_3 \psi_{1,k} + \frac{a_3^2}{\Phi_k} (\psi_{1,k}^2 + \psi_{2,k}^2) - \Phi_{r,k+1} \end{pmatrix} \\ &= K_1 z_{1,k} \end{aligned}$$

with  $K_1 = \text{diag}\{k_{11}, k_{12}\}$  Schur, one works out the solutions

$$\psi_{r,1,k} = -\frac{a_0}{a_3} \Phi_k \pm \frac{1}{a_3} \sqrt{\Delta_k}$$

$$\psi_{r,2,k} = \frac{1}{a_2} \left( \omega_{r,k+1} - \omega_k + \frac{\delta}{J} C_{L,k} + k_{11}(\omega_k - \omega_{r,k}) \right)$$

$$\Delta_k = (\Phi_{r,k+1} + k_{12}(\Phi_k - \Phi_{r,k})) \Phi_k - a_3^2 \psi_{r,2,k}^2.$$

With this choice for  $\psi_{r,1,k}, \psi_{r,2,k}$ , the explicit expression of  $I_{r,k} = \kappa_2$  remains determined when  $\Phi_k \neq 0$

$$I_{r,k} = \kappa_2 = \frac{1}{\Phi_k} (\phi_k \quad \mathfrak{S} \phi_k) \begin{pmatrix} \psi_{r,1,k} \\ \psi_{r,2,k} \end{pmatrix} = e^{\theta_{\phi,k} \mathfrak{S}} \Psi_{r,k} \quad (29)$$

with  $\Psi_{r,k} = (\psi_{r,1,k} \quad \psi_{r,2,k})^T$ , i.e.  $I_{r,k}$  is  $\Psi_{r,k}$  rotated of  $\theta_{r,k}$  counterclockwise. Moreover, it is easy to check by induction that, starting from finite values of  $\omega_k, \Phi_k$  at  $k = 0$ ,  $\Psi_{r,k}$  remains bounded. As a consequence, also  $I_{r,k}$  remains bounded. Furthermore, imposing

$$\begin{aligned} z_{2,k+1} &= x_{2,k+1} - I_{r,k+1} \\ &= (1 - \delta \gamma) I_k + \delta \beta (\alpha \mathbb{I}_d - p \omega_k \mathfrak{S}) \phi_k + \frac{\delta}{\sigma} u_k - I_{r,k+1} \\ &= K_2 z_{2,k} \end{aligned}$$

with  $K_2 = \text{diag}\{k_{21}, k_{22}\}$  Schur, one gets

$$\begin{aligned} u_{k,eq} &= \frac{\sigma}{\delta} \left( -(1 - \delta \gamma) I_k - \delta \beta (\alpha \mathbb{I}_d - p \omega_k \mathfrak{S}) \phi_k \right. \\ &\quad \left. + I_{r,k+1} + K_2 (I_k - I_{r,k}) \right). \end{aligned}$$

The fulfillment of the control objective derives from Section 2. In fact, the system is diffeomorphic to system (13), with  $q = 2$

$$z_{0,k+1} = f_0(z_{0,k}, z_{1,k} + x_{r,1,k}, z_{2,k} + x_{r,2,k}, w_k)$$

$$z_{1,k+1} = K_1 z_{1,k} + \Delta f_1$$

$$z_{2,k+1} = f_2(z_{0,k}, z_{1,k} + x_{r,1,k}, z_{2,k} + x_{r,2,k}) + B_2 u_k$$

and

$$\begin{aligned} \Delta f_1 &= f_1(z_{0,k}, z_{1,k} + x_{r,1,k}, z_{2,k} + x_{r,2,k}) \\ &\quad - f_1(z_{0,k}, z_{1,k} + x_{r,1,k}, z_{2,k}). \end{aligned}$$

Hence, the bounded control (16) steers  $S_k = z_{2,k} = I_k - I_{r,k}$  to zero after a finite time interval. Hence,  $\Delta f_1$  tends to zero as well. The sliding mode on  $S_k = 0$  are given by

$$z_{0,k+1} = f_0(z_{0,k}, z_{1,k} + x_{r,1,k}, x_{r,2,k}, w_k)$$

$$z_{1,k+1} = K_1 z_{1,k}.$$

Since  $z_{1,k+1} = K_1 z_{1,k}$  are asymptotically stable, the residual dynamics are  $z_{0,k+1} = f_0(z_{0,k}, x_{r,1,k}, x_{r,2,k}, w_k)$ .

### 3.2. Reduced order nonlinear observer

Under the assumption (H.1) for  $C_L$  (see Appendix), the following reduced order nonlinear observer can be used for rotor flux and load torque estimation

$$\hat{\omega}_{k+1} = \omega_k + a_2 I_k^T \hat{\phi}_k - \frac{\delta}{J} \hat{C}_{L,k} + \lambda_1 (\omega_k - \hat{\omega}_k)$$

$$\hat{C}_{L,k+1} = \hat{C}_{L,k} + \lambda_2 (\omega_k - \hat{\omega}_k)$$

$$\hat{\phi}_{k+1} = e^{\Delta\theta_k} \left( a_0 \hat{\phi}_k + a_3 I_k \right).$$

This reduced observer can be obtained from (24) considering only the dynamics of the component  $\omega_k$  of  $\eta_k$ . Obviously, here  $d_{\eta,k} = C_{L,k}$  and  $\zeta_k = \phi_k$ . The observation errors  $e_{1,k} = \omega_k - \hat{\omega}_k$ ,  $e_{2,k} = \phi_k - \hat{\phi}_k$ ,  $e_{3,k} = C_{L,k} - \hat{C}_{L,k}$  have the following dynamics

$$\begin{pmatrix} e_{1,k+1} \\ e_{3,k+1} \end{pmatrix} = \begin{pmatrix} -\lambda_1 & -\frac{\delta}{J} \\ -\lambda_2 & 1 \end{pmatrix} \begin{pmatrix} e_{1,k} \\ e_{3,k} \end{pmatrix} + \begin{pmatrix} a_2 I_k^T \hat{\phi}_k \\ 0 e_{2,k} \end{pmatrix} \quad (30)$$

$$e_{2,k+1} = a_0 e^{\Delta\theta_k} e_{2,k}.$$

A Lyapunov stability analysis will prove the exponential stability of the origin. For, let us consider the Lyapunov function  $V_k = e_{2,k}^T e_{2,k}$ . Since  $e^{\Delta\theta_k}$  is orthogonal, one checks that

$$V_{k+1} = -(1 - a_0^2) \|e_{2,k}\|^2 = -(1 - a_0^2) V_k.$$

Since  $a_0 = e^{-\alpha\delta}$ , with  $\alpha > 0$ , for every sampling period  $\delta$  one has that  $V_{k+1} < 0$  and the exponential convergence of  $e_{2,k}$  to zero is verified. Finally, using the Jury criterion (Åström & Wittenmark, 1997), for

$$\lambda_2 < 0, \quad \frac{\delta}{J} \lambda_2 + \lambda_1 + 1 > 0, \quad \frac{\delta}{J} \lambda_2 + 2\lambda_1 - 2 < 0$$

one checks that the first equation of (30) is input-to-state stable, and therefore  $e_{1,k}$ ,  $e_{3,k}$  tend to zero exponentially. Note that physically  $I_k$  is bounded, and moreover it is common in applications to use devices to limit the current amplitude. Hence the function  $f_\eta(\eta_k) = a_2 I_k^T \hat{\phi}_k$  can be considered globally bounded.

The controller so determined is hence,

$$\hat{u}_k = \begin{cases} \hat{u}_{k,eq} & \text{for } \|\hat{u}_{k,eq}\| \leq u_{\max} \\ \frac{\hat{u}_{k,eq}}{\|\hat{u}_{k,eq}\|} u_{\max} & \text{for } \|\hat{u}_{k,eq}\| > u_{\max} \end{cases} \quad (31)$$

where  $u_{\max}$  is chosen to fulfill Assumption (A.2), and the equivalent control  $\hat{u}_{k,eq}$  is

$$\hat{u}_{k,eq} = \frac{\sigma}{\delta} \left( -(1 - \delta\gamma) I_k - \delta\beta(\alpha I_d - p\omega_k \hat{\phi}_k) \right) + \hat{I}_{r,k+1} + K_2(I_k - \hat{I}_{r,k})$$

with

$$\hat{I}_{r,k} = \frac{1}{\hat{\phi}_k} \left( \hat{\phi}_k \hat{\phi}_k \right) \begin{pmatrix} \hat{\psi}_{r,1,k} \\ \hat{\psi}_{r,2,k} \end{pmatrix}, \quad \hat{\phi}_k = \hat{\phi}_k^T \hat{\phi}_k$$

$$\hat{\psi}_{r,1,k} = -\frac{a_0}{a_3} \hat{\phi}_k \pm \frac{1}{a_3} \sqrt{\hat{\Delta}_k} \quad (32)$$

$$\hat{\psi}_{r,2,k} = \frac{1}{a_2} \left( \omega_{r,k+1} - \omega_k + \frac{\delta}{J} \hat{C}_{L,k} + k_{11}(\omega_k - \omega_{r,k}) \right)$$

$$\hat{\Delta}_k = \left( \phi_{r,k+1} + k_{12}(\hat{\phi}_k - \phi_{r,k}) \right) \hat{\phi}_k - a_3^2 \hat{\psi}_{r,2,k}^2.$$

It is worth noting that the initial condition  $\hat{\phi}_k(0)$  for the flux observer has to be chosen different from zero in order to avoid singularities in (32).

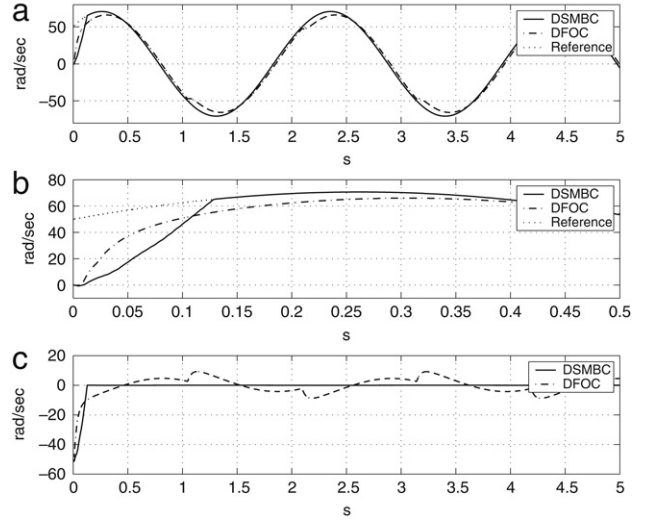


Fig. 1. (a) Angular velocity: comparison between DSMBC and sampled DFOC; (b) Zoom of (a); (c) Tracking errors  $z_{11,k} = \omega_k - \omega_{r,k}$  for the DSMBC and the sampled DFOC.

## 4. Simulations

To show the effectiveness of the proposed control law, simulations have been carried out on a three-phase, two-pole machine, with a stator-referred rotor. The plate parameter values of the motor are  $P_{nom} = 0.25$  Hp,  $\omega_{nom} = 1600$  rpm,  $V_{nom} = 220$  V,  $I_{nom} = 1.0$  A for power, velocity, voltage and current, while the parameters are  $R_s = 14 \Omega$ ,  $R_r = 10.1 \Omega$ ,  $L_s = 400$  mH,  $L_m = 377$  mH,  $L_r = 412.9$  mH,  $p = 2$ ,  $J = 0.01$  kg m<sup>2</sup>. The nominal load torque is  $C_{L,n} = 1.1$  N m.

It is worth mentioning that the induction motor has been simulated as a continuous time system, in order to consider a more realistic condition. Moreover, in order to obtain better current transients, the reference current components have been bounded by five times  $I_{nom}$ , which is an admissible value during transients.

A “worst case”-like scenario is considered, in which the unknown load torque is supposed a square signal, ranging in the interval  $[-C_{L,n}, C_{L,n}]$ , as shown in Fig. 4c. Moreover, we suppose that the rotor velocity tracks the sampled sinusoidal signal  $\omega_{r,k} = 70 \sin 3k\delta$ ,  $k = 0, 1, 2, \dots$ , and the flux magnitude tracks a constant signal  $\Phi_{r,k} = 0.2$  Wb<sup>2</sup>.

The parameters used in the control law and the observer are  $\delta = 500 \mu\text{s}$ ,  $u_{\max} = 220$  V,  $k_{11} = 0.1$ ,  $k_{12} = 0.9$ ,  $\lambda_1 = 0.7$ ,  $\lambda_2 = -0.7$ ,  $\hat{\phi}_0 = (0 \ 0.1)^T$  Wb.

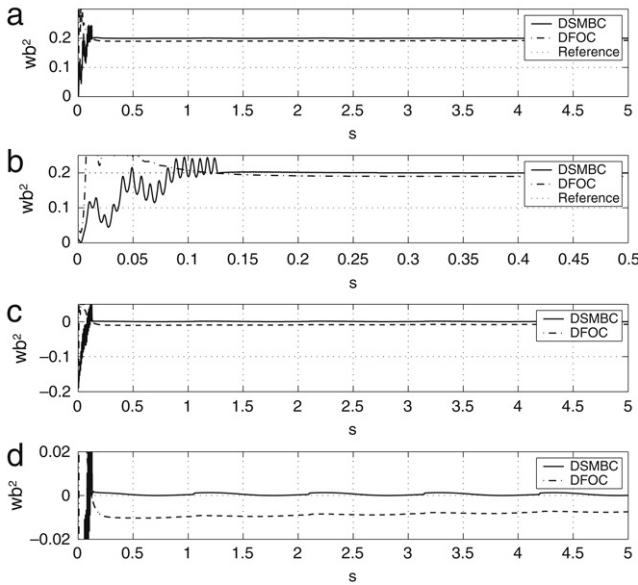
Figs. 1 and 2 show the output tracking results for the controller (31), compared with the sampled version of a direct field oriented control (DFOC) used in Fekih and Chowdhury (2004).

Fig. 3 summarizes the behavior of the voltage  $u_k$ , the flux  $\phi_k$ , and the current  $I_k$ . It is clear the better performance obtained with control (31).

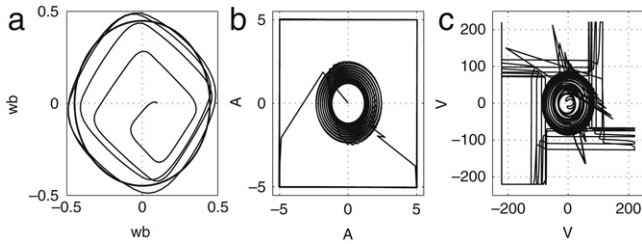
Finally, in Fig. 4 the estimation errors  $\phi_{\alpha,k} - \hat{\phi}_{\alpha,k}$ ,  $\phi_{\beta,k} - \hat{\phi}_{\beta,k}$ , the load torque  $C_{L,k}$  and its estimate  $\hat{C}_{L,k}$  are shown. It is worth noting that, despite the load torque is assumed piece-wise constant, the observer follows well square shape signals and its response is fast for step changes in  $C_L$ .

## 5. Conclusions

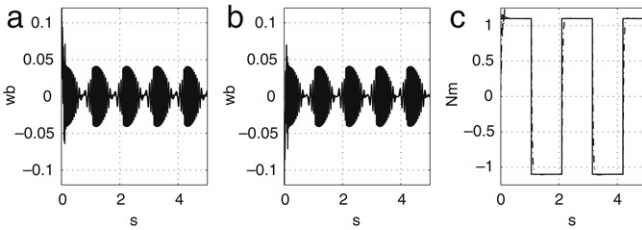
In this work, a sliding mode control has been proposed for output reference tracking, based on a general decomposition method for discrete time nonlinear systems. A reduced order observer has been designed for the unmeasured states and



**Fig. 2.** (a) Rotor squared flux norm: comparison between DSMBC and sampled DFOC; (b) Zoom of (a); (c) Tracking errors  $z_{12,k} = \Phi_k - \Phi_{r,k}$  for the DSMBC and the sampled DFOC; (d) Zoom of (c).



**Fig. 3.** (a)  $\phi_{\alpha,k}$  vs.  $\phi_{\beta,k}$ ; (b)  $i_{\alpha,k}$  vs.  $i_{\beta,k}$ ; (c)  $u_{\alpha,k}$  vs.  $u_{\beta,k}$ .



**Fig. 4.** (a) Estimation error  $\phi_{\alpha,k} - \hat{\phi}_{\alpha,k}$ ; (b) Estimation error  $\phi_{\beta,k} - \hat{\phi}_{\beta,k}$ ; (c)  $C_{L,k}$  (solid) and  $\hat{C}_{L,k}$  (dash-dot).

perturbation, ensuring the fulfillment of the control objective. This controller has been applied for the control of induction motors. An approximated discrete time model for induction motors has been derived. The resulting dynamic controller has been simulated and compared with a classical field oriented control, showing satisfactory steady state and transient performance, and showing at the same time the capability of the control system of rejecting modeled disturbances.

**Appendix. Approximated sampled model for induction motors**

In this appendix, an approximated sampled model for induction motors is derived. Under the assumptions of equal mutual inductance and a linear magnetic circuit, a sixth-order induction

motor model is given as (Marino et al., 1993)

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= \mu I^T \Im \phi - \frac{1}{J} C_L \\ \dot{\phi} &= -\alpha \phi + p\omega \Im \phi + \alpha L_m I \\ \dot{I} &= \alpha \beta \phi - p\beta \omega \Im \phi - \gamma I + \frac{1}{\sigma} u \end{aligned} \tag{A.1}$$

where  $\theta$  and  $\omega$  are the rotor position and angular velocity respectively,  $\phi = (\phi_\alpha \ \phi_\beta)^T$ ,  $I = (i_\alpha \ i_\beta)^T$ ,  $u = (u_\alpha \ u_\beta)^T$  are the rotor flux, stator current and voltage vectors,  $C_L$  is the load torque,  $J$  is the rotor moment of inertia, and  $\alpha = \frac{R_r}{L_r}$ ,  $\beta = \frac{L_m}{\sigma L_r}$ ,  $\gamma = \frac{L_m^2 R_r}{\sigma L_r^2} + \frac{R_s}{\sigma}$ ,  $\sigma = L_s - \frac{L_m^2}{L_r}$ ,  $\mu = \frac{3}{2} \frac{L_m p}{J L_r}$ , and  $L_s, L_r, L_m$  are the stator, rotor and mutual inductance respectively,  $R_s$  and  $R_r$  are the stator and rotor resistance respectively, and  $p$  is the number of pole pairs.

The first three Eqs. (A.1) constitute the so-called current-fed model, in which  $I$  can be regarded as the control input. It is easy to check that the current-fed model cannot be exactly discretized (Monaco & Normand-Cyrot, 1985). Necessary and sufficient conditions for exact discretizability under coordinate transformation and input feedback can be found in Castillo-Toledo et al. (2008), Monaco and Di Giamberardino (1996) and Monaco, Di Giamberardino, and Normand-Cyrot (2006). Nevertheless, an approximated sampled model can be derived considering constant the current  $I$  in the third Eq. (A.1) over the sampling period  $\delta$ , and under the following assumption.

(H.1). *The load torque  $C_L$  is piece-wise constant over the sampling period  $\delta$ .*  $\triangleleft$

Assumption (H.1) holds for all cases in which  $C_L$  is slowly varying with respect to the electric dynamics.

As far as the approximated sampled version of current-fed model is concerned, let us use the following globally defined change of coordinate

$$\begin{pmatrix} \theta \\ \omega \\ Y \\ X \end{pmatrix} = \begin{pmatrix} \theta \\ \omega \\ e^{-p\theta \Im} \phi \\ e^{-p\theta \Im} I \end{pmatrix} \tag{A.2}$$

where

$$\frac{d}{dt} e^{-p\theta \Im} = -p\omega \Im e^{-p\theta \Im}, \quad e^{-p\theta \Im} \Im = \Im e^{-p\theta \Im}.$$

Considering  $X$  as new input, so neglecting for the moment its dynamics, one obtains the following bilinear model

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= \mu X^T \Im Y - \frac{1}{J} C_L \\ \dot{Y} &= -\alpha Y + \alpha L_m X. \end{aligned} \tag{A.3}$$

The advantage of considering (A.2) is that the equation regarding  $Y$  becomes linear. Under the following simplifying assumption this last equation can be easily discretized.

(H.2). *The new control  $X$  in the dynamics of  $Y$  in (A.3) can be considered piece-wise constant over the sampling period  $\delta$ .*  $\triangleleft$

Assumption (H.2) can be considered valid for small sampling periods  $\delta$ . Thanks to this assumption, from the third equation of (A.3) one gets

$$Y(t) = e^{-\alpha(t-k\delta)} Y_k + L_m (1 - e^{-\alpha(t-k\delta)}) X_k \tag{A.4}$$

where  $Y_k = Y(k\delta)$ ,  $X_k = X(k\delta)$ ,  $t \in [k\delta, (k+1)\delta)$ . Now, under (H.1), the integration of the second of (A.3) gives

$$\omega(t) = \omega_k + \mu X_k^T \Im \int_{k\delta}^t Y(\xi) d\xi - \frac{1}{J} C_{L,k} \int_{k\delta}^t d\xi \tag{A.5}$$





