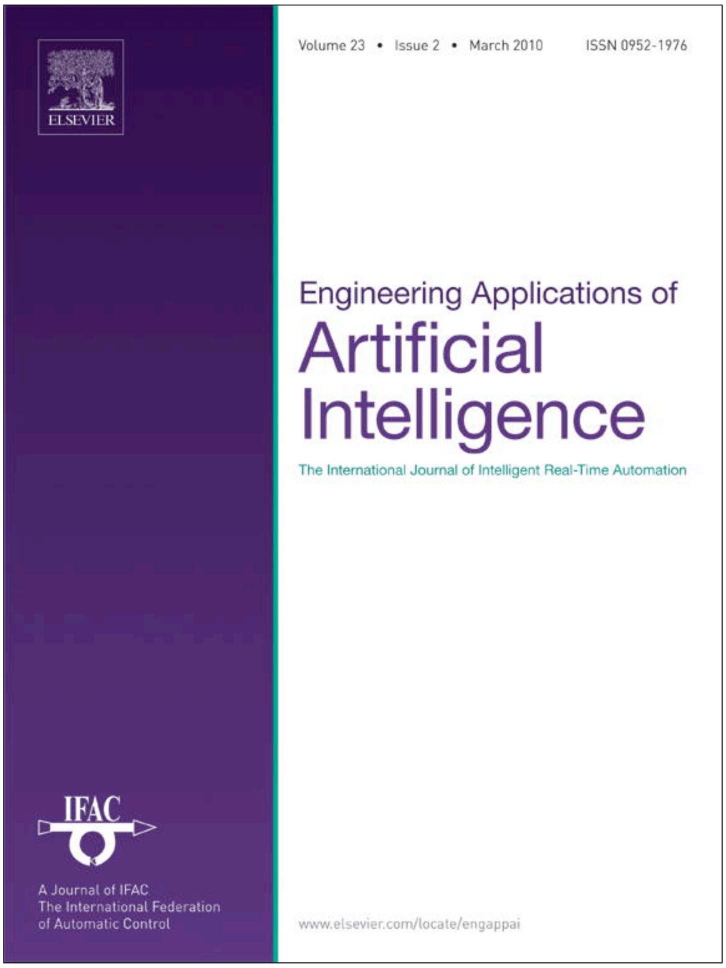
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Stabilization for a class of nonlinear systems: A fuzzy logic approach ☆

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ABSTRACT

In this paper, the problem of stabilization for the class of continuous time nonlinear systems which are discretized in closed form is addressed. By using the Takagi–Sugeno model approach, a discrete controller capable of stabilizing the discrete Takagi–Sugeno model and the continuous model as well, is obtained. This scheme allows using a digital controller for stabilizing an analog plant.

1. Introduction

The stabilization problem of dynamical systems constitutes an interesting problem in control theory. Many approaches have been proposed, and continue to appear, to tackle this problem for different kinds of systems—linear or nonlinear, continuous or discrete. In general, the control law has the same nature of the system, namely, for continuous systems, continuous controllers are designed to guarantee the stability of the closed-loop system.

The use of faster digital computers has motivated the design of sampled-data controllers for continuous time plants. Depending on the holders—usually being zero order—the dynamics of the sampled nonlinear system, or discretized system for short, are usually only an approximation (Monaco and Normand-Cyrot, 1985, 1987, 1988, 2007; Nesic and Teel, 2004). Furthermore, once the (approximated) discretized system dynamics have been obtained, and the digital controller has been designed fulfilling certain design requirements, the controller performance may not be necessarily satisfactory when applied to the continuous system. This is due to the fact that the control signals differ from those of a continuous controller designed for the continuous system with the same requirements, because of the sampling and holding operations.

Several methods for approximate discretization can be found in the literature. Obviously, the performance of a controller designed on the basis of the approximate discretization depends on the degree of approximation. For example, when using the simple Euler method, it is possible that the controller does not guarantee the stability of the closed loop system (Monaco and Normand-Cyrot, 1997, 2001, 2007). A way to overcome this situation would be the determination of a discretization in closed form. Clearly, this might not be always possible. Some methods for the discretization in closed form have appeared recently in the literature and, moreover, some results point out that it is possible to induce the "discretizability property in closed form" by a suitable feedback (Di Giamberardino et al., 2000). An example of such systems are the systems completely linearizable by feedback or the class of systems transformable to a polynomial triangular form.

On the other hand, recent results on fuzzy modeling, in particular the Takagi–Sugeno (TS) modelization approach, can be fruitfully used in the problem of stabilization of nonlinear systems (Takagi and Sugeno, 1985). The main feature of the TS models is that they represent in a certain region the local dynamics of the system, by means of a linear model. The complete fuzzy model is then obtained by a fuzzy aggregation of these linear models. In this way, it can be shown that the TS fuzzy models are universal approximators of many nonlinear systems, and the design of the controller can be made on the basis of the linear subsystems describing locally the aggregate nonlinear TS model (Tanaka and Wang, 2001).

In this paper, supposing to be able to determine the discretized dynamics in closed form of a continuous time system, we

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determine a TS discrete time model. This TS model is then used to design a discrete time controller, stabilizing the TS discrete time model. Moreover, we show that this discrete controller, when implemented via a zero order holder and under certain conditions, stabilizes the continuous time system as well. The main advantage of the proposed technique relies in the simplicity of the controller design. In fact, the discrete time controller results from the fuzzy aggregation of the digital controllers, each designed for a discrete time linear system of the TS model.

2. Some facts about the discretization of dynamical systems

Consider a linear time invariant system described by

$$\dot{x} = A_c x + B_c u, \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$. It is well known that the discretization of this linear system with a sampling interval δ is given by

$$x_{k+1} = A_d x_k + B_d u_k$$

where

$$x_{k+1} = x(k\delta + \delta), \quad x_k = x(k\delta), \quad u_k = u(k\delta),$$

$$A_d = e^{A_c\delta}, \quad B_d = \int_0^\delta e^{A_c s}\, ds\, B_c.$$

For nonlinear systems, however, finding the solution of the differential equations is difficult in many cases. Hence, various authors consider approximate discretizations. As a result, at the sampling instants the solutions of the differential and approximate discretized systems do not coincide, and poor accuracy may result. Also, relatively large sampling period may cause instability or undesired behavior.

However, some cases of discretization in closed form of nonlinear systems have been studied in the literature (Monaco et al., 1996), (Di Giamberardino et al., 2006), (see also Monaco and Normand-Cyrot, 2007, and references therein). These schemes allow expressing the discretization process as a power Lie series, and discretization in closed form can be obtained if some condition on the residuals holds. To precise these ideas, let us consider the nonlinear system

$$\dot{x} = f(x,u).$$

Expanding its solutions $x(t)$ around $t=0$ we get

$$\begin{aligned} x(t) &= \left[\frac{x(t)}{0!}\right]_{t=0} + \left[\frac{\dot{x}(t)}{1!}\right]_{t=0} t + \left[\frac{\ddot{x}(t)}{2!}\right]_{t=0} t^2 + \cdots \\ &= x(0) + f(x(0),u(0))t + \frac{1}{2!}\left[\dot{f}(x,u)\right]_{t=0} t^2 + \cdots \\ &= x(0) + \sum_{i=1}^{\infty} \frac{t^i}{i!}\left[f^{(i)}(x,u,\ldots u,\ldots,u^{(i-1)})\right]_{t=0}, \end{aligned} \tag{2}$$

where the operator $f^{(i)}(x,u,\dot{u},\ldots,u^{(i-1)})$ is defined as

$$f^{(1)}(x,u) = f(x,u),$$

$$\begin{aligned} f^{(i)}(x,u,\ldots,u^{(i-1)}) &= \frac{\partial f^{(i-1)}(x,u,\dot{u},\ldots,u^{(i-2)})}{\partial x} f(x,u) \\ &+ \frac{\partial f^{(i-1)}(x,u,\dot{u},\ldots,u^{(i-1)})}{\partial u}\dot{u} + \cdots \\ &+ \frac{\partial f^{(i-1)}(x,u,\dot{u},\ldots,u^{(i-1)})}{\partial u^{(i-2)}} u^{(i-1)}. \end{aligned}$$

Taking the solution (2) around $t = k\delta$ and considering a piecewise constant input u_k for $k\delta \le t < (k+1)\delta$, we can write the discrete solution as

$$x_{k+1} = \sum_{i=0}^{\infty} \frac{\delta^i}{i!}\left[L^i_{f(x,u)}(x)\right]_{\substack{x=x_k\\u=u_k}} = e^{\delta L^i_{f(x,u)}}(x)\big|_{\substack{x=x_k\\u=u_k}}, \tag{3}$$

where $L^i_{f(x,u)}(\cdot)$ is defined as

$$L^i_{f(x,u)}(x) = \frac{\partial L^{i-1}_{f(x,u)}}{\partial x} f(x,u), \quad L^0_{f(x,u)}(x) = x.$$

From the previous expression, if for a finite i the term $L^i_{f(x,u)}$ is zeroed, namely the nilpotency condition is fulfilled, and the discretization becomes in closed form. Otherwise, only an approximation up to a certain degree can be obtained. The condition of nilpotency is a sufficient condition for discretization in closed form.

3. The Takagi–Sugeno fuzzy model

Let us consider a continuous time nonlinear system described by

$$\dot{x} = f(x,u), \tag{4}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$. It is well known that it is possible to describe, at least in a certain region of interest, the behavior of the nonlinear system (4) by a suitable aggregation of local linear subsystems. One of these approaches is the Takagi–Sugeno modelization (Takagi and Sugeno, 1985). This model is described by a set of fuzzy IF–THEN rules representing local linear input–output dynamics of a nonlinear system. The most interesting feature in the TS model is that the local dynamics can be described by linear submodels and the aggregated model is obtained by fuzzy blending of the linear submodels. These linear models can be derivated directly from the nonlinear model by a sector nonlinearity approach, in which case the aggregated model describes exactly the nonlinear dynamics in a regions $\mathcal{D}$ of interest, or in an approximate way. In the later case, the number of the IF–THEN rules may be reduced, but the controller calculated on the basis of the approximated TS model cannot guarantee a priori the stability of the original system. A robust scheme could be a way of overcome this situation. In this work, the sector nonlinearity approach will be taken, namely, we will assume that the TS model obtained is exact with respect to the original nonlinear dynamics in a region $\mathcal{D}$ (Tanaka and Wang, 2001).

To be more precise, let us have the local subsystems defined as follows:

Plant rule i: IF z_j is $F_{ji}, j = 1,\ldots,p$
THEN Σ: $\dot{x} = A_i x + B_i u, \quad i = 1,\ldots,r$

where $z_1,\ldots,z_p$ are measurable premise variables, which may coincide with the state vector or with a partial set of this vector through the output signals y_i. Moreover, F_{1i} are the corresponding fuzzy sets. Usually, these linear subsystems are obtained from some knowledge of the process dynamics or by their linearization about some point of interest.

For a given pair $(x(\cdot),u(\cdot))$, the aggregate fuzzy model is obtained by using a singleton fuzzifier, a product inference and a center of gravity defuzzifier, giving a Continuous Fuzzy Model (*CFM*) described by

$$\dot{x} = \sum_{i=1}^{r} \mu_i(z) A_i x + \sum_{i=1}^{r} \mu_i(z) B_i u \tag{5}$$

with $z = (z_1 \cdots z_p)^T$, where $\mu_i(z)$ is the normalized weight for each rule calculated from the membership functions for z_j in F_{ji}, and such that $\mu_i(z) \ge 0$, and

$$\sum_{i=1}^{r} \mu_i(z) = 1.$$

The sampled version of system (4), using zero order holders, is given by

$$x_{k+1} = f(x_k, u_k). \tag{6}$$

Also in the discrete time context one can consider a description of the sampled system (6) by means of an aggregation of linear subsystems which, in the Takagi–Sugeno modelization, are defined as

Plant rule i : IF $z_{j,k}$ is F_{ji}, $j=1,\dots,p$

THEN Σ: $x_{k+1}=A_i^d x_k+B_i^d u_k$, $i=1,\dots,r$.

With an analogous procedure one can get a Discrete Fuzzy Model (*DFM*), described by

$$\text{DFM:}\ x_{k+1}=\sum_{i=1}^{r}\mu_i(z_k)A_i^d x_k+\sum_{i=1}^{r}\mu_i(z_k)B_i^d u_k. \tag{7}$$

The *CFM*, *DFM* modelizations (5), (7) allow studying nonlinear systems by introducing tools valid for the linear setting. In particular, several results are known for stabilization of (5) and (7). Sufficient conditions for the asymptotic stability of the equilibrium of the aggregate fuzzy models *CFM*, *DFM* are given by the following (Tanaka and Wang, 2001):

Theorem 1. *Let us assume that the pairs A_i, B_i of (5) are stabilizable, $i=1,\dots,r$, namely there exist matrices K_i such that $A_i+B_iK_i$ is Hurwitz. Then, the equilibrium of the continuous fuzzy control system (5) is globally asymptotically stable if there exist a common positive definite matrix P such that*

$$P(A_i+B_iK_i)+(A_i+B_iK_i)^TP<0 \tag{8}$$

for $i=1,2,\dots,r$, and

$$G_{ij}^TP+PG_{ij}<0 \tag{9}$$

for $i<j$, and $G_{ij}=((A_i+B_iK_j)+(A_j+B_jK_i))/2$.

Analogously, let us assume that the pairs A_i^d, B_i^d of (7), are stabilizable, $i=1,\dots,r$, namely there exist matrices K_i^d such that $A_i^d+B_i^dK_i^d$ is Schur. Then, the equilibrium of the discrete fuzzy control system (7) is globally asymptotically stable if there exist a common positive definite matrix P such that

$$(A_i^d+B_i^dK_i^d)^TP(A_i^d+B_i^dK_i^d)-P<0 \tag{10}$$

for $i=1,2,\dots,r$, and

$$(G_{ij}^d)^TPG_{ij}^d-P<0 \tag{11}$$

for $i<j$ and $G_{ij}^d=((A_i^d+B_i^dK_j^d)+(A_j^d+B_j^dK_i^d))/2$.

Theorem 1 expresses the fact that if each linear subsystem can be stabilized, and if there exists a matrix P satisfying the matrix Lyapunov equations (8), (9) in the continuous time case, or (10), (11) in the discrete time case, then the continuous fuzzy controller

$$u=\left(\sum_{i=1}^{r}\mu_j(x)K_j\right)x \tag{12}$$

stabilizes the fuzzy system (5), and the discrete fuzzy controller

$$u_k=\left(\sum_{j=1}^{r}\mu_j(x_k)K_j^d\right)x_k \tag{13}$$

stabilizes the fuzzy system (7).

An interesting question arises at this point: if the discrete system (6) is the discretization in closed form of the corresponding continuous system (4), and the fuzzy system (7) is the exact description of system (6), will the controller (13) stabilize the continuous system (5) as well? This question will be studied in the following section.

4. The fuzzy discrete stabilization problem

Let us consider the nonlinear system (4), defined globally in a region $\mathcal{D}$, and let us suppose to determine its discretization in closed form (Monaco and Normand-Cyrot, 2001)

$$x_{k+1}=A_dx_k+B_du_k+f_{2d}(\delta,x_k,u_k) \tag{14}$$

with f_{2d} the map corresponding to the nonlinear terms of second order or greater. We also assume that it its possible to obtain an exact TS fuzzy model (7) of (14), defined in the same region $\mathcal{D}$. The problem of global discretized stabilization can be formulated as follows.

In the following $x_{zh}(t)$ denotes the piece-wise constant function obtained as output of a zero order holder, having the discrete time function x_k as input.

Global discretized stabilization problem (*GDSP*). Given a nonlinear system (4), the GDSP consists of finding a discrete system (7) and a piece-wise constant controller

$$u(t)=\left(\sum_{j=1}^{r}\mu_j(x_k)K_j^d\right)x_{zh}(t),\quad t\in[k\delta,(k+1)\delta) \tag{15}$$

obtained from (13) by means of a zero order holder, such that for any initial condition $x_0=x(0)\in\mathcal{D}$ the solution of the closed-loop system

$$\dot{x}=f\left(x,\sum_{j=1}^{r}\mu_j(x_k)K_j^dx_{zh}(t)\right)$$

satisfies $\lim_{t\to\infty}x(t)=0$.

The main result of this paper expresses a condition for the existence of a solution to the GDSP.

Theorem 2. *Assume the following assumptions hold*:

(H_0) *The nonlinear system (4) is discretizable in closed form (14), and the nonlinear discrete system (14) admits an exact Takagi–Sugeno fuzzy model.*

(H_1) *The solution $x(t)=\phi(x_0,u(t))$ of the closed loop continuous system (5)–(12) exists globally on a region $\mathcal{D}$ and can be expressed as*

$$x(k\delta+\theta)=F(x_k,\theta)$$

with F the solution of the closed loop continuous system (5)–(12), such that

$$F(0,\theta)=0$$

where $t=k\delta+\theta$ and $\theta\in[0,\delta)$ a linear function.

(H_2) *There exist matrices $P>0$, and K_i^d such that the matrix inequalities (8), (9) hold.*

Under (H_0), (H_1), (H_2) *the controller (15) solves the GDSP.*

Proof. From the previous discussion, by (H_2) the controller (13) stabilizes the system

$$x_{k+1}=\sum_{i=1}^{r}\mu_i(x_k)A_i^dx_k+\sum_{i=1}^{r}\mu_i(x_k)B_i^d\left(\sum_{j=1}^{r}\mu_j(x_k)K_j^d\right)x_k.$$

Hence, since the discrete TS model describes exactly the dynamics of the system (14) in a region $\mathcal{D}$, then (15) stabilizes also the system discretized in closed form (14) and then we have that

$$\lim_{k\to\infty}x(k\delta)=\lim_{k\to\infty}x_k=0.$$

Now, by assumption (H_1), if x_k goes to zero, then since $F(0,\theta)=0$, by continuity, it follows that

$$\lim_{t\to\infty}x(t)=\lim_{t\to\infty}x(k\delta+\theta)=0.\qquad\square$$

Remark 1. Assumption (H_0) is necessary in order to compute the linear systems which form the TS model. Moreover, note that assumption (H_1) is the extension to the nonlinear setting of the respective result for linear systems. In fact, recalling that the solution of a linear system can be written as

$$x(k\delta+\theta)=e^{A\theta}x_k+\int_{k\delta}^{k\delta+\theta}e^{A(k\delta+\theta-s)}Bu(s)\,ds$$

if a piece-wise constant controller $u(t)=Kx_k$ is used in the interval $t\in[k\delta,(k+1)\delta)$, then the solution takes the form

$$x(k\delta+\theta)=\left(e^{A\theta}+\int_{k\delta}^{k\delta+\theta}e^{A(k\delta+\theta-s)}\,ds\,BK\right)x_k$$
$$=\left(e^{A\theta}+\int_0^{\theta}e^{A\tau}\,d\tau\,BK\right)x_k$$

and from this it follows that if the linear discrete system is stabilized, then the linear continuous system is stabilized as well. It is worth noting that when taking an approximate solution, this property is no longer guaranteed, as pointed out before. It is also worth noting that (H_1) is verified, for instance, when the function f is Lipschitz with respect to x and u, and a stabilizing Lipschitz feedback $u=\alpha(x)$ exists. In particular, this happens when f in (4) is affine in the input, namely

$$f(x,u)=f_0(x)+g(x)u$$

the maps $f_0(x)$, $g(x)$ are Lipschitz, and a stabilizing Lipschitz feedback $u=\alpha(x)$ exists. Assumption (H_3) could be also replaced by a less conservative condition for stability of the discrete closed-loop system, for example taking a parameter-dependent Lyapunov function among other choices.

4.1. The particular case of a class of feedback discretizable nonlinear systems

In general there is no special form for a nonlinear system to guarantee that there exists a discretization in closed form. For particular cases, however, it is possible to show that there exists a discretization in closed form. For example, the class of systems described by

$$\dot{x}_1=f_1(x_2,x_3,\ldots,x_n,u),$$
$$\dot{x}_2=f_2(x_3,x_4,\ldots,x_n,u),$$
$$\vdots$$
$$\dot{x}_{n-1}=f_{n-1}(x_n,u),$$
$$\dot{x}_n=f_n(u), \tag{16}$$

where $f_i(\cdot)$ are polynomials, can be discretized in closed form.

More in general, a special class of systems for which discretization in closed form can be obtained is that of the nonlinear system (Monaco et al., 1996)

$$\dot{x}=f(x)+g(x)u_k,\quad g(x)=(g_1(x)\ \cdots\ g_m(x))^T, \tag{17}$$

where $x\in\mathbb{R}^n$, $f(x)$, $g_1(x),\ldots,g_m(x)$ are real analytic vector fields on $\mathbb{R}^n$, and the control $u_k\in\mathbb{R}^m$ is piece-wise constant over the sampling period δ. The sampled version of (17) is described by (Monaco and Normand-Cyrot, 1985; Monaco et al., 1996)

$$x_{k+1}=e^{f(x)+g(x)u_k}(I_d)\Big|_{x_k}=\sum_{k=0}^{\infty}L^k_{f(x)+g(x)u_k}(I_d)\Big|_{x_k}. \tag{18}$$

This expression is the formal exponential function, calculated as an infinite series in x_k. Thus, the problem is to determine the function $F(\delta,x_k,u_k)\colon\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$, expressing the sampled closed form and sum of the infinite exponential series. If the sum of (18) is finite, i.e. if there exists a $\bar{k}$ such that for any $(x,u)\in\mathbb{R}^n\times\mathbb{R}^m$

$$x_{k+1}=\sum_{k=0}^{\bar{k}}L^k_{f(x)+g(x)u_k}(I_d)\Big|_{x_k} \tag{19}$$

then $F(\delta,x_k,u_k)$ is the finite discretization of the continuous system (17) (Monaco et al., 1996), i.e. the desired discretization in closed form. This finite discretization is not *coordinate free*, but in closed form. These concepts are related to the nilpotency of the Lie algebra associated to the continuous system. In fact, if this algebra is nilpotent and of dimension n at some point, then there exists locally a sampled closed form and, in *suitable coordinates*, the system is finite discretizable. Hence, following Monaco et al. (1996), one tries to induce the finite discretizability property by applying first a continuous feedback, as indicated in the following assumption.

(H_3) There exist a feedback

$$u=\alpha(x)+\beta(x)v$$

and a transformation

$$z=\Phi(x)$$

such that the system (4) is transformed into a nonlinear system

$$\dot{z}=\tilde{f}(z,v),$$
$$y=h(\Phi^{-1}(z)), \tag{20}$$

which is discretizable in closed form.

This condition relaxes assumption (H_0) in the sense that it covers a broader set of nonlinear systems. The following corollary can be thus derived.

Corollary 1. *Assume conditions (H_1), (H_2) and (H_3) hold. Then the GDEP is solved.*

In this case, the controller to be implemented is an hybrid controller, given by

$$u(t)=\alpha(x)+\beta(x)\left(\sum_{j=1}^{r}\mu_j(\Phi^{-1}(z_k))\tilde{K}_j^d\right)\Phi^{-1}(z_k),$$

where the membership functions $\mu_j(\Phi^{-1}(z_k))$ and the gains $\tilde{K}_j^d$ are calculated for the transformed system (20).

5. Illustrative examples

5.1. A simple system

Let us consider the system given by

$$\dot{x}_1=x_2+x_3^2,$$
$$\dot{x}_2=x_3,$$
$$\dot{x}_3=u,$$

$x=(x_1\ x_2\ x_3)^T$, which is in the special form (16). Its discretization is

$$x_{k+1}=\begin{pmatrix}1 & \delta & \delta x_3+\delta^2\dfrac{(1+2u_k)}{2}\\ 0 & 1 & \delta\\ 0 & 0 & 1\end{pmatrix}x_k+\begin{pmatrix}\delta^3\dfrac{(1+2u_k)}{6}\\ \dfrac{\delta^2}{2}\\ \delta\end{pmatrix}u_k.$$

To obtain the TS discrete fuzzy model, we follow the sector nonlinearity approach and rewrite this system as

$$x_{k+1} = \begin{pmatrix} 1 & \delta & z_1 \\ 0 & 1 & \delta \\ 0 & 0 & 1 \end{pmatrix} x_k + \begin{pmatrix} z_2 \\ \dfrac{\delta^2}{2} \\ \delta \end{pmatrix} u_k$$

$x_k = (x_{1,k}\ \ x_{2,k}\ \ x_{3,k})^T$, where the minimum and maximum values for z_1 and z_2 are chosen as

$$z_{1,\min} = -2.211875, \quad z_{1,\max} = 2.28125 \quad z_{2,\min} = 0, \quad z_{2,\max} = 0.0052$$

and $u \in [-1,1]$, $x_3 \in [-1,1]$, $\delta = 0.25$ s. The membership functions are obtained as in Tanaka and Wang (2001), namely

$$M_1(z_1) = \frac{z_1 + 2.21875}{4.5}, \quad M_2(z_1) = \frac{-z_1 + 2.21875}{4.5},$$

$$N_1(z_2) = \frac{z_2}{0.0052}, \quad N_1(z_2) = \frac{-z_2 + 0.0052}{0.0052},$$

$$\mu_1(x_k) = M_1 N_1, \quad \mu_2(x_k) = M_2 N_1,$$

$$\mu_1(x_k) = M_1 N_2, \quad \mu_1(x_k) = M_2 N_2.$$

The exact aggregate model is thus written as

$$x_{k+1} = \sum_{i=1}^{4} \mu_i(x_k) A_i^d x_k + \sum_{i=1}^{4} \mu_i(x_k) B_i^d u_k,$$

where

$$A_1^d = A_3^d = \begin{pmatrix} 1 & \delta & 2.28125 \\ 0 & 1 & \delta \\ 0 & 0 & 1 \end{pmatrix}, \quad A_2^d = A_4^d = \begin{pmatrix} 1 & \delta & -2.28125 \\ 0 & 1 & \delta \\ 0 & 0 & 1 \end{pmatrix},$$

$$B_1^d = B_2^d = \begin{pmatrix} 0.0052 \\ \dfrac{\delta^2}{2} \\ \delta \end{pmatrix} \qquad B_3^d = B_4^d = \begin{pmatrix} 0 \\ \dfrac{\delta^2}{2} \\ \delta \end{pmatrix}.$$

The initial conditions are $x_1(0) = x_{1,0} = 0.5$, $x_2(0) = x_{2,0} = 0.6$, $x_3(0) = x_{3,0} = -0.8$. The results of the application of the discrete stabilizer (13), with

$$K_1^d = (0.0010\ \ 0.0307\ \ 0.2080), \quad K_2^d = (0.0010\ \ 0.0133\ \ 0.2107),$$
$$K_3^d = (0.0010\ \ 0.0307\ \ 0.2080), \quad K_4^d = (0.0010\ \ 0.0133\ \ 0.2108)$$

are shown in Figs. 1 and 2. Fig. 1 shows the response of the discretized system with the discrete controller, while Fig. 2 shows the response of the continuous time nonlinear system when driven by the digital controller. As can be observed, the continuous systems is stabilized by the proposed discrete digital.

5.2. Synchronous motor

The equations of a permanent magnet (PM) synchronous motor, expressed in the so-called (d,q)-frame, are (Leonhard, 1985)

$$\frac{di_d}{dt} = -\frac{R_s}{L_d} i_d + p\frac{L_q}{L_d}\omega i_q + \frac{1}{L_d} v_d,$$

$$\frac{di_q}{dt} = -\frac{R_s}{L_q} i_q - p\frac{L_d}{L_q}\omega i_d - \frac{k_m}{L_q}\omega + \frac{1}{L_q} v_q,$$

$$\dot{\omega} = \frac{k_m}{J} i_q + p\frac{L_d - L_q}{J} i_d i_q - \frac{F}{J}\omega - \frac{1}{J} C_l, \qquad (21)$$

in which R_s is the stator winding resistance; L_d, L_q are the d, q inductances; $k_m = p\Phi_m$ is the motor torque constant (p is the number of pole pairs and Φ_m is the flux of the permanent magnets); i_d, i_q are the currents and v_d, v_q are the applied voltages;

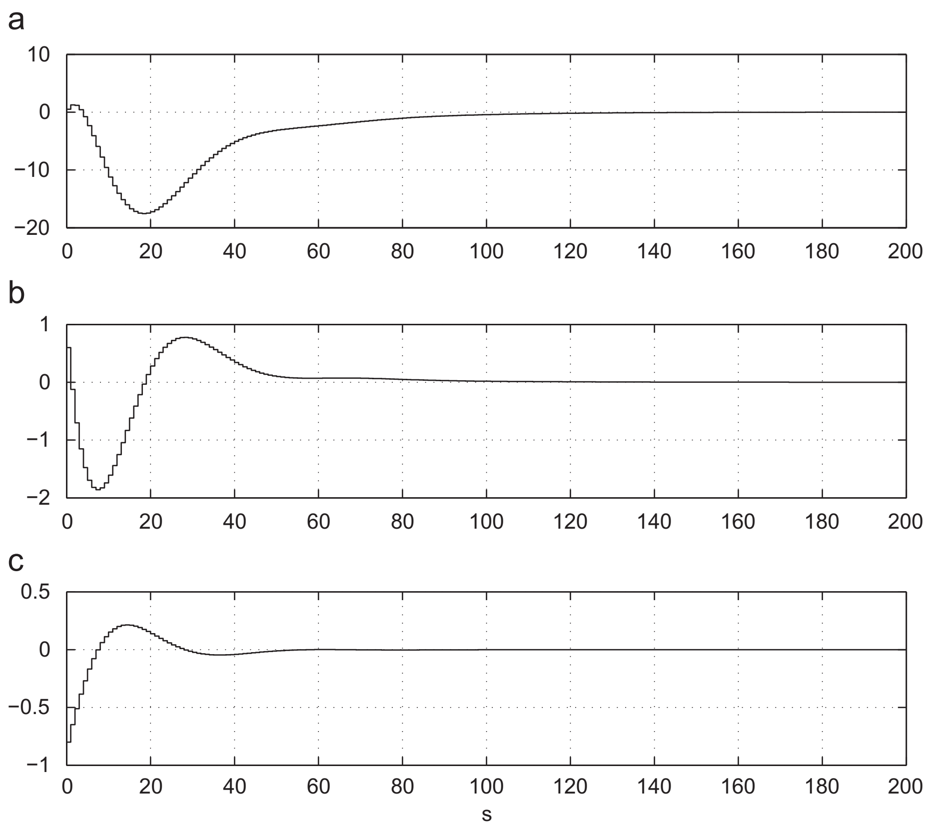

Fig. 1. Response of the discrete controller applied to the discretized system. (a) $x_{1,k}$; (b) $x_{2,k}$; (c) $x_{3,k}$.

ω is the rotor angular velocity; J is the rotor moment of inertia; F is the viscous friction coefficient; C_l is the load torque.

The variable to be controlled are the i_d component of the current and the angular velocity ω. The references to be tracked are commonly a piece-wise constant angular velocity reference ω_r (this is a physically acceptable hypothesis, since the mechanical dynamics is much slower compared with the magnetic ones) and a reference i_{dr} is usually taken equal to zero when ω_r is less or equal to the nominal speed. Moreover, C_l is a disturbance load torque acting on the motor. One can re-write the motor's equations as follows:

$$\dot{x} = f(x,w,u) = Ax + Bu + Dd + \varphi(x,w)$$

with $x = (i_d\ i_q\ \omega)^T$, $u = (v_d\ v_q)^T$, $d = C_l$

$$A = \begin{pmatrix} -\dfrac{R_s}{L_d} & 0 & 0 \\ 0 & -\dfrac{R_s}{L_q} & -\dfrac{k_m}{L_q} \\ 0 & \dfrac{k_m}{J} & -\dfrac{F}{J} \end{pmatrix}, \quad B = \begin{pmatrix} \dfrac{1}{L_d} & 0 \\ 0 & \dfrac{1}{L_q} \\ 0 & 0 \end{pmatrix}$$

and

$$D = \begin{pmatrix} 0 \\ 0 \\ -\dfrac{1}{J} \end{pmatrix}, \quad \varphi(x,w) = \begin{pmatrix} p\dfrac{L_q}{L_d}\omega i_q \\ -p\dfrac{L_d}{L_q}\omega i_d \\ p\dfrac{L_d - L_q}{J} i_d i_q \end{pmatrix}.$$

The discrete time model of the synchronous motor is obtained considering first the continuous feedback

$$v_d = -pL_q \omega i_q + u_d,$$

$$v_q = pL_d \omega i_d + k_m \omega + u_q,$$

which puts (21) in the form

$$\frac{di_d}{dt} = -\frac{R_s}{L_d} i_d + \frac{1}{L_d} u_d,$$

$$\frac{di_q}{dt} = -\frac{R_s}{L_q} i_q + \frac{1}{L_q} u_q,$$

$$\dot{\omega} = \frac{k_m}{J} i_q + p\frac{L_d - L_q}{J} i_d i_q - \frac{F}{J}\omega - \frac{1}{J} C_l$$

from which a sampled version is easily obtained. In fact,

$$i_d(t) = e^{\lambda_1 (t-t_0)} i_d(t_0) + \frac{1}{L_d}\int_{t_0}^{t} e^{\lambda_1 (t-\tau)} u_d(\tau)\, d\tau,$$

$$i_q(t) = e^{\lambda_2 (t-t_0)} i_q(t_0) + \frac{1}{L_q}\int_{t_0}^{t} e^{\lambda_2 (t-\tau)} u_q(\tau)\, d\tau,$$

$$\omega(t) = e^{\lambda_3 (t-t_0)} \omega(t_0) + \frac{k_m}{J}\int_{t_0}^{t} e^{\lambda_3 (t-\xi)} i_q(\xi)\, d\xi + p\frac{L_d - L_q}{J}\int_{t_0}^{t} e^{\lambda_3 (t-\xi)} i_d(\xi) i_q(\xi)\, d\xi - \frac{1}{J}\int_{t_0}^{t} e^{\lambda_3 (t-\xi)} C_l(\xi)\, d\xi,$$

$\lambda_1 = -R_s/L_d$, $\lambda_2 = -R_s/L_q$, $\lambda_3 = -F/J$. Setting $t_0 = k\delta$, $t = (k+1)\delta$, $\delta = 2 \times 10^{-4}$ s and considering that

$$u_d(t) = u_d(k\delta) = u_{d,k}, \quad u_q(t) = u_q(k\delta) = u_{q,k}$$

are constant for $t \in [k\delta, (k+1)\delta)$, and assuming that also $C_l(t) = C_l(k\delta) = C_{l,k}$ is constant over the same time interval, one works out

$$i_{d,k+1} = e^{\lambda_1 \delta} i_{d,k} - \frac{1 - e^{\lambda_1 \delta}}{\lambda_1 L_d} u_{d,k},$$

$$i_{q,k+1} = e^{\lambda_2 \delta} i_{q,k} - \frac{1 - e^{\lambda_2 \delta}}{\lambda_2 L_q} u_{q,k},$$

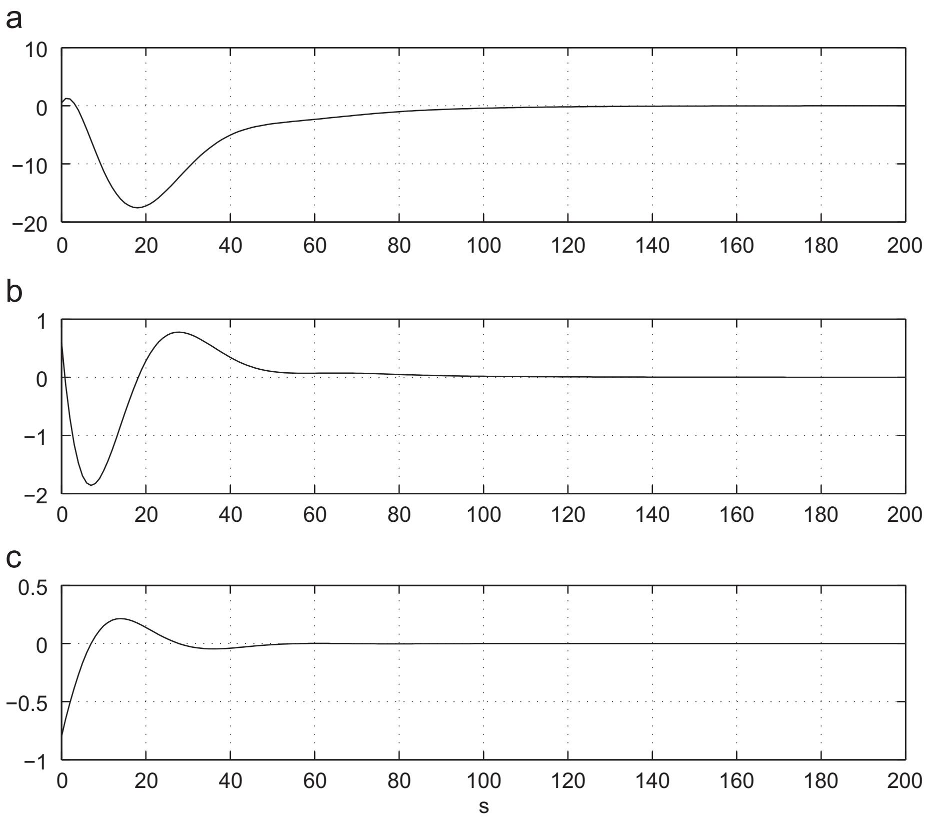

Fig. 2. Response of the discrete controller applied to the continuous system. (a) x_1; (b) x_2; (c) x_3.

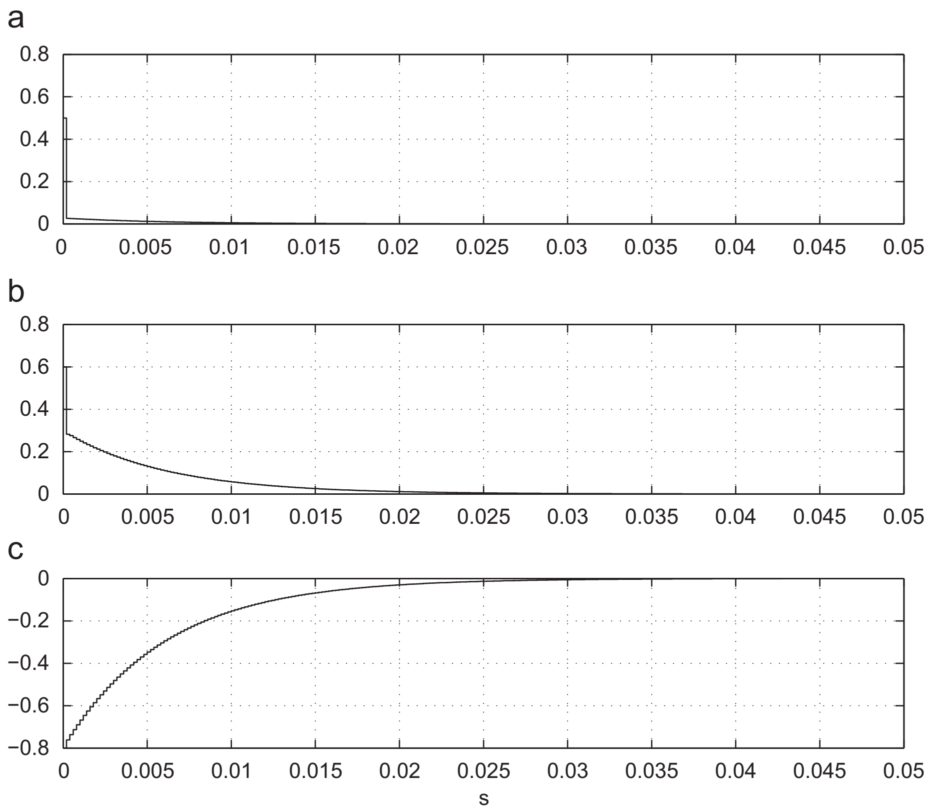

Fig. 3. Response of the discrete controller applied to the discretized system. (a) $i_{d,k}$ (A vs. s); (b) $i_{q,k}$ (A vs. s); (c) ω_k (rad/s vs. s).

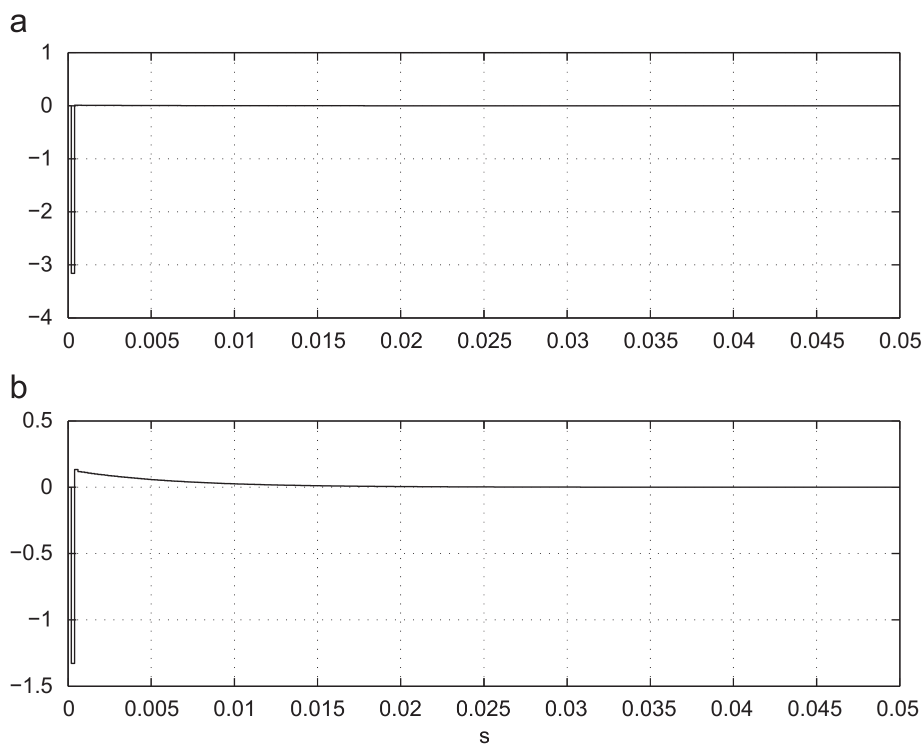

Fig. 4. Inputs of the discrete controller applied to the discretized system. (a) $v_{d,k}$ (V vs. s); (b) $v_{q,k}$ (V vs. s).

$$\omega_{k+1} = e^{\lambda_3\delta}\omega_k + \frac{k_m}{J}\int_{k\delta}^{(k+1)\delta} e^{\lambda_3((k+1)\delta-\xi)} i_q(\xi)\,d\xi + p\frac{L_d - L_q}{J}\int_{k\delta}^{(k+1)\delta} e^{\lambda_3((k+1)\delta-\xi)} i_d(\xi) i_q(\xi)\,d\xi - \frac{1}{J}\int_{k\delta}^{(k+1)\delta} e^{\lambda_3((k+1)\delta-\xi)} C_l(\xi)\,d\xi$$

and after tedious calculations

$$i_{d,k+1} = a_{11} i_{d,k} + b_1 u_{d,k},$$

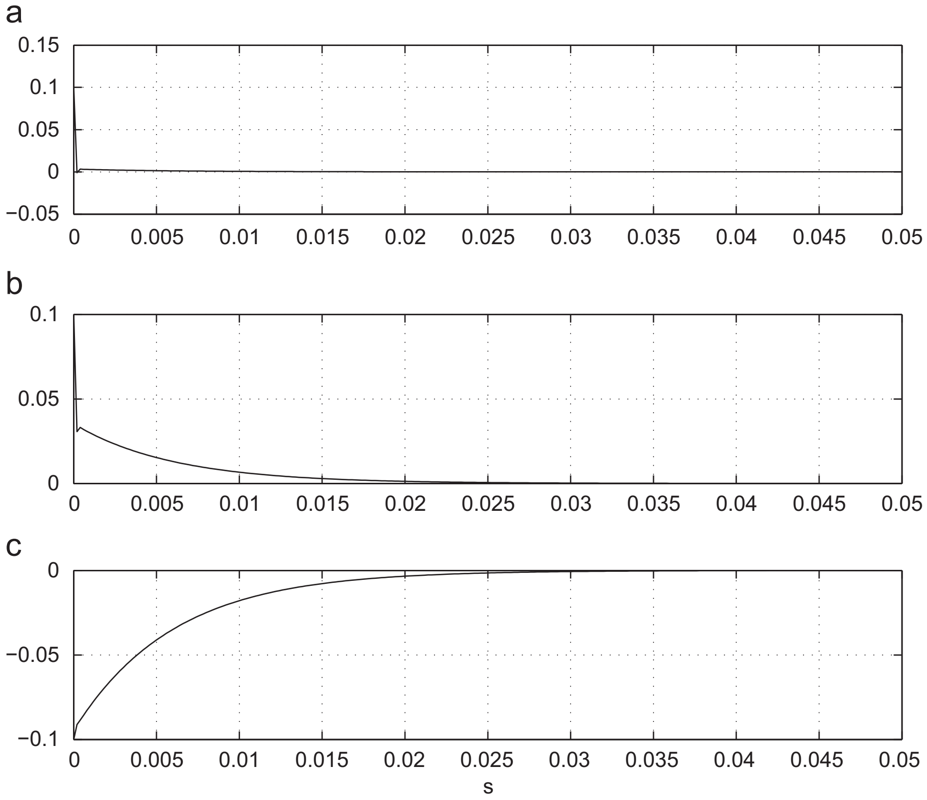

Fig. 5. Response of the discrete controller applied to the continuous system. (a) i_d (A vs. s); (b) i_q (A vs. s); (c) ω (rad/s vs. s).

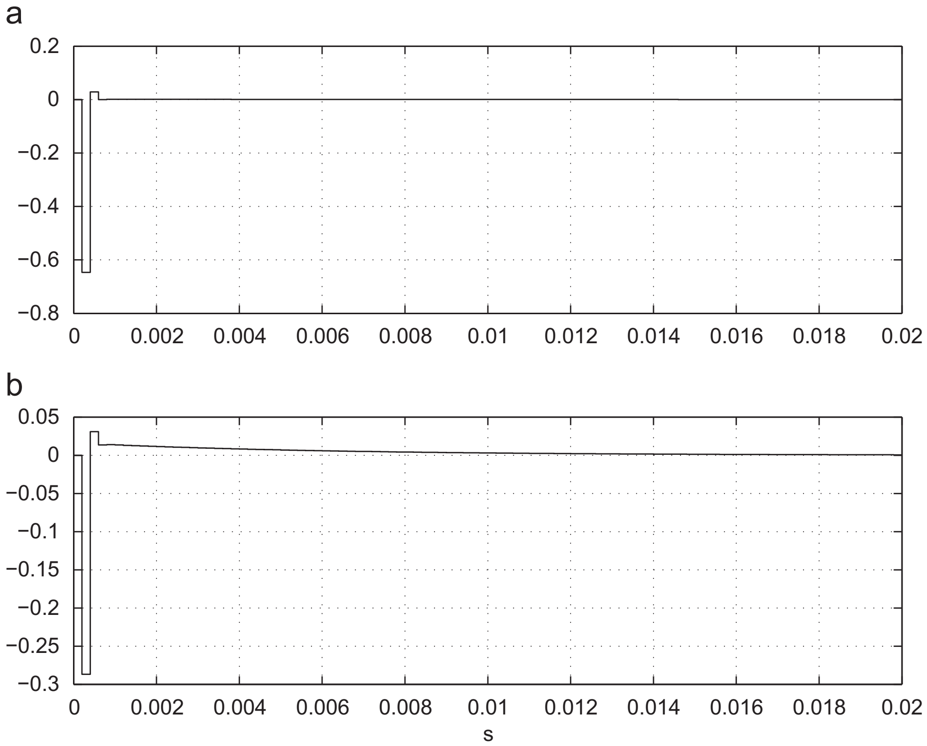

Fig. 6. Inputs of the discrete controller applied to the continuous system. (a) v_d (V vs. s); (b) v_q (V vs. s).

$$i_{q,k+1} = a_{22} i_{q,k} + b_2 u_{q,k},$$

$$\omega_{k+1} = a_{32} i_{q,k} + a_{33}\omega_k + a_{34} i_{d,k} i_{q,k} + b_3 u_{q,k} + c_1 i_{d,k} u_{q,k} + c_2 i_{q,k} u_{d,k} + c_3 u_{d,k} u_{q,k} - b_0 C_{l,k},$$

where

$$a_{11} = e^{\lambda_1 \delta}, \quad b_0 = -\frac{1-e^{\lambda_3 \delta}}{\lambda_3 J},$$

$$a_{22}=e^{\lambda_2\delta},\quad b_1=-\frac{1-e^{\lambda_1\delta}}{\lambda_1 L_d},$$

$$a_{33}=e^{\lambda_3\delta},\quad b_2=-\frac{1-e^{\lambda_2\delta}}{\lambda_2 L_q},$$

$$a_{32}=\frac{k_m}{J}d_{23},\quad b_3=\frac{k_m}{J}\frac{1}{\lambda_2 L_q}(d_{23}+d_{30}),$$

$$a_{34}=p\frac{L_d-L_q}{J}\,d_1,$$

$$c_1=p\frac{L_d-L_q}{J}\;\frac{1}{\lambda_2 L_q}(d_1-d_{13}),$$

$$c_2=p\frac{L_d-L_q}{J}\;\frac{1}{\lambda_1 L_d}(d_1-d_{23}),$$

$$c_3=p\frac{L_d-L_q}{J}\;\frac{1}{\lambda_1\lambda_2 L_d L_q}(d_1-d_{13}-d_{23}-d_{30}),$$

$$d_1=\frac{e^{(\lambda_1+\lambda_2)\delta}-e^{\lambda_3\delta}}{\lambda_1+\lambda_2-\lambda_3},\quad d_{13}=\frac{e^{\lambda_1\delta}-e^{\lambda_3\delta}}{\lambda_1-\lambda_3},$$

$$d_{23}=\frac{e^{\lambda_2\delta}-e^{\lambda_3\delta}}{\lambda_2-\lambda_3},\quad d_{30}=\frac{1-e^{\lambda_3\delta}}{\lambda_3}.$$

The PM synchronous motor parameters used in the simulations are the following:

$$R_s=0.6\,\Omega,\quad F=0.0014\,\mathrm{N\,m\,s},$$

$$J=0.0011\,\mathrm{kg\,m^2},\quad L_d=0.0014\,\mathrm{H},$$

$$L_q=0.0011\,\mathrm{H},\quad p=4,$$

$$k_m=0.48\,\mathrm{Wb}.$$

The load torque is assumed constant and equal to $C_l=14.4\,\mathrm{N\,m}$. The references considered are $i_{dr}=0$, $\omega_r=80\,\mathrm{rad/s}$. Let us define

$$z_1=i_{q,k}\in(0,40),\quad z_2=u_{q,k}\in(0,90)$$

and

$$z_1^-=\min z_1=0,\quad z_1^+=\max z_1=40,$$
$$z_2^-=\min z_2=0,\quad z_2^+=\max z_2=90.$$

The membership functions for each fuzzy set are

$$M_1(z_1)=\frac{z_1-z_1^-}{z_1^+-z_1^-},\quad M_2(z_1)=\frac{-z_1+z_1^+}{z_1^+-z_1^-},$$

$$N_1(z_2)=\frac{z_2-z_2^-}{z_2^+-z_2^-},\quad N_2(z_2)=\frac{-z_2+z_2^+}{z_2^+-z_2^-},$$

while the membership functions for each subsystem are

$$\mu_1(z)=M_1(z_1)\times N_1(z_2)\to(A_1^d,B_1^d),$$
$$\mu_2(z)=M_1(z_1)\times N_2(z_2)\to(A_2^d,B_2^d),$$
$$\mu_3(z)=M_2(z_1)\times N_1(z_2)\to(A_3^d,B_3^d),$$
$$\mu_4(z)=M_2(z_1)\times N_2(z_2)\to(A_4^d,B_4^d).$$

Hence, one may take

$$A=\begin{pmatrix}a_{11} & 0 & 0\\ 0 & a_{22} & 0\\ a_{34}z_1+c_1z_2 & a_{32} & a_{33}\end{pmatrix},\quad B=\begin{pmatrix}b_1 & 0\\ 0 & b_2\\ c_2z_1+c_3z_2 & b_3\end{pmatrix}$$

and

$$A_1^d=A(z_1^+,z_2^+),\quad A_2^d=A(z_1^+,z_2^-),\quad A_3^d=A(z_1^-,z_2^+),\quad A_4^d=A(z_1^-,z_2^-),$$
$$B_1^d=B(z_1^+,z_2^+),\quad B_2^d=B(z_1^+,z_2^-),\quad B_3^d=B(z_1^-,z_2^+),\quad B_4^d=B(z_1^-,z_2^-).$$

The gains are obtained as

$$K_1^d=\begin{pmatrix}6.7035 & -2.5559 & -0.0306\\ 0.0011 & 10.0297 & 0.0638\end{pmatrix},$$
$$K_2^d=\begin{pmatrix}6.7040 & -1.4690 & -0.0176\\ 0.0007 & 10.01920 & 0.0636\end{pmatrix},$$

$$K_3^d=\begin{pmatrix}6.7041 & -2.4991 & -0.0299\\ 0.0003 & 10.0307 & 0.0638\end{pmatrix},$$
$$K_4^d=\begin{pmatrix}6.7040 & -3.3429 & -0.0400\\ 0.0001 & 10.0307 & 0.0638\end{pmatrix}.$$

The initial conditions are $i_d(0)=i_{d,0}=0.1\,\mathrm{A}$, $i_q(0)=i_{q,0}=0.1\,\mathrm{A}$, $\omega(0)=\omega_0=-0.1\,\mathrm{rad/s}$. Figs. 3 and 4 show the simulation results obtained from the application of the discrete controller of the discretized model. The hybrid controller calculated on the basis of the discretized Takagi–Sugeno fuzzy model is then applied to the continuous time models. Figs. 5 and 6 show that this controller indeed stabilizes the continuous model.

We conclude this section pointing out that other approaches have been proposed in the literature for the design of digital controllers for synchronous motors, on the basis of exact or approximate discretization of the continuous time nonlinear system (see Georgiou et al., 1992; Chelouah et al., 1993; Di Gennaro and Tursini, 1994; Monaco et al., 1997; Monaco and Normand-Cyrot, 1997; Castillo-Toledo, 2008, among the others). However, the main advantage of the proposed controller is the simplicity of the design. In fact, the digital controller results the fuzzy aggregation of the digital controllers, each designed for a digital linear system of the TS model.

6. Conclusions

In this paper a scheme that guarantees the global stabilization of a continuous time nonlinear system by means of a digital controller has been proposed. This scheme is based on the existence of a discretization in closed form and an exact discrete Takagi–Sugeno fuzzy model for which a discrete stabilizer is calculated. This result can be extended to the class of nonlinear systems whose model can be transformed by feedback to a description which may be discretized in closed form. The proposed scheme can be seen as an extension of the well known result for linear time invariant systems and may be viewed as a possible way of stabilizing nonlinear systems by a digital controller calculated on the basis of a discrete Takagi–Sugeno fuzzy model. The application to an illustrative example and to an synchronous motor suggests the effectiveness of the proposed control scheme.

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