Adaptive integrated vehicle control using active front steering and rear torque vectoring

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Abstract: This work studies the combination of Active Front Steering (AFS) with Rear Torque Vectoring (RTV) actuators in an integrated controller to guarantee vehicle stability. Adaptive feedback technique has been used to design the controller. The feedback linearisation is applied to cancel the nonlinearities in the input–output dynamics of the vehicle. Parameter adaptation then is used to robustify the exact cancellation of the nonlinear terms. The results show tracking and stabilisation capabilities when important parameters, like tyre stiffness and tyre characteristics, are affected by estimation errors.

Keywords: integrated vehicle control; AFS; active front steering; RTV; rear torque vectoring; non-linear control; adaptive control.

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Biographical notes: Domenico Bianchi followed a course in scientific studies and took a degree and Master Degree from University of L'Aquila in Computer Science and Automatic Control. In his thesis he evaluated the possibility of combining active differential with active front steering actuators in an integrated controller to improve vehicle stability and limit

handling capability, with less need for the brakes based controller support. After a short work period in ICT field, he started PhD in 2008 in the Department of Electrical Engineering and Computer Science in L'Aquila. His fields of research interest are vehicle dynamics control, control of hybrid electric vehicles and applications of nonlinear control.

Alessandro Borri received his Bachelor of Engineering degree and his Master of Engineering degree in Control Systems and Computer Science, from the University of L'Aquila (Italy), in 2004 and 2007, respectively. He spent six months in 2006–2007 at the Ford Research Center Aachen (Germany), working on Vehicle Control. In the academic year 2008–2009, he also spent three months as a Visiting Research Student at the University of California, Berkeley (USA), working on hybrid modelling of two-people interaction. At present, he is a PhD Student at the University of L'Aquila, Italy (advisor: Prof. Maria Domenica Di Benedetto). His research interests focus on modelling and control of nonlinear and hybrid systems. Applications of his work are in vehicle control, cognitive radio networks, body sensor networks.

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Gilberto Burgio received his Master in Electronic Engineering in 1992 from the University of Rome 'La Sapienza'. He joined Magneti Marelli Powertrain (Fiat Group) in Bologna where he has been working in control algorithms for gasoline engines. In 2001 he joined the Ford Research Center in Aachen where he is currently working as Researcher in the area of vehicle dynamics controls development.

1 Introduction

Today's vehicles are equipped with many electronically controlled systems, whose integration is rising in complexity with the increasing number of available customer features and technologies. A way to solve the integration problem can be the introduction of a hierarchical control structure, where all control commands are computed in parallel in one core algorithm, and where the control has to take into account the interactions among the vehicle subsystems, driver and vehicle.

The key element of integrated vehicle control is that the behaviour of the various vehicle subsystems has to be coordinated, i.e., subsystems have to behave as cooperatively as possible in performing the desired vehicle functions. Clearly, the fully integrated controller will be more complex than the sum of the stand alone ones, but it will guarantee increased performance and robustness.

Active Safety Systems Integration is one of the main research topics in vehicle control area. In order to maintain safe handling characteristics of the vehicle, several active system technologies (active braking, active steering, active differential, active suspension, etc.) have been developed. All these technologies modify the vehicle dynamics imposing forces or moments to the vehicle body, which can be generated in different ways (see e.g., Burgio and Zegelaar, 2006; Karbalaei et al., 2007; Baslamisli et al., 2007; Ackermann et al., 1995; Malan et al., 1994). An important design factor to be considered in the standalone or integrated controller design is the actuator saturations, which limits the maximum obtainable performance. In an integrated control structure more power is available for control, thus potentially limiting the saturation occurrences.

In Burgio and Zegelaar (2006), a fully integrated vehicle controller with steering and brakes is proposed, using the exact feedback linearisation method. The feedback linearisation control method amounts to canceling the nonlinearities in a nonlinear system so that the closed–loop dynamics is in a linear form. The main drawback of this methodology is that performance deteriorates in the presence of parameter uncertainty or unmodelled dynamics.

Other works on integrated active chassis systems propose integrated control systems with AFS and direct yaw moment control based on fuzzy logic control (Karbalaei et al., 2007). In Baslamisli et al. (2007), an active steering control design method is proposed in order to preserve vehicle stability in extreme handling situations. Active chassis control systems, guaranteeing the stability performance in the presence of parameter uncertainty/variation, can be found in Ackermann et al. (1995) and Malan et al. (1994), where issues on robustness of active steering systems are addressed.

In this paper we propose an adaptive integrated chassis control for a rear–wheel drive vehicle, equipped with Active Front Steering (AFS) and Rear Torque Vectoring (RTV) devices, in the presence of parameter uncertainties. While the AFS provides additional steering angle over the driver defined one, the RTV gives asymmetric left/right wheel torque on the rear axle. The uncertain parameters are the lateral tyre stiffness.

The adaptive linearising controller is designed as follows. Following Borri (2007) and Bianchi et al. (2009), we first design an adaptive feedback linearisation control of the vehicle dynamics. Then, making use of a classical control scheme, we will impose this control as a reference to the actuators, which actively impose the linearising control. The advantage of the proposed control scheme with respect to a linearising of the whole dynamics (vehicle plus actuator) is, essentially, that it is less complex to implement. This results in an easier control structure without the need of the so–called overparametrisation. Moreover, it is not necessary to measure or evaluate the derivative of the wheel angle imposed by the driver.

Another goal of the paper is to evaluate the possibility of combining AFS with RTV to improve vehicle stability in a variety of situations, not only in the presence of deviation of the vehicle parameters from the nominal values, but also in situations of rapid variations of road conditions (dry, wet, iced). This is accomplished by using the adaptive feedback linearisation technique, where parameter adaptation is used to robustify the exact cancellation of nonlinear terms.

The paper is organised as follows. In Section 2, the mathematical model of a vehicle is recalled, and the control problem is set. In Section 3, the adaptive feedback linearisation technique is applied, and the adaptive controller is designed. In Section 4, the proposed controller is tested with simulations. Some comments conclude the paper.

2 Problem setting

The vehicle motion can be in general described as a rigid body moving in the free space, with 6 degrees of freedom, connected with the ground surface through tyres and suspensions. This results in a model with high non-linear behaviour and high coupling effects. For simplicity, we consider the model of a rear–wheel driven vehicle. The actuators considered in this work are

- Active Front Steer (AFS), which imposes an incremental steer angle on top of the driver's input. The control is then actuated through the front axle tyre characteristic.
- Rear Torque Vectoring (RTV), which distributes the torque in the rear axle, usually to improve vehicle traction, handling and stability. The control is then actuated through the rear axle tyre characteristics.

The mathematical model is derived under the following assumptions, which are verified in a large number of situations and which mitigate the complexity of the vehicle dynamics

- The vehicle moves on a horizontal plane
- The longitudinal velocity is constant, so that vehicle shaking/pitch motions can be neglected
- The vehicle has stiff suspensions, so that the vehicle roll can be neglected
- The steering system is rigid, so that the angular position of the front wheels is uniquely determined by the steering wheel position
- The wheels masses are much lower than the vehicle one, so the steering action does not affect the position of the centre of mass of the entire vehicle
- The vehicle takes large radius bends and the road wheel angles are 'small' (less than 10°), i.e., the bend curve radius is much higher than the vehicle width
- The aerodynamic resistance and the wind lateral thrust are not considered
- The tyre vertical loads are constant
- The actuators are ideally modelled.

As a consequence of the previous assumptions, the height of each point of the vehicle is kept constant, and the vehicle motion is planar. The resulting model has two degrees of freedom

$$\begin{aligned} m(\dot{v}_y + v_x\omega_z) &= F_{y,f} + F_{y,r} \\ J\dot{\omega}_z &= F_{y,f}l_f - F_{y,r}l_r + M_z \end{aligned} \tag{1}$$

where the state variables are the vehicle lateral velocity v_y (m/s), and the vehicle yaw velocity ω_z (rad/s), and where we have denoted

m	Vehicle mass (kg)
J	Vehicle inertia momentum (kg m^2)
v_x	Vehicle longitudinal velocity (m/s)
l_f, l_r	Distance from the centre of gravity to the front and rear axle (m)
$F_{y,f}, F_{y,r}$	Front, rear tyre lateral forces (N)
M_z	RTV yaw moment (N m).

It is worth noting that the simplification of using a single track model of the form (1), without the rolling dynamics, to approach the design of a handling and stability controller is very used in the literature and in the industry applications. This is justified by the fact that, despite its simplicity, it well captures the real vehicle major characteristics of interest for the controller design (like the steady state and dynamic responses of the yaw rate, lateral acceleration, lateral velocity). Obviously, model (1) with the front/rear axle force characteristics, must be fitted to the vehicle experimental data.

The front/rear lateral forces

$$F_{y,f} = F_{y,f}(\alpha_f), \quad F_{y,r} = F_{y,r}(\alpha_r)$$

depend on the front/rear tyre slip angles (rad)

$$\alpha_f = \delta - \frac{v_y + l_f\omega_z}{v_x}, \quad \alpha_r = -\frac{v_y - l_r\omega_z}{v_x}$$

with $\delta = \delta_c + \delta_d$ the road wheel angle (rad), sum of the AFS angle δ_c (rad) and the driver angle δ_d (rad).

The tyre lateral behaviour can be represented by some functions describing the dependence on the slip angle (see, for instance Pacejka, 2005). The tyres determine a force in the direction of the slip angles, which contrasts the drift of the wheel, but this force decreases after a certain value of the slip angle, as in the following function

$$\begin{aligned} F_{y,f}(\alpha_f) &= C_{y,f}F_{yn,f}(\alpha_f) \\ F_{y,r}(\alpha_r) &= C_{y,r}F_{yn,r}(\alpha_r) \end{aligned}$$

where

$$\begin{aligned} F_{yn,f}(\alpha_f) &= \sin(A_{y,f}\arctan(B_{y,f}\alpha_f)) \\ F_{yn,r}(\alpha_r) &= \sin(A_{y,r}\arctan(B_{y,r}\alpha_r)) \end{aligned}$$

are normalised tyre functions, and with $A_{y,f}$, $B_{y,f}$, $C_{y,f}$, $A_{y,r}$, $B_{y,r}$, $C_{y,r}$ positive experimental parameters. Note that the normalised functions $F_{yn,f}(\alpha_f)$, $F_{yn,r}(\alpha_r)$ take into account the variation of the local tyre stiffness with the slip angles, which is therefore considered in the control design.

The inputs of the vehicle dynamics (1) are the AFS angle δ_c and the RTV yaw moment M_z. The former can be computed inverting the function $F_{y,f}$. This can be done up to the tyre saturation point $\alpha_{f,\mathrm{sat}}$ and saturating the inverse function elsewhere, i.e.,

$$\delta_c = \begin{cases} -\delta_d + \dfrac{v_y + l_f\omega_z}{v_x} + F_{y,f}^{-1}(F_0), & |F_0| \leq F_{y,f}(\alpha_{f,\mathrm{sat}}) \\ -\delta_d + \dfrac{v_y + l_f\omega_z}{v_x} + \alpha_{f,\mathrm{sat}}, & \text{otherwise} \end{cases} \tag{2}$$

for a given value F_0. This allows considering as control input the difference

$$\Delta_{y,f} = F_{yn,f}(\alpha_f) - F_{yn,f}(\alpha_{f,d})$$
$$\alpha_{f,d} = \delta_d - \frac{v_y + l_f\omega_z}{v_x}$$

instead of δ_c, so that equations (1) become

$$\begin{aligned} m(\dot{v}_y + v_x\omega_z) &= C_{y,f}F_{yn,f}(\alpha_f) + C_{y,r}F_{yn,r} + C_{y,f}\Delta_{y,f} \\ J\dot{\omega}_z &= C_{y,f}F_{yn,f}(\alpha_f)l_f - C_{y,r}F_{yn,r}l_r + C_{y,f}l_f\Delta_{y,f} + M_z. \end{aligned} \tag{3}$$

Therefore, the vehicle dynamics are

$$\begin{aligned} \dot{v}_y &= -v_x\omega_z + \frac{1}{m}\Big(C_{y,f}F_{yn,f}(\delta_d, v_y, \omega_z) + C_{y,r}F_{yn,r}(v_y, \omega_z)\Big) + \frac{1}{m}C_{y,f}\Delta_{y,f} \\ \dot{\omega}_z &= \frac{1}{J}\Big(C_{y,f}F_{yn,f}(\delta_d, v_y, \omega_z)l_f - C_{y,r}F_{yn,r}(v_y, \omega_z)l_r\Big) \\ &\quad + \frac{1}{J}C_{y,f}l_f\Delta_{y,f} + \frac{1}{J}M_z. \end{aligned} \tag{4}$$

In $F_{yn,f}, F_{yn,r}$ we put in evidence the dependence on v_y, ω_z, and $\delta_d(t)$.

The AFS and the RTV are imposed by means of appropriate actuators. We have already noted that the AFS is actuated through the front axle tyres characteristic, while the RTV is actuated through the rear axle tyres characteristic. The AFS actuator can be modelled by a simple first-order equation

$$\dot{\delta}_c = -\frac{1}{\tau_m}\delta_c + \frac{1}{\tau_m}\delta_m \tag{5}$$

where τ_m is the actuator time constant (s), and δ_m is its input (rad).

The RTV is imposed by means of rear left and right tyre longitudinal slips

$$k_{r,l} = \frac{\omega_{r,l}r_w - v_x}{v_x}, \quad k_{r,r} = \frac{\omega_{r,r}r_w - v_x}{v_x}$$

with

$$J_w \dot{\omega}_{r,l} = T_{r,l} - T_a - r_w F_{x,rl}$$
$$J_w \dot{\omega}_{r,r} = T_{r,r} + T_a - r_w F_{x,rr}$$

the rear left/right wheel dynamics, and

r_w, J_w	Wheel radius (m), inertia momentum (kg m^2)
$F_{x,rl}, F_{x,rr}$	Rear left/right tyre longitudinal forces (N)
T_a	Wheel differential torque (N m)
$T_{r,l}, T_{r,r}$	Left/right wheel tractions (N m)
ω_r	Rear wheel angular velocity (rad/s).

In the following we assume v_x constant, so that $T_{r,l} \simeq 0$, $T_{r,r} \simeq 0$. Hence, since $F_{x,rl}, F_{x,rr}$ are odd functions, it is easy to show that $k_{r,l} = -k_{r,r}$. Finally, denoting $k_r = k_{r,l}$, and $F_{x,r}(\alpha_r, k_r)$ the rear tyre longitudinal force, we obtain the following simplified RTV dynamics

$$\begin{aligned} \dot{k}_r &= -\frac{r_w^2}{J_w v_x} F_{x,r}(\alpha_r, k_r) + \frac{r_w}{J_w v_x} T_a \\ M_z &= F_{x,r}(\alpha_r, k_r) t_v \end{aligned} \tag{6}$$

and with t_v the vehicle track. Note that in steady state, $T_a = r_w M_z / t_v$. The combined tyre characteristic $F_{x,r}(\alpha_r, k_r)$ is assumed to admit the following factorisation

$$F_{x,r}(\alpha_r, k_r) = p_x(\alpha_r)\varphi_x(k_r)$$

with $\varphi_x(k_r) = F_{x,r}(0, k_r)$ given by

$$\varphi_x(k_r) = C_{x,r} F_{xn,r}(k_r), \quad F_{xn,r} = \sin(A_{x,r} \arctan(B_{x,r} k_r))$$

and where $p_x(\alpha_r)$ is a penality function. Hence, equation (6) can be rewritten as follows

$$\begin{aligned} \dot{k}_r &= -\frac{C_{x,r} r_w^2}{J_w v_x} p_x(v_y, \omega_z) F_{xn,r}(k_r) + \frac{r_w}{J_w v_x} T_a \\ M_z &= C_{x,r} p_x(v_y, \omega_z) F_{xn,r}(k_r) t_v \end{aligned} \tag{7}$$

where the dependence on v_y, ω_z in p_x has been evidenced.

Equations (5) and (7) constitute the actuator dynamics which, along with equation (4), determine the mathematical model of the system.

The control problem is the following: Given the system (4), (5), (7), and two target functions $v_{y,\text{ref}}(t)$, $\omega_{z,\text{ref}}(t)$ for lateral velocity and yaw rate, find a control law ensuring global asymptotic tracking, i.e.

$$\lim_{t\to\infty} v_y = v_{y,\text{ref}}, \quad \lim_{t\to\infty} \omega_z = \omega_{z,\text{ref}}$$

for all initial conditions $v_y(0)$, $\omega_z(0)$, and in presence of uncertainties on the parameters $C_{y,f}$, $C_{y,r}$.

The target or reference signals $v_{y,\mathrm{ref}}(t)$, $\omega_{z,\mathrm{ref}}(t)$ are desired behaviours of the vehicle, and are bounded functions with bounded derivatives. In the following we will assume that also the driver angle δ_d is bounded.

3 Adaptive feedback linearisation

The parameters of the vehicle dynamics that can be known with uncertainty, or may be subject to variations are the stiffness $C_{y,f}$, $C_{y,r}$ of the front/rear tyres.

In this section we first design an adaptive feedback linearisation control of the dynamics (4). Then, we will impose this control as reference for the actuator dynamics.

3.1 Adaptive feedback linearisation of the vehicle dynamics

Introducing the parameters

$$\theta_1 = C_{y,f}, \qquad \theta_2 = C_{y,r}$$

equations (4) rewrite

$$\begin{aligned} \dot{v}_y &= -v_x\omega_z + \frac{\theta_1}{m}F_{yn,f}(\delta_d, v_y, \omega_z) + \frac{\theta_2}{m}F_{yn,r}(v_y, \omega_z) + \frac{\theta_1}{m}\Delta_{y,f} \\ \dot{\omega}_z &= \frac{\theta_1 l_f}{J}F_{yn,f}(\delta_d, v_y, \omega_z) - \frac{\theta_2 l_r}{J}F_{yn,r}(v_y, \omega_z) + \frac{\theta_1 l_f}{J}\Delta_{y,f} + \frac{1}{J}M_z. \end{aligned} \tag{8}$$

or, equivalently,

$$\begin{pmatrix} \dot{v}_y \\ \dot{\omega}_z \end{pmatrix} = f_0(v_y, \omega_z) + f_1^T(t, v_y, \omega_z)\theta + B(\theta)\begin{pmatrix} \Delta_{y,f} \\ M_z \end{pmatrix} \tag{9}$$

where

$$f_0(v_y, \omega_z) = \begin{pmatrix} -v_x\omega_z \\ 0 \end{pmatrix}, \quad B(\theta) = \begin{pmatrix} \frac{1}{m}\theta_1 & 0 \\ \frac{l_f}{J}\theta_1 & \frac{1}{J} \end{pmatrix}$$

$$f_1^T(t, v_y, \omega_z) = \begin{pmatrix} \frac{1}{m}F_{yn,f} & \frac{1}{m}F_{yn,r} \\ \frac{l_f}{J}F_{yn,f} & -\frac{l_r}{J}F_{yn,r} \end{pmatrix}, \quad \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \in \mathbb{R}^2.$$

If θ is known, the law which linearises and decouples the vehicle dynamics (8) is (Isidori, 1995)

$$\begin{pmatrix} \Delta_{y,f}(t, v_y, \omega_z, \theta) \\ M_z(t, v_y, \omega_z, \theta) \end{pmatrix} = \Big(B(\theta)\Big)^{-1}\Big(v - f_0(v_y, \omega_z) - f_1(t, v_y, \omega_z)\theta\Big) \tag{10}$$

yields to the linear input–state dynamics

$$\begin{pmatrix} \dot{v}_y \\ \dot{\omega}_z \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v$$

where the new input v can be chosen as

$$\begin{aligned} v &= \begin{pmatrix} \dot{v}_{y,\mathrm{ref}} \\ \dot{\omega}_{z,\mathrm{ref}} \end{pmatrix} + Ae \\ A &= \begin{pmatrix} -k_{p1} & 0 \\ 0 & -k_{p2} \end{pmatrix}, \quad e = \begin{pmatrix} v_y - v_{y,\mathrm{ref}} \\ \omega_z - \omega_{z,\mathrm{ref}} \end{pmatrix} \end{aligned} \tag{11}$$

$k_{p1}, k_{p2} > 0$, ensuring the exponential tracking of the bounded references with bounded derivatives. Note that A is Hurwitz. Note also that the control (10) exists when $\theta_1/(mJ) \neq 0$, i.e., when, $C_{y,f} \neq 0$, as in real cases if the front axle is not saturating. Note that, up to now, no limitation is assumed on the RTV actuation M_z.

If θ is uncertain, the adaptive linearisation technique can be exploited (Sastry and Isidori, 1989), in which θ is replaced with its estimate $\hat{\theta}$, yielding to the control

$$\begin{aligned} \begin{pmatrix} \Delta_{y,f}(t, v_y, \omega_z, \hat{\theta}) \\ M_z(t, v_y, \omega_z, \hat{\theta}) \end{pmatrix} &= \begin{pmatrix} \hat{\Delta}_{y,f,\mathrm{ref}} \\ \hat{M}_{z,\mathrm{ref}} \end{pmatrix} \\ &= \left(B(\hat{\theta})\right)^{-1} \left(v - f_0(v_y, \omega_z) - f_1(t, v_y, \omega_z)\hat{\theta}\right) \end{aligned} \tag{12}$$

and an appropriate adaption rule $\dot{\hat{\theta}}$ is designed. Since the control (12) exists when $\hat{\theta}_1/(mJ) \neq 0$, such an adaptation rule has to constrain these estimations to positive values. This mathematical condition does not have a direct correspondence on the estimates of the real parameters, $C_{y,f}$, $C_{y,r}$ since from $\hat{\theta}_1$, $\hat{\theta}_2$ it is not possible to determine univocally estimations of the real parameters.

The adaption scheme will improve the performance of equation (10) in the presence of parameter uncertainty, ensuring the asymptotic stability of the tracking errors, as shown in what follows. This adaption scheme will be designed under the standard hypothesis that the $\dot{\theta} = 0$, namely the parameters may be piece–wise constant or vary slowly with respect to the plant dynamics. Since

$$v = f_0(v_y, \omega_z) + f_1(t, v_y, \omega_z)\hat{\theta} + B(\hat{\theta}) \begin{pmatrix} \hat{\Delta}_{y,f,\mathrm{ref}} \\ \hat{M}_{z,\mathrm{ref}} \end{pmatrix}$$

using equation (12) in equation (9) one gets

$$\begin{aligned} \begin{pmatrix} \dot{v}_y \\ \dot{\omega}_z \end{pmatrix} &= v + f_1^T(t, v_y, \omega_z)\tilde{\theta} + \left(B(\theta) - B(\hat{\theta})\right) \begin{pmatrix} \hat{\Delta}_{y,f,\mathrm{ref}} \\ \hat{M}_{z,\mathrm{ref}} \end{pmatrix} \\ &= v + \Lambda(t, v_y, \omega_z, \hat{\theta})\tilde{\theta} \end{aligned}$$

where $\tilde{\theta} = \theta - \hat{\theta}$, and

$$\Lambda(t, v_y, \omega_z, \hat{\theta}) = \begin{pmatrix} \overset{\lambda_{11}}{\dfrac{1}{m} F_{yn,r}(v_y, \omega_z)} & \overset{\lambda_{12}}{-\dfrac{l_r}{J} F_{yn,r}(v_y, \omega_z)} \end{pmatrix}$$

$$\lambda_{11} = \frac{1}{m}\big(F_{yn,f}(\delta_d, v_y, \omega_z) + \hat{\Delta}_{y,f,\mathrm{ref}}\big)$$
$$\lambda_{12} = \frac{l_f}{J}\big(F_{yn,f}(\delta_d, v_y, \omega_z) + \hat{\Delta}_{y,f,\mathrm{ref}}\big).$$

Considering the expression (11) of the new input, we can rewrite

$$\dot{e} = Ae + \Lambda(t, v_y, \omega_z, \hat{\theta})\tilde{\theta}. \tag{13}$$

To study the stability of the origin of the feedback system (9), (12), (11), let us consider the candidate Lyapunov function

$$V(e, \tilde{\theta}) = \frac{1}{2}e^T P e + \frac{1}{2}\tilde{\theta}^T \Gamma^{-1}\tilde{\theta}$$

with $P = P^T > 0$ solution of the equation

$$PA + A^T P = -2Q$$

for a chosen $Q = Q^T > 0$. Differentiating along the trajectories of the system, one gets

$$\dot{V} = e^T P\dot{e} + \tilde{\theta}^T\Gamma^{-1}\dot{\tilde{\theta}} = -e^T Qe + \tilde{\theta}^T\left(\Lambda^T(t, v_y, \omega_z, \hat{\theta})Pe - \Gamma^{-1}\dot{\hat{\theta}}\right)$$

since $\dot{\theta} = 0$. Therefore, setting

$$\dot{\hat{\theta}} = \Gamma\Lambda^T(t, v_y, \omega_z, \hat{\theta})Pe$$

one gets

$$\dot{V} = -e^T Qe \leq -\lambda^Q_{\min}\|e\|^2 \tag{14}$$

with $\lambda^Q_{\min}$ the minimum eigenvalue of Q. This ensures that the tracking error e and the parameter error $\tilde{\theta}$ are bounded, i.e., $e, \tilde{\theta} \in L_\infty$.

To prove the convergence of e to zero, integrating equation (14) from t_0 to t one obtains

$$V(t) - V(t_0) \leq -\lambda^Q_{\min}\int_{t_0}^{t}\|e(\tau)\|^2\,\mathrm{d}\tau$$

and since $V(t) > 0$

$$\int_{t_0}^{t}\|e(\tau)\|^2\,d\tau \leq \frac{V(t_0) - V(t)}{\lambda^Q_{\min}} \leq \frac{V(t_0)}{\lambda^Q_{\min}}.$$

Therefore, for $t \to \infty$

$$\int_{t_0}^{\infty}\|e(\tau)\|^2\,d\tau \leq \frac{V(t_0)}{\lambda^Q_{\min}}$$

i.e., $e \in L_2$. Finally, to prove that e is uniformly continuous, one notes that

$$\dot{e} = Ae + \Lambda(t, v_y, \omega_z, \hat{\theta})\tilde{\theta} \in L_\infty$$

since $e, \tilde{\theta} \in L_\infty$, as already noted, and δ_d, the references, and their derivatives are in L_∞, by assumption. Applying Barbalat lemma (Marino and Tomei, 1996), one deduces that

$$\lim_{t\to\infty} e = 0$$

i.e., the tracking error converges asymptotically to zero.

3.2 Design of an adaptive control scheme considering actuator dynamics

The inputs $\hat{\Delta}_{y,f,\mathrm{ref}}$, $\hat{M}_{z,\mathrm{ref}}$ given in equation (12) ensure, if applied to the dynamics (4), the exact asymptotic input–output linearisation and decoupling of these dynamics. However, to impose such inputs, we need to use the AFS and RTV actuators described by the dynamics (5), (7). A simple control scheme will be designed forcing $\Delta_{y,f}$, M_z to $\hat{\Delta}_{y,f,\mathrm{ref}}$, $\hat{M}_{z,\mathrm{ref}}$.

The design of the control scheme is obtained considering the following Lyapunov candidate

$$\mathcal{V} = V + \frac{1}{2}e_\delta^2 + \frac{1}{2}e_k^2$$

with

$$e_\delta = \delta_c - \delta_{c,\mathrm{ref}}, \quad e_k = k_r - k_{r,\mathrm{ref}}$$

and where the reference angle $\delta_{c,\mathrm{ref}}$ is obtained from

$$\delta_{c,\mathrm{ref}} = \left[\hat{\Delta}_{y,f,\mathrm{ref}} + F_{y,f}\left(\delta_d - \frac{v_y + l_f\omega_z}{v_x}\right)\right]^{-1} - \left(\delta_d - \frac{v_y + l_f\omega_z}{v_x}\right)$$

while the reference longitudinal slip is obtained from second equation of (7)

$$k_{r,\mathrm{ref}} = F_{xn,r}^{-1}\left(\frac{\hat{M}_{z,\mathrm{ref}}}{C_{x,r}p_x(v_y,\omega_z)t_v}\right).$$

Deriving $\mathcal{V}$ along the system dynamics, one obtains

$$\dot{\mathcal{V}} \le -\lambda_{\min}^Q\|e\|^2 - k_1e_\delta^2 - k_2e_k^2 + e_\delta\left(k_1e_\delta - \frac{1}{\tau_m}\delta_c + \frac{1}{\tau_m}\delta_m - \dot{\delta}_{c,\mathrm{ref}}\right)$$
$$+ e_k\left(k_2e_k - \frac{C_{x,r}r_w^2}{J_wv_x}p_x(v_y,\omega_z)F_{xn,r}(k_r) + \frac{r_w}{J_wv_x}T_a - \dot{k}_{r,\mathrm{ref}}\right).$$

Hence, setting

$$\delta_m = \delta_c + \tau_m\Big(- k_1(\delta_c - \delta_{c,\text{ref}}) + \dot{\delta}_{c,\text{ref}}\Big)$$
$$T_a = C_{x,r} r_w p_x(v_y,\omega_z) F_{xn,r}(k_r) + \frac{J_w v_x}{r_w}\Big(\dot{k}_{r,\text{ref}} - k_2(k_r - k_{r,\text{ref}})\Big) \tag{15}$$

one has

$$\dot{\mathcal{V}} \le -\lambda^Q_{\min}\|e\|^2 - k_1 e_\delta^2 - k_2 e_k^2.$$

To prove the convergence of e, e_δ, e_k to zero, with the same arguments of the previous section, one obtains

$$\int_{t_0}^{\infty} \|e(\tau)\|^2 \, d\tau + \int_{t_0}^{\infty} \|e_\delta(\tau)\|^2 \, d\tau + \int_{t_0}^{\infty} \|e_k(\tau)\|^2 \, d\tau \le \frac{\mathcal{V}(t_0)}{\ell_{\min}}$$

with $\ell_{\min} = \min\{\lambda^Q_{\min}, k_1, k_2\}$, i.e., $e, e_\delta, e_k \in L_2$. Finally, e, e_δ, e_k are uniformly continuous since, from equation (13) and from equations (5), (7) with equation (15), one has that $\dot{e}, \dot{e}_\delta, \dot{e}_k \in L_\infty$. Using Barbalat lemma (Marino and Tomei, 1996), one deduces that

$$\lim_{t\to\infty} e = 0, \quad \lim_{t\to\infty} e_\delta = 0, \quad \lim_{t\to\infty} e_k = 0.$$

In the remaining of this section we will show the advantages of the proposed control scheme with respect to an adaptive linearising control based on the whole dynamics, i.e. of the vehicle plus actuators, see equations (4), (5), (7)

$$\dot{v}_y = -v_x\omega_z + \frac{1}{m}\Big(\theta_1 F_{yn,f}(\delta_d, v_y, \omega_z) + \theta_2 F_{yn,r}(v_y,\omega_z)\Big) + \frac{\theta_1}{m}\Delta_{y,f}$$
$$\dot{\omega}_z = \frac{1}{J}\Big(\theta_1 F_{yn,f}(\delta_d, v_y,\omega_z) l_f - \theta_2 F_{yn,r}(v_y,\omega_z) l_r\Big) + \frac{\theta_1 l_f}{J}\Delta_{y,f} + \frac{C_{x,r} t_v}{J} p_x(v_y,\omega_z) F_{xn,r}(k_r)$$
$$\dot{\delta}_c = -\frac{1}{\tau_m}\delta_c + \frac{1}{\tau_m}\delta_m$$
$$\dot{k}_r = -\frac{C_{x,r} r_w^2}{J_w v_x} p_x(v_y,\omega_z) F_{xn,r}(k_r) + \frac{r_w}{J_w v_x} T_a.$$

Taking v_y, ω_z as outputs, it is easy to check that (see Appendix for details)

$$\begin{pmatrix} \ddot{v}_y \\ \ddot{\omega}_z \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} + \mathcal{A}\begin{pmatrix} \delta_m \\ T_a \end{pmatrix}$$

where the so-called decoupling matrix

$$\mathcal{A} = \begin{pmatrix} \dfrac{\theta_1}{m\tau_m}\dfrac{\partial \Delta_{y,f}}{\partial \delta_c} & 0 \\ \dfrac{\theta_1 l_f}{J\tau_m}\dfrac{\partial \Delta_{y,f}}{\partial \delta_c} & \dfrac{C_{x,r} t_v r_w}{J J_w v_x} p_x(v_y,\omega_z)\dfrac{dF_{xn,r}}{dk_r} \end{pmatrix}$$

is invertible for $C_{y,f} \neq 0$. The same conditions should be verified also for a (non adaptive) version of the controller (10), (15).

With the same arguments of Section 3.1, one works out that the linearising feedback is (see Borri, 2007; Bianchi, 2008)

$$\begin{pmatrix} \delta_m \\ T_a \end{pmatrix} = \mathcal{A}^{-1}\left[-\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} + w \right] \tag{16}$$

with the new input

$$w = \begin{pmatrix} \ddot{v}_{y,\mathrm{ref}} - k_{d1}(\dot{v}_y - \dot{v}_{y,\mathrm{ref}}) - k_{p1}(v_y - v_{y,\mathrm{ref}}) \\ \ddot{\omega}_{z,\mathrm{ref}} - k_{d2}(\dot{\omega}_z - \dot{\omega}_{z,\mathrm{ref}}) - k_{p2}(\omega_z - \omega_{z,\mathrm{ref}}) \end{pmatrix}$$

$k_{p1}, k_{p2}, k_{d1}, k_{d2} > 0$, $v_{y,\mathrm{ref}}, \dot{v}_{y,\mathrm{ref}}, \ddot{v}_{y,\mathrm{ref}}, \omega_{z,\mathrm{ref}}, \dot{\omega}_{z,\mathrm{ref}}, \ddot{\omega}_{z,\mathrm{ref}}, \delta_d, \dot{\delta}_d$ bounded. Moreover, its adaptive version is

$$\begin{aligned} \begin{pmatrix} \hat{\delta}_m \\ \hat{T}_a \end{pmatrix} &= \hat{\mathcal{A}}^{-1}\left[-\begin{pmatrix} \hat{F}_1 \\ \hat{F}_2 \end{pmatrix} + \hat{w} \right] \\ \dot{\hat{\theta}} &= \bar{\Gamma}\bar{\Lambda}^T \bar{P} \begin{pmatrix} v_y - v_{y,\mathrm{ref}} \\ \dot{v}_y - \dot{v}_{y,\mathrm{ref}} \\ \omega_z - \omega_{z,\mathrm{ref}} \\ \dot{\omega}_z - \dot{\omega}_{z,\mathrm{ref}} \end{pmatrix} \end{aligned} \tag{17}$$

(note that also w depends on unknown parameters, so that an estimate is necessary), which exists when $\hat{\theta}_1 \neq 0$.

$$F(\theta) - F(\hat{\theta}) + \big(\mathcal{A}(\theta) - \mathcal{A}(\hat{\theta})\big) \begin{pmatrix} \hat{\delta}_m \\ \hat{T}_a \end{pmatrix} = \bar{\Lambda}\tilde{\theta}$$

and $\bar{\Gamma} = \bar{\Gamma}^T > 0$ is a gain matrix, $\bar{P} = \bar{P}^T$ is an appropriate positive definite matrix.

With respect to equations (10), (11), (15), the controller (17) has some drawbacks

- It is more complex to implement
- An overparametrisation is necessary, namely also $\theta_4 = \theta_1^2$, $\theta_5 = \theta_1\theta_2$, $\theta_6 = \theta_2^2$, $\theta_7 = \theta_1 C_{x,r}$, $\theta_8 = \theta_1 C_{x,r}$, $\theta_9 = {C_{x,r}}^2$ have to be introduced and estimated, in order to write the terms linearly with respect the estimation errors
- The control law depends also on $\dot{\delta}_d$, which has to be measured or numerically determined.

This justifies the use of the controller (10), (11), (15).

We conclude this section observing that the proposed controller (12), (11), (15) depends on the lateral velocity v_y. Indeed, this is difficult to estimate. Various observers for the lateral velocity v_y have been proposed in the literature (see Ungoren et al. (2006), among the others), and can be considered for the implementation of the controller.

4 Simulation results

Simulations have been carried out to evaluate the performance of the adaptive integrated control (12), (11), (15).

The parameters of the vehicle are

$$m = 1870 \text{ kg}, \quad J = 3630\,\text{kg}\,\text{m}^2$$

$$l_f = 1.37 \text{ m}, \quad l_r = 1.52 \text{ m}, \quad C_{x,r} = 6400.$$

The real values of the parameters subject to variation/uncertainty are

$$C_{y,f} = 7153, \quad C_{y,r} = 6845$$

with superimposed an additive white Gaussian noise d of amplitude 0.02. Their nominal values are

$$C_{y,f,0} = 8941, \quad C_{y,r,0} = 8556.$$

The following test manoeuvres have been considered

- a step steer of 60° performed at 100 km/h, in non-nominal conditions of tyres and in presence of a wind blast;
- a double step steer of 100° at 100 km/h, in non-nominal conditions of tyres.

The references $v_{y,\text{ref}}$, $\omega_{z,\text{ref}}$ to be tracked have been generated using a standard single track model, identified according to the plant response, and modified using the adapted parameters, which have been initialised to the nominal values.

Figures 1–6 refer to the first test manoeuvre. A lateral wind blast from $t = 5$ s to $t = 10$ s is considered and vehicle parameters are subject to variations of the stiffness $C_{y,f}$, $C_{y,r}$ of the front/rear tyres. It can be noticed that the parameters estimation

Figure 1 Trajectory in the plane: reference (solid), actual trajectory (dash–dot)

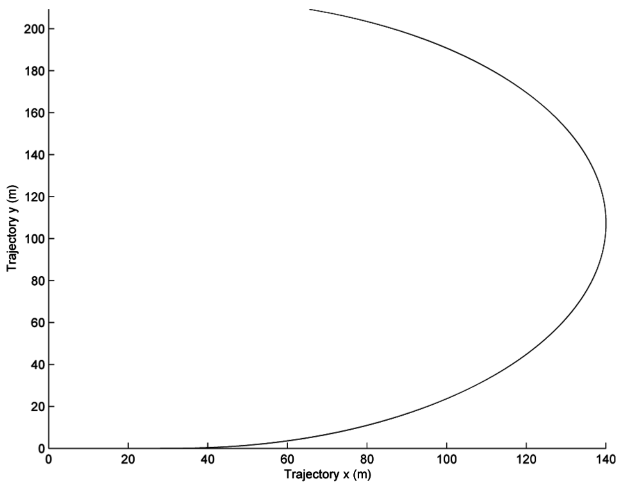
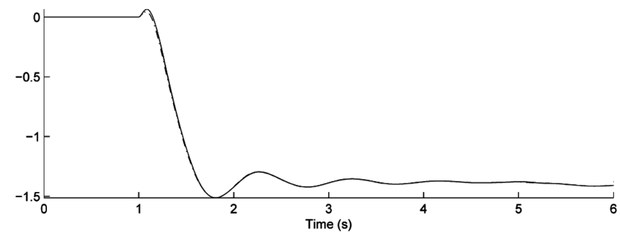
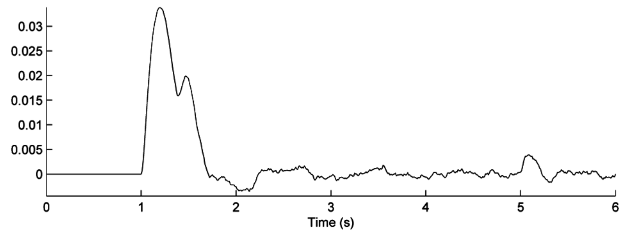


Figure 2 (a) $v_{y,\mathrm{ref}}$ (solid), v_y (dash–dot) and (b) $v_y - v_{y,\mathrm{ref}}$ [m/s]

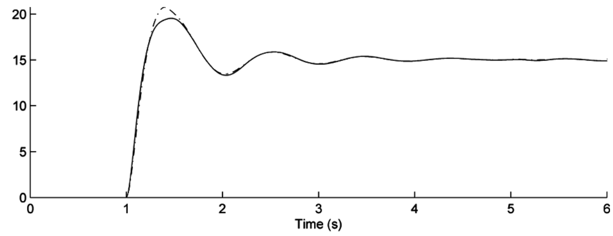


(a)

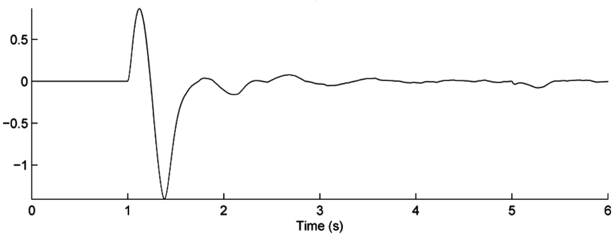


(b)

Figure 3 (a) $\omega_{z,\mathrm{ref}}$ (solid), ω_z (dash–dot) and (b) $\omega_z - \omega_{z,\mathrm{ref}}$ [deg/s]



(a)



(b)

is good (Figure 5), and the output tracking is very good (Figures 2 and 3): high yaw rate values and an agile cornering behaviour. The AFS action (Figure 4(b) goes in the direction of the driver's angle (Figure 4(a), on the contrary, the RTV input (Figure 4(c) contrasts driver's cornering action, reaching maximum absolute values of about 800 N m and a steady–state value of about 500 N m. It is worth noting that the magnitude of T_a, defined in equation (15), is physically limited by the tyre forces. The same limits are the ones which apply to a torque vectoring device. The maximum differentials torques are typically applied when the differential locks, and in this case one could reach the limits given by tyres saturations.

Figure 4 (a) δ_d [deg]; (b) δ_c [deg] and (c) T_a [N m]

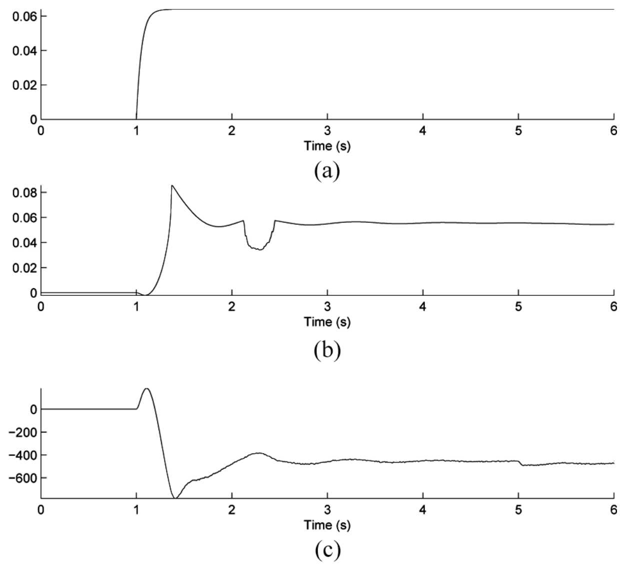
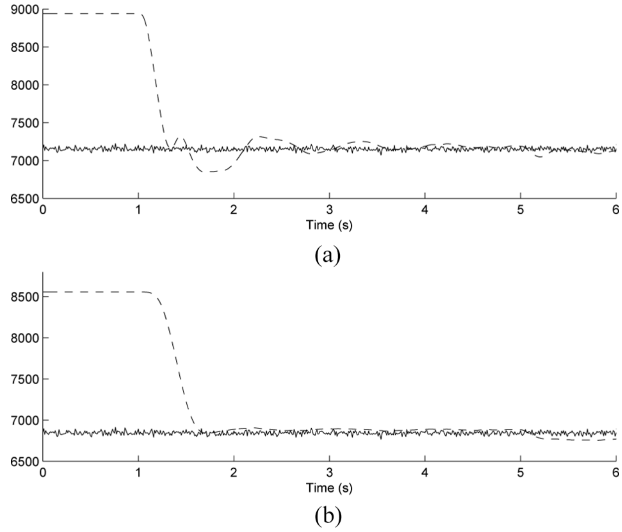
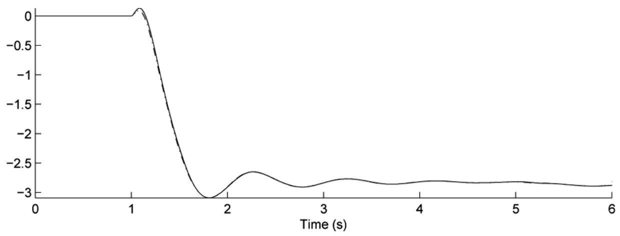


Figure 5 (a) θ_1 (solid), $\hat{\theta}_1$ (dashed) and (b) θ_2 (solid), $\hat{\theta}_2$ (dashed)

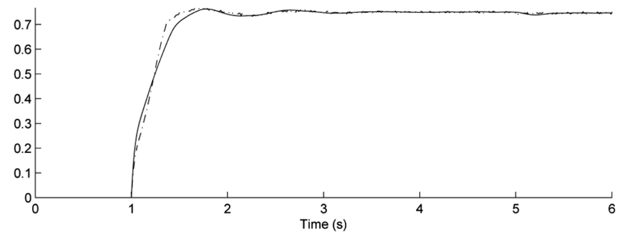


In second case, refer to Figures 7–12. Vehicle parameters are subject to variations of the stiffness $C_{y,f}$, $C_{y,r}$ of the front/rear tyres. While the non–controlled plant goes unstable at the first step of the steering manoeuvre (Figure 7, dashed), the adaptive controller keeps the vehicle stable (Figure 7, dash–dot) and ensures good tracking (Figures 8 and 9). The parameters are well estimated (Figure 11), and the vehicle reaches a lateral acceleration of about 0.95 g and the vehicle slip angle is not very high (Figure 12). The RTV action is against the direction of steering, while the AFS goes in the direction of the manoeuvre to balance the AD effect (Figure 10).

Figure 6 (a) Vehicle slip angle: reference (solid), adaptive control (dash–dot) [deg] and (b) lateral acceleration: reference (solid), adaptive control (dash–dot) [g]

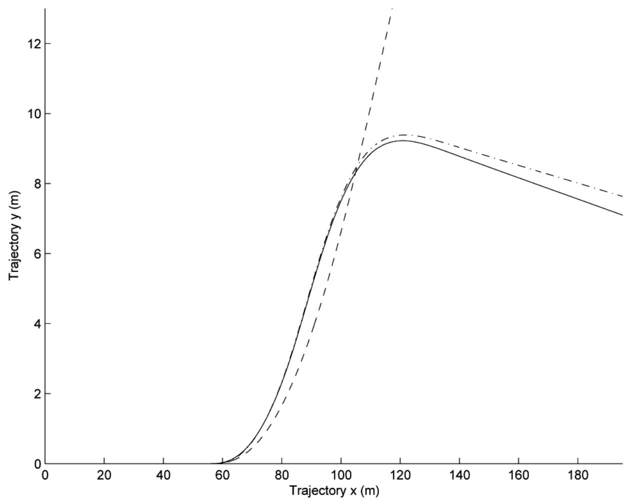


(a)



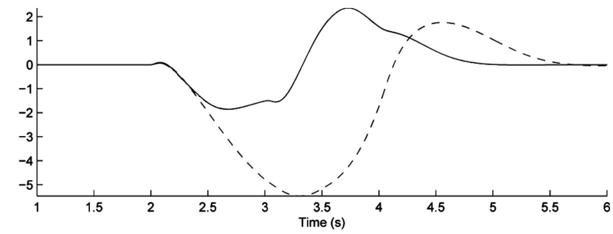
(b)

Figure 7 Trajectory in the plane: reference (solid), adaptive control trajectory (dash–dot), non-controlled trajectory (dashed)

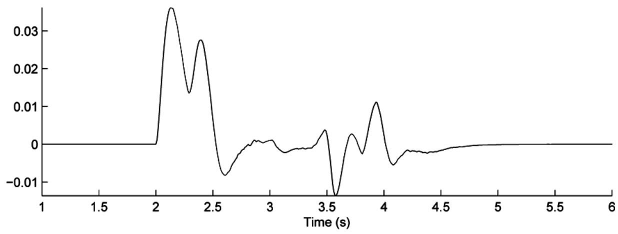


Since the solved control problem regards the tracking of lateral and angular velocity references, these simulations show that indeed these errors tend exponentially to zero, as proved analytically. The controller ensures also vehicle stability, namely the boundedness of the system state variables. This is consequence of the exponential tracking and of the boundedness of the chosen references. As a matter of fact, and as shown in the simulations, the first manoeuvre does not highlight a stability problem but, on the contrary, shows the capability of the proposed controller to robustly perform (i.e., exponential reference tracking and state variable boundedness), thanks

Figure 8 (a) $v_{y,\mathrm{ref}}$ (solid), adaptive control v_y (dash–dot), non controlled v_y (dashed) and (b) v_y error of controlled vehicle [m/s]

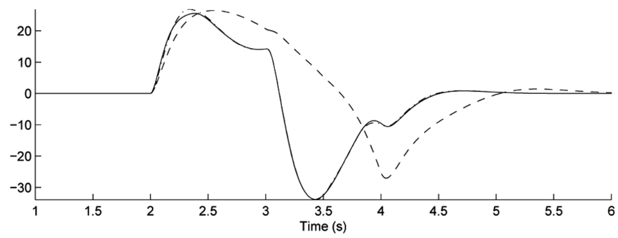


(a)

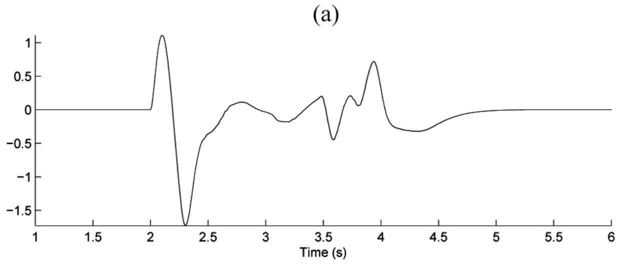


(b)

Figure 9 (a) $\omega_{z,\mathrm{ref}}$ (solid), adaptive control ω_z (dash–dot), non controlled ω_z (dashed) and (b) ω_z error of controlled vehicle [deg/s]



(a)



(b)

to the parameters estimation. Analogously, the second manoeuvre indeed shows the controller exponential reference tracking and state variable boundedness for the vehicle in a double step steer manoeuvre.

With the second manoeuvre we are showing with the CAE tool that the vehicle responds to an aggressive standard driver input showing indeed characteristics of agility (responsiveness of the state) and stability (the states do not diverge). CAE tools are nowadays very important, they are gaining always bigger role in the

Figure 10 (a) δ_d [deg]; (b) δ_c [deg] and (c) T_a [N m]

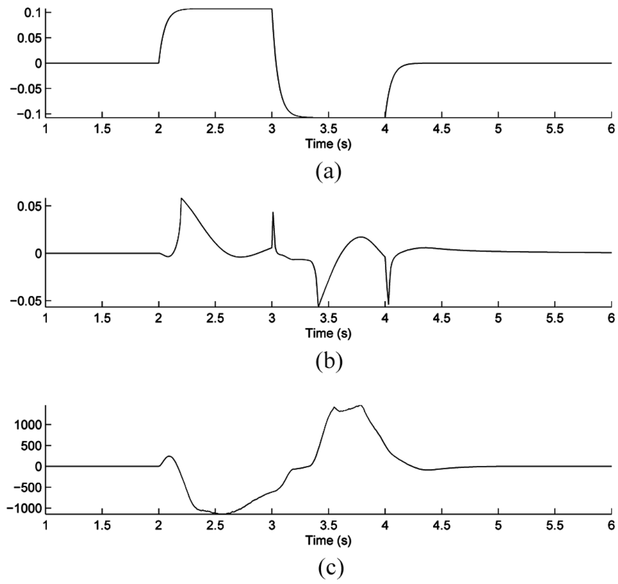
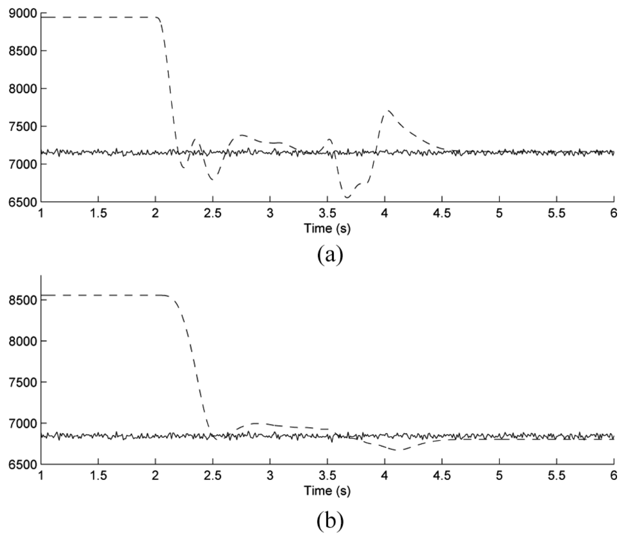


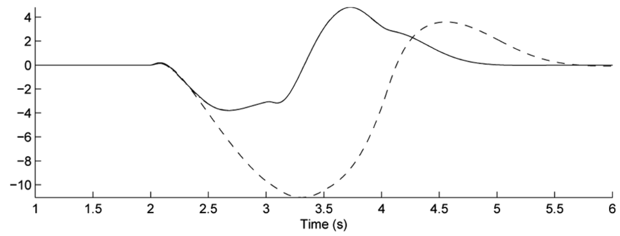
Figure 11 (a) θ_1 (solid), $\hat{\theta}_1$ (dashed) and (b) θ_2 (solid), $\hat{\theta}_2$ (dashed)



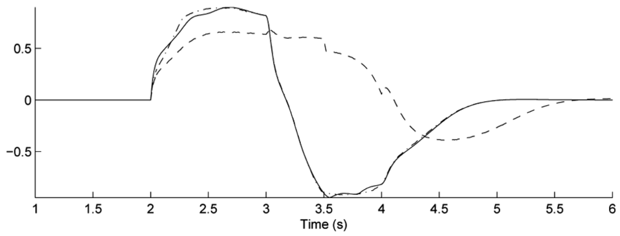
design process of chassis control systems. Sure, after having refined the results in CAE, it is always necessary to test the performance/adjust the design in a prototype.

We conclude observing that the use of adaptation, ensuring also parameter identification, is particularly useful because it allows the generation of more effective reference signals, taking into account the real force that the tyres can actually exert.

Figure 12 (a) Vehicle slip angle: reference (solid), adaptive control (dash–dot), non controlled vehicle (dashed) [deg] and (b) lateral acceleration: reference (solid), adaptive control (dash–dot), non controlled vehicle (dashed) [g]



(a)



(b)

5 Conclusions

In this paper an integrated chassis control system has been proposed to improve vehicle handling and stability. The chosen actuators are active steering and RTV, even if the proposed approach can be generalised and extended to other actuator configurations. In this work, it has been shown that AFS and RTV actuators can be effectively used in conjunction in an adaptive integrated controller, ensuring tracking of desired references and robustness in presence of parameter variations and external disturbances.

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Appendix

The expressions appearing in Section 3.2 are given by

$$
\begin{aligned}
F_1 &= \frac{\theta_1}{m}\frac{\partial F_{yn,f}}{\partial \delta_d}\dot{\delta}_d + \left(\frac{\theta_1}{m}\frac{\partial F_{yn,f}}{\partial v_y} + \frac{\theta_2}{m}\frac{\partial F_{yn,r}}{\partial v_y} + \frac{\theta_1}{m}\frac{\partial \Delta_{y,f}}{\partial v_y}\right) \\
&\quad \phi_1 + \left(-v_x + \frac{\theta_1}{m}\frac{\partial F_{yn,f}}{\partial \omega_z} + \frac{\theta_2}{m}\frac{\partial F_{yn,r}}{\partial \omega_z} + \frac{\theta_1}{m}\frac{\partial \Delta_{y,f}}{\partial \omega_z}\right)\phi_2 - \frac{\theta_1}{m\tau_m}\frac{\partial \Delta_{y,f}}{\partial \delta_c}\delta_c \\
F_2 &= \frac{\theta_1 l_f}{J}\frac{\partial F_{yn,f}}{\partial \delta_d}\dot{\delta}_d + \left(\frac{\theta_1 l_f}{J}\frac{\partial F_{yn,f}}{\partial v_y} - \frac{\theta_2 l_r}{J}\frac{\partial F_{yn,r}}{\partial v_y}\right. \\
&\quad \left. + \frac{\theta_1 l_f}{J}\frac{\partial \Delta_{y,f}}{\partial v_y} + \frac{C_{x,r}t_v}{J}\frac{\partial p_x}{\partial v_y}F_{xn,r}\right)\phi_1 \\
&\quad + \left(\frac{\theta_1 l_f}{J}\frac{\partial F_{yn,f}}{\partial \omega_z} - \frac{\theta_2 l_r}{J}\frac{\partial F_{yn,r}}{\partial \omega_z} + \frac{\theta_1 l_f}{J}\frac{\partial \Delta_{y,f}}{\partial \omega_z} + \frac{C_{x,r}t_v}{J}\frac{\partial p_x}{\partial \omega_z}F_{xn,r}\right)\phi_2 \\
&\quad - \frac{\theta_1 l_f}{J\tau_m}\frac{\partial \Delta_{y,f}}{\partial \delta_c}\delta_c - \frac{{C_{x,r}}^2 t_v r_w^2}{J J_w v_x}p_x^2\frac{\partial F_{xn,r}}{\partial k_r}F_{xn,r} \\
\phi_1 &= -v_x\omega_z + \frac{1}{m}\Big(\theta_1 F_{yn,f} + \theta_2 F_{yn,r}\Big) + \frac{\theta_1}{m}\Delta_{y,f} \\
\phi_2 &= \frac{1}{J}\Big(\theta_1 F_{yn,f} l_f - \theta_2 F_{yn,r} l_r\Big) + \frac{\theta_1 l_f}{J}\Delta_{y,f} + \frac{C_{x,r}t_v}{J}p_x F_{xn,r}.
\end{aligned}
$$