The identified parameter variations are shown in Fig. 4 (with delay alignment) and Fig. 5 (without delay alignment). The proposed delay alignment shows good estimation results and, contrarily, the estimation approach without delay alignment fails. In Fig. 5, the variations and errors are so large that the method without delay alignment does not make any sense.

Compared with the results in [9], where no delay alignment technique was used and a very high gain had to be adopted (at the order of 10^{10}) to achieve an acceptable result, the proposed delay alignment method is able to achieve a similar performance at a relatively small gain (at the order of 10^6). The advantage is that a smaller gain is numerically more stable and provides better computation efficiency in practice.

The estimated $[\Delta\hat{A}|\Delta\hat{B}]$ are listed in the second column of Table I. For comparison, *n4sid* is also evaluated, where only the input-output pairs $[u, y]$ are used to identify the parameters. Then the parameter variations are calculated as the parameter estimates minus their known nominal values. The third column shows the estimation errors in percentage. The percentage error is defined as $|(a_{ij} - \hat{a}_{ij})/a_{ij}| \times 100\%$ and $|(b_j - \hat{b}_j)/b_j| \times 100\%$, where a_{ij}, b_j are the elements of ΔA, ΔB and $\hat{a}_{ij}$ and $\hat{b}_j$ are their estimates, respectively. It can be seen that the proposed delay alignment scheme improves the parameter identification performance.

VI. Conclusion

Due to the dynamics of HGO and its nonzero phase response, the time delay appears which affects the performance of parameter identification. This technical note analyzes the properties of disturbance estimation in the HGO and proves that the estimation delay is independent of the parameter variations. Thus the delay can be computed accurately from the TFMs and be compensated (in the sense of identification) by aligning all the variables by the same delay. When the delay is presented in the HGO, the proposed algorithm improves the performance significantly. This has been verified by the simulation results on a gas turbine engine model.

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On the Observability of Continuous-Time Switched Linear Systems Under Partially Unknown Inputs

David Gómez-Gutiérrez, *Student Member, IEEE*, Antonio Ramírez-Treviño, *Member, IEEE*, Javier Ruiz-León, and Stefano Di Gennaro

***Abstract*—This technical note addresses the problem of observability of continuous—time Switched Linear Systems (*SLS*) subject to unknown disturbances, unknown switching signals and unconstrained nonzero dwell time. For this class of hybrid systems, a geometric characterization of the observability properties is presented. In particular, attention is focused on the estimation of the *SLS* state from the continuous measurements.**

Using the well known concept of (A, B)—invariant subspaces, three main problems are addressed and solved, namely the observability for every nonzero state trajectory, the observability for *almost every* control input, and the observability for *almost every* state trajectory. It is also shown that the approach herein presented subsumes some of the classical results on observability of *SLS* previously reported in the literature, and allows expressing them in a common framework.

***Index Terms*—Distinguishability, observability, switched systems.**

I. Introduction

Nowadays, controllers must handle complex systems (like biological networks, fault tolerant systems, chemical processes, etc.) where discrete and continuous dynamics appear and are combined in such a way that new phenomena, not encountered in purely continuous-time systems, arise in the system behaviors [1]. In order to interact with such systems, it is necessary to develop frameworks to systematically analyze them. One of these frameworks is the continuous-time Switched Linear Systems (*SLS*) paradigm [2], where the systems are modeled

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D. Gómez-Gutiérrez, A. Ramírez-Treviño, and J. Ruiz-León are with the Research and Advanced Studies Center, National Polytechnic Institute, Zapopan, 45019, Jalisco, Mexico (e-mail: dgomez@gdl.cinvestav.mx; art@gdl.cinvestav.mx; jruiz@gdl.cinvestav.mx).

S. Di Gennaro is with the Department of Electrical and Information Engineering and the Center of Excellence DEWS, University of L'Aquila, L'Aquila 67100, Italy (e-mail: stefano.digennaro@univaq.it).

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by a collection of Linear time-invariant Systems (*LS*) together with a switching signal determining, at each time instant, the evolving *LS*. For this class of systems many contributions have been reported concerning basic system properties as stability [3], controllability [4] and observability [5]. With respect to the latter, different observability problems for this kind of systems arise depending on the system's parameters that are considered. These parameters are for instance, the dwell time (i.e. the time that a *LS* remains active) which could be unconstrained or restricted by a fixed minimum/maximum value, the inputs that could be known or partially unknown (because of noisy inputs, disturbances, etc.) and the switching signal that could be known or unknown.

In this technical note we focus on solving a set of observability problems that arise in the case of *SLS* under partially unknown inputs and unknown switching signal, where no constraints are considered on the minimum/maximum value of the dwell time, apart from the assumption of non-zenoness. For instance, such a switching system arises in the fault identification of systems in which the safety is of paramount importance (chemical/nuclear reactors, avionic systems, etc.), and the control reconfiguration, necessary to face such events, needs to be very short. In this case, since the *SLS* is subject to arbitrary and unpredictable switchings, it is necessary to ensure the estimation of the *SLS* state before another switching occurrence, or even if no switching occurs at all. Thus, in order to estimate the continuous and discrete state, every *LS* of the *SLS* must be observable with unknown inputs. However, this condition is not sufficient, since it is also required to determine which *LS* is evolving.

Since the conditions for the observability with unknown inputs of the individual *LS* have been already established [6], the main problem is then to infer, from the knowledge of the continuous measurements, which *LS* is evolving. This is usually referred to as the distinguishability problem. This problem together with the *SLS* observability have been extensively studied in the literature for the case of known inputs ([7]–[11]). However, to the best of our knowledge, the observability of continuous-time *SLS* subject to unknown disturbances and unknown switching signals is still an open problem. For instance, although the recent work [12] considers unknown disturbances, the authors assume a known switching signal.

In the context of unknown switching signals, the concept of observability of *SLS* becomes more complex than in the *LS* case. The complexity comes from the two following reasons: First, because the autonomous case [7] becomes non equivalent to the non autonomous case [8]–[10], since the input plays a central role. This gives rise to different observability notions, since one can be interested in ensuring the computation of the state for every input [10] or in finding an input [8] or a generic set of inputs [9], allowing to infer the state (these two latter cases result equivalent [9]). Second, because while in an unobservable *LS* is always impossible to recover the initial state, in *SLS* with unknown switching signal, the set of continuous state trajectories for which the evolving systems cannot be recovered, have initial conditions from a particular subspace. Thus if this subspace is not the complete state space, there exist state trajectories for which the evolving system can still be inferred. This gives rise to another two different notions, since one can be interested in recovering the state (or part of it) for every nonzero state trajectory [7], [10] or for a generic set of state trajectories [9]. Furthermore, under the assumption of unknown switching signal, the observability in *SLS* can be presented separately for continuous and discrete states [9]. A geometric characterization of the main results presented in [7]–[10] for *SLS* with unknown switching signal and no maximum dwell time is reported in [5].

If on the contrary a maximum dwell time is considered, then the *SLS* state does not need to be recovered before the first switching time, but after a finite number of switches, either by taking advantage of the underlying discrete event system [8] or by investigating the distinguishability between *LS* [10]. This problem has been considered in [13] for autonomous systems and in [8] and [10] for the known inputs case.

Similar problems have been considered for discrete-time *SLS* also in the presence of unknown switching signal and unconstrained dwell time, for both with known inputs [14] and with unknown but bounded disturbances [15]–[17]. Nevertheless, as pointed out in [9], continuous and discrete—time *SLS* have some notable differences which requires to study these two classes of systems separately.

A different set of problems for *SLS* arises when the switching signal is known [4] or observable from the discrete measurements [18], in this situation the main problem is to compute the continuous state. If no maximum dwell time is set, then this problem reduces to the observability of each *LS*. On the other hand if a maximum dwell time is set the problem translates into recovering the continuous state after a finite number of switches, in this case it has been shown that the observability of each *LS* is not necessary [4].

In the present work, we encompass the observability problems that arise in the study of *SLS* subject to unknown switching signals and unconstrained dwell time into three ones, namely observability for every nonzero state trajectory, observability for "almost every" control input and observability for almost every state trajectory, which will be presented separately for continuous and discrete states. For these observability notions we will derive necessary and sufficient conditions for *SLS* under partially unknown inputs.

It is also shown in this technical note that the approach herein presented subsumes some of the well known results on observability of *SLS* under known inputs. In particular, as shown later, if the disturbance is absent, the results herein presented reduce to those presented in [7]–[10] for *SLS* with unconstrained dwell time. When considering unknown inputs, the results of those papers remain only necessary. This approach also subsumes our previous work [5] for *SLS* under fully unknown inputs.

This technical note is organized as follows. In Section II, we present some preliminaries on *LS* and introduce basic concepts on measure theory. In Section III the problem formulation and the different observability notions are presented. These notions are completely characterized in Section IV, where the main contributions of this work are contained. Finally, in Section V, the conclusions and the future work are presented.

II. Preliminaries

A. Basic Concepts on Measure Theory

Through this technical note, the concept from measure theory of "almost everywhere" or for "almost every" is used to express that the observability property is virtually certain to hold. When referring to $\mathbb{R}^n$, a property is said to hold "almost everywhere" or "for almost every" if it only fails on a set of Lebesgue measure zero, also known as "null set" [19]. An extension of these concepts to function spaces has been proposed in [20], where a property on a function space is said to hold "for almost every" (in the sense of [20]) if the set of exceptions is a shy set, also known as a Haar null set [20], [21].

Let V be a complete metric linear space, possible infinite—dimensional (e.g. $\mathbb{R}^n$ and $L_p(\mathcal{U})$).[1] Although we do not report here the formal definition of a shy set $S \subset V$, the following important properties give an insight on its meaning [20]:

- A set $S \subset \mathbb{R}^n$ is shy if and only if it has Lebesgue measure zero. Thus, the concept of shy set is a natural extension of the concept of null set because both concepts coincides in $\mathbb{R}^n$.
- A shy set has no interior. Thus, "almost every" implies dense.

[1] $L_p(\mathcal{U})$, $1 \leq p \leq \infty$ is the set of all piecewise continuous functions $u : \mathbb{R}_{\geq 0} \to \mathcal{U}$ satisfying $|u(\cdot)|_p = (\int_0^\infty \|u(t)\|_p^p dt)^{1/p} < \infty$ [22].

- Every countable set in V and every proper subspace of V are shy with respect to V.

In particular, this last property will be used to demonstrate that under the conditions derived in Section IV, the observability property holds for almost every control input and for almost every state trajectory.

B. Preliminaries on Linear Systems

The notation and terminology on *LS* used through this work is taken mainly from [23]. The next lines review some of the basic geometric concepts on *LS*.

A *LS* Σ is represented by

$$\dot{x}(t) = Ax(t) + Bu(t) + Sd(t),$$
$$y(t) = Cx(t)$$

where $x \in \mathcal{X} = \mathbb{R}^n$ is the state vector, $u \in \mathcal{U} = \mathbb{R}^g$ is the control input, $y \in \mathcal{Y} = \mathbb{R}^q$ the output signal, $d \in \mathcal{D} = \mathbb{R}^l$ the disturbance and A, B, C and S are constant matrices of appropriate dimensions. The input function space, denoted by $\mathcal{U}_f$, is considered to be $L_p(\mathcal{U})$. Through the technical note $\mathcal{B}$ stands for $\operatorname{Im} B$, $\mathcal{S}$ for $\operatorname{Im} S$ and $\mathcal{K}$ for $\ker C$.

A subspace $\mathcal{T} \subset \mathcal{X}$ is called A—invariant if $A\mathcal{T} \subset \mathcal{T}$. The supremal A—invariant subspace contained in $\mathcal{K}$ is $\mathcal{N} = \bigcap_{i=1}^{n} \ker(CA^{i-1})$. The subspace $\mathcal{N}$ is known as the unobservable subspace of the *LS* Σ. A subspace $\mathcal{V} \subset \mathcal{X}$ is said to be (A, B)—invariant if there exists a state feedback $u = Fx$ such that $(A + BF)\mathcal{V} \subset \mathcal{V}$ or, equivalently, if $A\mathcal{V} \subset \mathcal{V} + \mathcal{B}$. The set of maps F for which $(A + BF)\mathcal{V} \subset \mathcal{V}$ holds is denoted as $\mathbf{F}(\mathcal{V})$. The set of (A, B)—invariant subspaces contained in a subspace $\mathcal{L} \subset \mathcal{X}$ is denoted by $\Im(A, B; \mathcal{L})$. It is well known that this set is closed under addition, then it contains a supremal element [22], [23] denoted $\sup \Im(A, B; \mathcal{L})$. Furthermore, $\sup \Im(A, B; \mathcal{L}) = \mathcal{V}^{(\nu)}$, where $\mathcal{V}^{(0)} = \mathcal{L}$

$$\mathcal{V}^{(i)} = \mathcal{L} \cap A^{-1}\left(\mathcal{B} + \mathcal{V}^{(i-1)}\right), \quad i = 1, \cdots, \nu \tag{1}$$

and $\nu = \dim(\mathcal{L})$ [22], [23].

Using this concept of (A, B)—invariant subspaces, the observability for *LS* under partially unknown inputs, which is a clear necessary condition in our setting, was completely characterized in [6], its main result is formally stated in the following theorem.

Theorem 1: [6] Let Σ be a *LS* with associated matrices A, B, C and S. Then the *LS* Σ is observable under partially unknown inputs if and only if the $\sup \Im(A, S; \mathcal{K})$ is trivial.

This theorem subsumes the classical result on observability of *LS* with known inputs, since $\sup \Im(A, 0; \mathcal{K}) = \mathcal{N}$. The following result will be used later.

Lemma 2: [22] Any state trajectory $x(t)$, $t \in [t_0, \tau]$ of Σ belongs to a subspace $\mathcal{L} \subseteq \mathcal{X}$ if and only if $x(t_0) \in \mathcal{L}$ and $\dot{x}(t) \in \mathcal{L}$ almost everywhere in $[t_0, \tau]$.

III. Problem Formulation

A. Switched Linear System's Model

A *SLS* is described as a tuple $\mathcal{G} = \langle \mathcal{F}, \sigma \rangle$, where $\mathcal{F} = \{\Sigma_1, \ldots, \Sigma_m\}$ is a collection of *LS* and $\sigma: [t_0, \infty) \to \{1, \ldots, m\}$ is the switching signal determining, at each time instant the evolving *LS* $\Sigma_\sigma \in \mathcal{F}$. The *SLS*'s state equation is represented by

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + S_{\sigma(t)}d(t), \quad x(t_0) = x_0$$
$$y(t) = C_{\sigma(t)}x(t), \quad \sigma(t_0) = \sigma_0. \tag{2}$$

We make the following assumptions on the *SLS*.

- The signals $y(t)$ and $u(t)$ are measurable, while $x(t)$, $d(t)$ and $\sigma(t)$ are unknown. Furthermore, $\sigma(t)$ is assumed to be generated exogenously, i.e. we do not assume any knowledge on an underlying discrete event system.
- Only a finite number of switches can occur in a finite interval, i.e. Zeno behavior is not possible.
- Neither a minimum nor a maximum dwell time is set.

We use the notation $x_i(x_0, u_{[t_0,\tau]}, d_{[t_0,\tau]})$, to emphasize that the state trajectory $x(t)$ is obtained when $\sigma(t) = i$ in the interval $[t_0, \tau]$ and the inputs $u_{[t_0,\tau]}$, $d_{[t_0,\tau]}$ (i.e. the restriction of the functions $u(t)$, $d(t)$ to $[t_0, \tau]$) are applied at the initial condition x_0, where $\tau = t_0 + \Delta t$ with $\Delta t > 0$. In a similar way, $y_i(x_0, u_{[t_0,\tau]}, d_{[t_0,\tau]})$ represents its output.

B. Distinguishability Between LS

Since no maximum dwell time is set and Zeno behaviors are excluded, the observability with unknown inputs is necessary to estimate the continuous state and these can be obtained if it is known which *LS* is currently active. Then the observability problem can be tackled by addressing a set of distinguishability problems. In general, if $x_i(x_0, u_{[t_0,\tau]}, d_{[t_0,\tau]})$ is evolving, the *LS* Σ_i can be distinguished from Σ_j by measuring $u_{[t_0,\tau]}$, $y_{[t_0,\tau]}$, if and only if for every x'_0 and $d'_{[t_0,\tau]}$

$$y(t) = y_i\left(x_0, u_{[t_0,\tau]}, d_{[t_0,\tau]}\right) \neq y_j\left(x'_0, u_{[t_0,\tau]}, d'_{[t_0,\tau]}\right). \tag{3}$$

Thus, if (3) does not hold for some x'_0 and $d'_{[t_0,\tau]}$, it is impossible to determine from the measured signals, if the state trajectory was generated by Σ_i or by Σ_j. In a similar way, it is impossible to determine if the continuous initial state is x_0 or x'_0. Clearly, (3) does not hold for some x'_0 and $d'(t)$ if x_0, $u_{[t_0,\tau]}$ and $d_{[t_0,\tau]}$ are all zero.

Different problems arise when dealing with the distinguishability problem, since one may be interested in that (3) holds for every x_0, $u_{[t_0,\tau]}$ and $d_{[t_0,\tau]}$ (except when they are all zero) or in finding a set of inputs $u_{[t_0,\tau]}$ ensuring that (3) holds despite x_0 and $d_{[t_0,\tau]}$. Moreover, as it will be shown later, (3) does not hold only when x_0 and x'_0 belong to the so called indistinguishability subspace, hereinafter introduced. Then, the systems can be distinguished even if they are not distinguishable for every state trajectory.

Thus, in order to solve the observability problem we mainly focus on solving the following set of distinguishability problems which form the basis for the different observability notions that will be characterized in Section IV.

Problem 3: (Distinguishability for every nonzero state trajectory) Given the *LS* Σ_i and Σ_j and observing the signals $y_{[t_0,\tau]}$ and $u_{[t_0,\tau]}$, under which conditions can Σ_i be distinguished from Σ_j for every nonzero state trajectory? Namely, under which conditions the set

$$\left\{x_0 : \exists x'_0, u_{[t_0,\tau]}, d_{[t_0,\tau]}, d'_{[t_0,\tau]},\ y_i\left(x_0, u_{[t_0,\tau]}, d_{[t_0,\tau]}\right) = y_j\left(x'_0, u_{[t_0,\tau]}, d'_{[t_0,\tau]}\right)\right\} \tag{4}$$

contains only the zero initial state? ◇

A solution to this problem provides the basis for determining the *SLS* state for every state trajectory of the system. This notion ensures the computation of the *SLS* state in every situation. However, in most applications this can be rather restrictive. Thus, the study can be focused on finding a control input allowing the estimation of the *SLS* state, we go further by showing that if such an input exists then the *SLS*'s state can be estimated for almost every control input. The solution to the following problem provides the basis for this observability notion.

Problem 4: (Distinguishability for almost every control input) Given the *LS* Σ_i and Σ_j and observing the signals $y_{[t_0,\tau]}$ and $u_{[t_0,\tau]}$, under which conditions can Σ_i be distinguished from Σ_j for almost every control input? Namely, under which conditions is

$$\left\{u_{[t_0,\tau]} : \exists x_0, x'_0, d_{[t_0,\tau]}, d'_{[t_0,\tau]}, y_i\left(x_0, u_{[t_0,\tau]}, d_{[t_0,\tau]}\right) = y_j\left(x'_0, u_{[t_0,\tau]}, d'_{[t_0,\tau]}\right)\right\} \quad (5)$$

a shy set? ⋄

Even if the state of the *SLS* cannot be determined in the sense of Problem 3 and Problem 4, it still may be possible to determine the state for *almost* every state trajectory, because, as it will be shown later, the set of initial conditions for which the evolving *LS* cannot be determined forms a subspace $\mathcal{R} \subseteq \mathcal{X}$. Thus, if this is a proper subspace, then almost every state trajectory (except for those starting in $\mathcal{R}$, which is a shy set) will allow the estimation of the evolving system.

The following problem provides the basis for the characterization of the observability for almost every state trajectory.

Problem 5: (Distinguishability for almost every state trajectory) Given the *LS* Σ_i and Σ_j and observing the signals $y_{[t_0,\tau]}$ and $u_{[t_0,\tau]}$, under which conditions can Σ_i be distinguished from Σ_j for *almost* every state trajectory? Namely, under which conditions is

$$\mathcal{R} = \left\{x_0 : \forall u_{[t_0,\tau]}, d_{[t_0,\tau]} \exists x'_0, d'_{[t_0,\tau]}, y_i\left(x_0, u_{[t_0,\tau]}, d_{[t_0,\tau]}\right) = y_j\left(x'_0, u_{[t_0,\tau]}, d'_{[t_0,\tau]}\right)\right\} \quad (6)$$

a shy set? ⋄

IV. Observability of SLS Under Partially Unknown Inputs

In order to solve the different distinguishability problems between Σ_i and Σ_j introduced in the previous section, we define an auxiliary extended *LS* which contains the dynamics of both constituting systems, but the outputs of the two systems are subtracted. This extended *LS* is denoted by $\hat{\Sigma}_{ij}$ and is formed with the matrices

$$\hat{A}_{ij} = \begin{bmatrix} A_i & 0 \\ 0 & A_j \end{bmatrix} \quad \hat{B}_{ij} = \begin{bmatrix} B_i \\ B_j \end{bmatrix}$$
$$\hat{C}_{ij} = [\,C_i \quad -C_j\,] \quad \hat{S}_{ij} = \begin{bmatrix} S_i & 0 \\ 0 & S_j \end{bmatrix}.$$

The state space of $\hat{\Sigma}_{ij}$ is denoted by $\hat{\mathcal{X}}_{ij}$. We denote by $\hat{x}_0$ the initial state, $\hat{x}(t)$ the state trajectory and $\hat{d}(t)$ the disturbance signal of the extended *LS* $\hat{\Sigma}_{ij}$.

Lemma 6: Let Σ_i and Σ_j be two *LS*. The equation $y_i(x_0, u_{[t_0,\tau]}, d_{[t_0,\tau]}) = y_j(x'_0, u_{[t_0,\tau]}, d'_{[t_0,\tau]})$ holds if and only if the extended *LS* $\hat{\Sigma}_{ij}$ produces a zero output for all $t \in [t_0, \tau]$, i.e. $y_{ij}(\hat{x}_0, u_{[t_0,\tau]}, \hat{d}_{[t_0,\tau]}) = 0$ with $\hat{x}_0 = [x_0^T \ x_0'^T]^T$, $u(t)$ and $\hat{d}(t) = [d^T(t) \ d'^T(t)]^T$.

Proof: The proof follows from a simple algebraic procedure. ■

Definition 7: Given two *LS* Σ_i, Σ_j, the indistinguishability subspace $\hat{\mathcal{W}}_{ij}$ of Σ_i, Σ_j is defined as

$$\hat{\mathcal{W}}_{ij} = \left\{\begin{bmatrix} x_0 \\ x'_0 \end{bmatrix} : \begin{matrix} \exists u_{[t_0,\tau]}, d_{[t_0,\tau]}, d'_{[t_0,\tau]}, \\ y_i\left(x_0, u_{[t_0,\tau]}, d_{[t_0,\tau]}\right) = y_j\left(x'_0, u_{[t_0,\tau]}, d'_{[t_0,\tau]}\right) \end{matrix}\right\}$$

The indistinguishability subspace represents the set of initial conditions that under a particular input makes impossible to determine which is the evolving system and the current discrete state. Let $\mathcal{Q}_i : \hat{\mathcal{X}}_{ij} \to \mathcal{X}$ be the projection of $\hat{\mathcal{X}}_{ij}$ over $\mathcal{X}$ such that $\mathcal{Q}_i([x^T \ x'^T]^T) = x$, then if the state trajectory $x_i(x_0, u_{[t_0,\tau]}, d_{[t_0,\tau]})$ evolves inside $\mathcal{Q}_i\hat{\mathcal{W}}_{ij}$ then it is impossible to determine, from the measurements, if the evolving state trajectory is $x_i(x_0, u_{[t_0,\tau]}, d_{[t_0,\tau]})$ or $x_j(x'_0, u_{[t_0,\tau]}, d'_{[t_0,\tau]})$, thus it is not possible to infer neither the continuous initial state which could be x_0 or x'_0 nor the discrete state which could be $\sigma(t) = i$ or $\sigma(t) = j$ $\forall t \in [t_0, \tau]$.

The following result characterizes the indistinguishability subspace.

Lemma 8: Let Σ_i and Σ_j be two *LS* and let $\mathbf{B}_{ij} = [\hat{B}_{ij} \ \hat{S}_{ij}]$. Then the indistinguishability subspace $\hat{\mathcal{W}}_{ij}$ is equal to the supremal $(\hat{A}_{ij}, \mathbf{B}_{ij})$-invariant subspace contained in $\hat{\mathcal{K}}_{ij} = \mathrm{Ker}\hat{C}_{ij}$, denoted as $\sup \Im(\hat{A}_{ij}, \mathbf{B}_{ij}; \hat{\mathcal{K}}_{ij})$.

Proof: Assume that $[x_0^T \ x_0'^T]^T \in \hat{\mathcal{W}}_{ij}$. Hence, there exist signals $u(t)$, $d(t)$ and $d'(t)$ such that (3) is satisfied, which implies, by virtue of Lemma 6, that $y_{ij}(\hat{x}_0, u_{[t_0,\tau]}, \hat{d}_{[t_0,\tau]}) = 0$ i.e.

$$\hat{C}_{ij} e^{\hat{A}_{ij}t}\left[\hat{x}_0 + \int_{t_0}^{t} e^{-\hat{A}_{ij}\tau}\mathbf{B}_{ij}\mathbf{u}(\tau)d\tau\right] = 0 \quad \forall t \geq t_0 \quad (7)$$

with $\hat{x}_0 = [x_0^T \ x_0'^T]^T$, $\mathbf{B}_{ij} = [\hat{B}_{ij} \ \hat{S}_{ij}]$ and $\mathbf{u}(t) = [u^T(t) \ \hat{d}^T(t)]^T$.

Note that, if $\hat{x}_0 \in \hat{\mathcal{W}}_{ij}$ then there exists $\mathbf{u}(t)$ such that $\hat{x}(t) \in \hat{\mathcal{W}}_{ij}$ for all $t \geq t_0$. Now, According to Lemma 2, for an arbitrary $t = \bar{t}$, $\dot{\hat{x}}(\bar{t}) = \hat{A}_{ij}\hat{x}(\bar{t}) + \mathbf{B}_{ij}u(\bar{t}) \in \hat{\mathcal{W}}_{ij}$, hence $\hat{A}_{ij}\hat{\mathcal{W}}_{ij} \subset \hat{\mathcal{W}}_{ij} + \mathrm{Im}\hat{\mathbf{B}}_{ij}$. Furthermore, since for every $t \geq t_0$ the extended system $\hat{\Sigma}_{ij}$ produces a zero output, then $\hat{\mathcal{W}}_{ij} \subset \hat{\mathcal{K}}_{ij}$ showing that $\hat{\mathcal{W}}_{ij} \in \Im(\hat{A}_{ij}, \mathbf{B}_{ij}; \hat{\mathcal{K}}_{ij})$.

In order to show that $\hat{\mathcal{W}}_{ij}$ is the supremal in $\Im(\hat{A}_{ij}, \mathbf{B}_{ij}; \hat{\mathcal{K}}_{ij})$, let $\hat{\mathcal{V}}_{ij} \in \Im(\hat{A}_{ij}, \mathbf{B}_{ij}; \hat{\mathcal{K}}_{ij})$ and $\hat{x}_0 \in \hat{\mathcal{V}}_{ij}$, then there exists $\mathbf{u}(t) = F\hat{x}(t)$ with $F \in \mathbf{F}(\hat{\mathcal{V}}_{ij})$ such that $(\hat{A}_{ij} + \mathbf{B}_{ij}F)\hat{\mathcal{V}}_{ij} \subset \hat{\mathcal{V}}_{ij}$ making the state trajectory $\hat{x}(t)$ to completely belong to $\hat{\mathcal{V}}_{ij}$. Since $\hat{\mathcal{V}}_{ij} \subset \hat{\mathcal{K}}_{ij}$ then $\hat{y}(t) = \hat{C}_{ij}\hat{x}(t) = 0$ for all $t \geq t_0$. Thus, (7) holds and $\hat{\mathcal{V}}_{ij} \subset \hat{\mathcal{W}}_{ij}$. ■

By means of the algorithm described by (1), the indistinguishability subspace can be computed. This subspace is fundamental in our approach, since the different observability notions will be derived from it.

A. Observability for Every Nonzero State Trajectory

In order to derive the conditions for the observability for every nonzero state trajectory, in this subsection we focus on solving Problem 3. The conditions for solving this problem are presented next based on the indistinguisability subspace.

Proposition 9: Let Σ_i and Σ_j be two *LS* and let $\hat{\mathcal{W}}_{ij}$ be as in Definition 7. Then Σ_i can be distinguished from Σ_j for every nonzero state trajectory if and only if $\mathcal{Q}_i\hat{\mathcal{W}}_{ij} = 0$, where $\mathcal{Q}_i : \hat{\mathcal{X}}_{ij} \to \mathcal{X}$ is such that $\mathcal{Q}_i([x^T \ x'^T]^T) = x$. Furthermore, Σ_i and Σ_j are distinguishable from each other for every nonzero state trajectories if and only if $\hat{\mathcal{W}}_{ij} = 0$.

Proof: (Sufficiency) Notice that $\mathcal{Q}_i\hat{\mathcal{W}}_{ij}$ is equal to the set in (4). Since $\mathcal{Q}_i\hat{\mathcal{W}}_{ij} = 0$, then (3) only holds when x_0, $u(t)$ and $d(t)$ are all equal to zero (without any restriction on x'_0 and $d'(t)$), thus Σ_i is indistinguishable from Σ_j only for the zero state trajectory. Clearly, if $\hat{\mathcal{W}}_{ij} = 0$ then (3) only holds with x_0, x'_0, $u(t)$, $d(t)$ and $d'(t)$ all equal to zero, and Σ_i and Σ_j are distinguishable from each other for every nonzero state trajectories.

(Necessity) It follows by noticing that $\mathcal{Q}_i\hat{\mathcal{W}}_{ij}$ is equal to the set in (4). ■

Notice that if the conditions of Proposition 9 hold, then the continuous state can also be determined, since $\mathcal{Q}_i\hat{\mathcal{W}}_{ij} = 0$ implies the observability with partially unknown inputs of the *LS* Σ_i, because otherwise there exists $x_0 \in \sup \Im(A_i, S_i; \mathcal{K}_i)$ such that $y_i(x_0, u_{[t_0,\tau]}, d_{[t_0,\tau]}) = 0$, and therefore $[x_0^T \ 0]^T \in \hat{\mathcal{W}}_{ij}$ and $x_0 \in \mathcal{Q}_i\hat{\mathcal{W}}_{ij}$. However, neither $\hat{\mathcal{W}}_{ij} = 0$ nor $\mathcal{Q}_i\hat{\mathcal{W}}_{ij} = 0$ are necessary for recovering the continuous initial and current states.

Definition 10: Let Σ_i and Σ_j be two *LS*. The indistinguishability subspace $\hat{\mathcal{W}}_{ij}$ is said to be symmetric if every vector $[x^T \ x'^T]^T \in \hat{\mathcal{W}}_{ij}$ is of the form $x = x'$.

The following proposition presents the conditions for recovering the continuous initial state.

Proposition 11: Let Σ_i and Σ_j be two *LS*. The continuous initial state can be uniquely determined for every state trajectory, if and only if their indistinguishability subspace $\hat{\mathcal{W}}_{ij}$ is symmetric.

Proof: Since the continuous initial state can be uniquely determined for every state trajectory, if and only if $y_i(x_0, u_{[t_0,\tau]}, d_{[t_0,\tau]}) = y_j(x'_0, u_{[t_0,\tau]}, d'_{[t_0,\tau]}) \Rightarrow x_0 = x'_0$. Thus, the result follows trivially from Definition 7. Since the indistinguishable trajectories are generated from the same continuous initial state, then it can be recovered even though the evolving system cannot. ■

By setting $S_i = 0, i = 1, \ldots, m$, in (2), Proposition 9 reduces to the conditions of Corollary 4.4 of [10] for the distinguishability between *LS*. Clearly, with $B_i = 0$ and $S_i = 0$, Proposition 9 reduces to the conditions for distinguishability of non autonomous *LS* of [7] and Proposition 11 is equivalent to Proposition 2 of [9] (see [5]).

Based on the previous results, we can easily characterize the following observability notions.

Theorem 12: Let $\mathcal{G} = \langle \mathcal{F}, \sigma \rangle$ be a *SLS* with partially unknown inputs and maximum dwell time $\tau = \infty$. Then

1) The discrete state σ_0 and $\sigma(t)$ of $\mathcal{G}$ are observable for every nonzero state trajectory if and only if $\forall \Sigma_i, \Sigma_j \in \mathcal{F}\ i \neq j$, $\hat{\mathcal{W}}_{ij} = 0$.
2) The continuous state x_0 and $x(t)$ of $\mathcal{G}$ are observable for every state trajectory if and only if $\forall \Sigma_i, \Sigma_j \in \mathcal{F}\ i \neq j$ their indistinguishability subspace $\hat{\mathcal{W}}_{ij}$ is symmetric.

The proof follows trivially from Proposition 9 and Proposition 11, respectively.

B. Observability for Almost Every Control Input

In this section we focus on solving Problem 4, that is, we are interested in deriving the conditions under which the *SLS*'s state can be estimated for almost every control input. In order to be able to distinguish Σ_i from Σ_j, these inputs need to be capable of steering the state trajectory outside $\mathcal{Q}_i \hat{\mathcal{W}}_{ij}$, i.e. the projection of the indistinguishability subspace $\hat{\mathcal{W}}_{ij}$ over $\mathcal{X}$. The following lemma will be used in deriving such conditions.

Lemma 13: Let $\hat{\mathcal{W}}_{ij} = \sup \Im(\hat{A}_{ij}, [\hat{L}_1 \hat{L}_2]; \hat{\mathcal{K}}_{ij})$ where $\hat{L}_1 : \mathbb{R}^{r_1} \to \hat{\mathcal{X}}_{ij}$ and $\hat{L}_2 : \mathbb{R}^{r_2} \to \hat{\mathcal{X}}_{ij}$ with $\hat{\mathcal{L}}_k = \mathrm{Im}\hat{L}_k$, $k = 1, 2$. Then $\hat{\mathcal{W}}_{ij}$ coincides with $\sup \Im(\hat{A}_{ij}, \hat{L}_1; \hat{\mathcal{K}}_{ij})$ if $\hat{\mathcal{L}}_2 \subseteq \hat{\mathcal{W}}_{ij} + \hat{\mathcal{L}}_1$.

Proof: Since $\mathrm{Im}[\hat{L}_1 \hat{L}_2] = \hat{\mathcal{L}}_1 + \hat{\mathcal{L}}_2$ then $\hat{A}_{ij}\hat{\mathcal{W}}_{ij} \subset \hat{\mathcal{W}}_{ij} + \hat{\mathcal{L}}_1 + \hat{\mathcal{L}}_2$. Hence for every $w_1 \in \hat{\mathcal{W}}_{ij}$ there exist $w_2 \in \hat{\mathcal{W}}_{ij}$, $z_1 \in \hat{\mathcal{L}}_1$ and $z_2 \in \hat{\mathcal{L}}_2$ such that $\hat{A}_{ij} w_1 = w_2 + \hat{L}_1 z_1 + \hat{L}_2 z_2$. Set $g = z_1 - h$ with h such that $\hat{L}_1 h + \hat{L}_2 z_2 = w_3 \in \hat{\mathcal{W}}_{ij}$, such a h always exists for every z_2 because by assumption $\hat{\mathcal{L}}_2 \subseteq \hat{\mathcal{W}}_{ij} + \hat{\mathcal{L}}_1$, thus $\hat{A}_{ij} w_1 = w_2 + w_3 + \hat{L}_1 g$ and $\hat{A}_{ij}\hat{\mathcal{W}}_{ij} \subset \hat{\mathcal{W}}_{ij} + \hat{\mathcal{L}}_1$. Hence $\hat{\mathcal{W}}_{ij} \in \Im(\hat{A}_{ij}, \hat{L}_1; \hat{\mathcal{K}}_{ij})$. Since, trivially, every element of $\Im(\hat{A}_{ij}, \hat{L}_1; \hat{\mathcal{K}}_{ij})$ is contained in $\hat{\mathcal{W}}_{ij}$ (since $\hat{\mathcal{W}}_{ij}$ is supremal) then $\hat{\mathcal{W}}_{ij}$ concides with $\sup \Im(\hat{A}_{ij}, \hat{L}_1; \hat{\mathcal{K}}_{ij})$. ■

In order to solve Problem 4, we first derive the conditions for the existence of an input allowing to distinguish Σ_i from Σ_j and then we show that if these conditions hold then (5) is a shy set.

Proposition 14: Let Σ_i and Σ_j be two *LS*. Then there exists a control input ensuring the distinguishability between the *LS* Σ_i and the *LS* Σ_j, i.e. there exists a control input $u_{[t_0,\tau]}$ such that $\forall x_0, x_0', d_{[t_0,\tau]}, d'_{[t_0,\tau]}$

$$y_i\left(x_0, u_{[t_0,\tau]}, d_{[t_0,\tau]}\right) \neq y_j\left(x_0', u_{[t_0,\tau]}, d'_{[t_0,\tau]}\right) \quad (8)$$

if and only if $\hat{\mathcal{B}}_{ij} \not\subseteq \hat{\mathcal{W}}_{ij} + \hat{\mathcal{S}}_{ij}$ or equivalently

$$\hat{\mathcal{B}}_{ij} \not\subseteq \sup \Im(\hat{A}_{ij}, \hat{\mathcal{S}}_{ij}; \hat{\mathcal{K}}_{ij}) + \hat{\mathcal{S}}_{ij}. \quad (9)$$

Proof: (Necessity) We next prove that if (9) does not hold, (8) does not hold as well, by showing that for every possible $u_{[t_0,\tau]}$ there exists an initial condition $\hat{x}_0$ and an unknown input $\hat{d}_{[t_0,\tau]}$ such that the extended *LS* $\hat{\Sigma}_{ij}$ produces a zero output for all $t \in [t_0, \tau]$.

Let $\mathcal{V}^* = \sup \Im(\hat{A}_{ij}, \hat{S}_{ij}; \hat{\mathcal{K}}_{ij})$ and set $\hat{d}(t) = F\hat{x}(t) + v(t)$ with $F \in \mathbf{F}(\mathcal{V}^*)$ and $v(t)$ such that for every $t \in [t_0, \tau]$

$$\hat{B}_{ij} u(t) + \hat{S}_{ij} v(t) \in \mathcal{V}^* \subset \hat{\mathcal{W}}_{ij}.$$

Notice that such a $v(t)$ always exists since (9) does not hold. Thus if $\hat{x}_0 \in \mathcal{V}^*$, then $\forall t \in [t_0, \tau] \dot{\hat{x}}(t) \in \mathcal{V}^*$, and $\hat{x}(t) \in \mathcal{V}^* \subset \hat{\mathcal{W}}_{ij}$ $\forall t \in [t_0, \tau]$ by Lemma 2, showing that (8) does not hold.

(Sufficiency) Suppose that (8) does not hold, then by Lemma 6 for every $u(t)$ there exist $\hat{x}_0$ and $\hat{d}(t)$ such that Σ_{ij} produces a zero output for all $t \in [t_0, \tau]$. Now, let $\mathbf{B}_{ij} = [\hat{B}_{ij}\ \hat{S}_{ij}]$, $\mathbf{u} = [u^T\ \hat{d}^T]^T$ and $\hat{\mathcal{W}}_{ij} = \sup \Im(\hat{A}_{ij}, \mathbf{B}_{ij}; \hat{\mathcal{K}}_{ij})$, and set $\mathbf{u}(t) = F\hat{x}(t) + \hat{v}(t)$, with $F \in \mathbf{F}(\hat{\mathcal{W}}_{ij})$ and $v(t) = [v_1^T(t)\ v_2^T(t)]^T$, notice that this is not a restriction over $u(t)$ since there are no constraints on $v_1(t)$.

Now, since for every $v_1(t)$ there exist $\hat{x}_0$ and $v_2(t)$ such that $\hat{x}(t) \in \hat{\mathcal{W}}_{ij}$, then by Lemma 2

$$\dot{\hat{x}}(t) = (\hat{A}_{ij} + \mathbf{B}_{ij} F)\hat{x}(t) + \hat{B}_{ij} v_1(t) + \hat{S}_{ij} v_2(t) \in \hat{\mathcal{W}}_{ij}.$$

This implies that for every $v_1(t)$ there exists $v_2(t)$ such that $\hat{B}_{ij} v_1(t) + \hat{S}_{ij} v_2(t) \in \hat{\mathcal{W}}_{ij}$. Hence $\hat{\mathcal{B}}_{ij} \subset \hat{\mathcal{W}}_{ij} + \hat{\mathcal{S}}_{ij}$ and by Lemma 13, $\hat{\mathcal{W}}_{ij}$ coincides with $\sup \Im(\hat{A}_{ij}, \hat{\mathcal{S}}_{ij}; \hat{\mathcal{K}}_{ij})$ and $\hat{\mathcal{B}}_{ij} \subset \sup \Im(\hat{A}_{ij}, \hat{\mathcal{S}}_{ij}; \hat{\mathcal{K}}_{ij}) + \hat{\mathcal{S}}_{ij}$ which completes the proof. ■

Thus, if $\hat{\mathcal{B}}_{ij} \not\subseteq \hat{\mathcal{W}}_{ij} + \hat{\mathcal{S}}_{ij}$ holds, not all the admissible values of $\dot{\hat{x}}(t)$ for the state trajectories starting at $\hat{\mathcal{W}}_{ij}$ belong to $\hat{\mathcal{W}}_{ij}$, thus it is possible, by a smooth input, to maintain $\dot{\hat{x}}(t)$ out of $\hat{\mathcal{W}}_{ij}$ for a finite time and reach points not belonging to $\hat{\mathcal{W}}_{ij}$, despite any disturbance. We show next that, if this condition hold then almost every input can be used to distinguish the *LS*. To this aim, recall first that if the distinguishability property only fails to hold in a shy set of inputs the distinguishability property is said to hold for almost every input. Based on the fact that every proper subspace of $\mathcal{U}_f$ is shy, we show that, if the conditions of Proposition 14 holds, then the set of exceptions (5) is a proper subspace of $\mathcal{U}_f$. Thus, almost every input can be used to distinguish the *LS*. Notice that due to the superposition property of the extended *LS*, the set of inputs in (5) is a subspace of $\mathcal{U}_f$, in which $u(t)$ takes values, since if $u_{[t_0,\tau]}$ and $u'_{[t_0,\tau]}$ are in (5), then according to Lemma 5, there exist combinations of initial conditions and inputs such that the extended *LS* produces zero output when these inputs are applied, i.e.

$$y_{ij}\left(\hat{x}_0, u_{[t_0,\tau]}, \hat{d}_{[t_0,\tau]}\right) = y_{ij}\left(\hat{x}_0', u'_{[t_0,\tau]}, \hat{d}'_{[t_0,\tau]}\right) = 0$$

for some $\hat{x}_0$, $\hat{x}_0'$, $\hat{d}_{[t_0,\tau]}$ and $\hat{d}'_{[t_0,\tau]}$. Due to the superposition property of the extended *LS* $\hat{\Sigma}_{ij}$, for each $\alpha, \alpha' \in \mathbb{R}$

$$y_{ij}\left(\alpha\hat{x}_0 + \alpha'\hat{x}_0, \alpha u_{[t_0,\tau]} + \alpha' u'_{[t_0,\tau]}, \alpha\hat{d}_{[t_0,\tau]} + \alpha'\hat{d}'_{[t_0,\tau]}\right) = 0.$$

Hence, by Lemma 6, (8) does not hold and therefore $\alpha u_{[t_0,\tau]} + \alpha' u'_{[t_0,\tau]}$ is in (5) and thus, (5) is a subspace of $\mathcal{U}_f$.

Since, we have shown the existence of a smooth input steering the state trajectory outside $\mathcal{W}_{ij}$, then (5) is a proper subspace of $\mathcal{U}_f$, and thus is shy set with respect to $\mathcal{U}_f$ [20]. Hence, almost every input $u_{[t_0,\tau]}$ can be used to guarantee the distinguishability between Σ_i and Σ_j.

The following remark relates the set of inputs in (5) with the system zeros [24], [25] of the extended *LS* $\hat{\Sigma}_{ij}$, because by Lemma 6 if the *LS* are indistinguishable then the extended *LS* produces a zero output.

Remark 15: Let $\mathcal{U} = \mathcal{U}_1 + \mathcal{U}_2$ *with* $\mathcal{U}_1 = \{u : \hat{B}_{ij} u \in \hat{\mathcal{W}}_{ij} + \hat{\mathcal{S}}_{ij}\}$ *and* $\mathcal{U}_1 \cap \mathcal{U}_2 = 0$. *Clearly, by an argument similar to the necessity in the proof of Proposition 14, every input* $u : \mathbb{R}_{\geq 0} \to \mathcal{U}_1$ *belongs to* (5) *for a suitable* $\hat{d}(t)$. *However, the inputs* $u : \mathbb{R}_{\geq 0} \to \mathcal{U}_2$ *that are contained in* (5) *are strictly linear combinations of vectors of the form* $u_{z_i} t^{m_k - 1} e^{z_i t}$, where z_i *is such that ker* $p(z_i) \neq 0$, *where*

$$P(z_i) = \begin{bmatrix} z_i I - \hat{A}_{ij} & -\hat{B}_{ij} & -\hat{S}_{ij} \\ \hat{C}_{ij} & 0 & 0 \end{bmatrix}$$

and the minimum value of m_k *is the algebraic multiplicity of* z_i. *The values of* z_i *such that* $P(z_i)$ *drops its normal rank are system zeroes of the extended LS* $\hat{\Sigma}_{ij}$. *For more details on system zeros, the reader may refer to* [24] *and* [25].

By setting $S_i = 0$, $i = 1, \cdots, m$, (9) reduces to $\hat{\mathcal{B}}_{ij} \not\subseteq \sup \Im(\hat{A}_{ij}, 0; \hat{\mathcal{K}}_{ij})$ where $\sup \Im(\hat{A}_{ij}, 0; \hat{\mathcal{K}}_{ij})$ is the unobservable subspace $\hat{\mathcal{N}}_{ij}$ of the extended *LS* $\hat{\Sigma}_{ij}$. Thus, by considering only known inputs, Proposition 14 is equivalent to Proposition 6 of [8] and Proposition 3 of [9] (see [5]).

In [26], an observer design for the case when $S_i = 0, i = 1, \cdots, m$, has been proposed.

From the previous proposition we can state the conditions for the observability for almost every control input, which is formally presented in the following theorem.

Theorem 16: Let $\mathcal{G} = \langle \mathcal{F}, \sigma \rangle$ be a *SLS* with partially unknown inputs and maximum dwell time $\tau = \infty$. Then

1) The discrete state σ_0 and $\sigma(t)$ of $\mathcal{G}$ are observable for almost every control input if and only if $\forall \Sigma_i, \Sigma_j \in \mathcal{F}\ i \neq j, \hat{\mathcal{B}}_{ij} \nsubseteq \hat{\mathcal{W}}_{ij} + \hat{\mathcal{S}}_{ij}$.
2) The continuous state x_0 and $x(t)$ of $\mathcal{G}$ are observable for almost every control input if and only if every *LS* $\Sigma_i \in \mathcal{F}$ is observable with partially unknown inputs and $\forall \Sigma_i, \Sigma_j \in \mathcal{F}\ i \neq j\ \hat{\mathcal{B}}_{ij} \nsubseteq \hat{\mathcal{W}}_{ij} + \hat{\mathcal{S}}_{ij}$ or $\hat{\mathcal{W}}_{ij}$ is symmetric.

C. Observability for Almost Every State Trajectory

In this section we focus on deriving the conditions for the observability for almost every state trajectory. There are two ways of achieving this. The first one, by avoiding the existence of a set of initial states $\mathcal{R}$ given by (6), such that Σ_i is indistinguishable from Σ_j for every state trajectory starting from that set. The necessary and sufficient condition for this case is

$$\mathrm{Im}[\hat{B}_{ij} \quad \hat{S}_i] \nsubseteq \hat{\mathcal{W}}_{ij} + \hat{\mathcal{S}}_j$$

with $\hat{S}_i = [S_i^T\ 0]^T$ and $\hat{S}_j = [0\ S_j^T]^T$.

The second one, by requiring that if this set exists then it is a shy set. In the former almost every input (control input or disturbance) allows to distinguish the *LS*, while in the latter almost every state trajectory except for those starting in $\mathcal{R}$ will be distinguishable.

Lemma 17: Let $\mathcal{V} = \sup \Im(\hat{A}_{ij}, \hat{\mathcal{S}}_j; \hat{\mathcal{K}}_{ij})$. If $\mathrm{Im}[\hat{B}_{ij}\hat{S}_i] \subseteq \hat{\mathcal{W}}_{ij} + \hat{\mathcal{S}}_j$, then the set $\mathcal{R}$ coincides with the subspace $\mathcal{Q}_i\mathcal{V}$.

Proof: The proof follows from Lemma 13. ■

Proposition 18: Let Σ_i and Σ_j be two *LS* and let $\mathcal{V} = \sup \Im(\hat{A}_{ij}, \hat{\mathcal{S}}_j; \hat{\mathcal{K}}_{ij})$. Then the system Σ_i is distinguishable from Σ_j for almost every state trajectory (i.e. $\mathcal{R}$ is a shy set) if and only if one of the following conditions hold

1) $\mathrm{Im}[\hat{B}_{ij}\hat{S}_i] \nsubseteq \hat{\mathcal{W}}_{ij} + \hat{\mathcal{S}}_j$ (equivalently $\mathrm{Im}[\hat{B}_{ij}\hat{S}_i] \nsubseteq \mathcal{V} + \hat{\mathcal{S}}_j$) or
2) $\dim(\mathcal{Q}_i\hat{\mathcal{W}}_{ij}) < \dim(\mathcal{X})$ (equivalently $\dim(\mathcal{Q}_i\mathcal{V}) < \dim(\mathcal{X})$).

Proof: (Sufficiency) Two cases are possible:

a) $\mathrm{Im}[\hat{B}_{ij}\hat{S}_i] \nsubseteq \mathcal{V} + \hat{\mathcal{S}}_j$, then it does not exist $d'(t)$ such that the effect of $u(t)$ and $d(t)$ does not show up at the output of the extended systems i.e. the set $\mathcal{R}$ is empty.
b) If $\mathrm{Im}[\hat{B}_{ij}\hat{S}_i] \subseteq \mathcal{V} + \hat{\mathcal{S}}_j$ but $\dim(\mathcal{Q}_i\mathcal{V}) < \dim(\mathcal{X})$ then by Lemma 17 $\mathcal{Q}_i\mathcal{V} = \mathcal{R}$, since $\mathcal{Q}_i\mathcal{V}$ is a subspace of $\mathcal{X}$ and $\dim(\mathcal{Q}_i\mathcal{V}) = \dim(\mathcal{R}) < \dim(\mathcal{X})$ then $\mathcal{R}$ is clearly a shy set.

(Necessity) Assume that $\mathrm{Im}[\hat{B}_{ij}\hat{S}_i] \subseteq \mathcal{V} + \hat{\mathcal{S}}_j$ and $\dim(\mathcal{Q}_i\mathcal{V}) = \dim(\mathcal{X})$ then by Lemma 17 $\mathcal{Q}_i\mathcal{V} = \mathcal{R} = \mathcal{X}$. Thus, $\mathcal{R}$ is not a shy set. ■

By setting $B_i = 0$ and $S_i = 0, i = 1, \cdots, m$, in (2) the conditions of Proposition 18 reduce to $\dim(\mathcal{Q}_i\hat{\mathcal{N}}_{ij}) < \dim(\mathcal{X})$ and it is equivalent to the condition of Proposition 1 of [9] (see [5]).

Based on these results we can easily characterize the following observability notions.

Theorem 19: Let $\mathcal{G} = \langle \mathcal{F}, \sigma(t) \rangle$ be a *SLS* with partially unknown inputs and maximum dwell time $\tau = \infty$. Then

1) The discrete states σ_0 and $\sigma(t)$ of the *SLS* $\mathcal{G}$ are observable for almost every state trajectory if and only if every pair of different *LS* are distinguishable for almost every state trajectory according to Proposition 18.
2) The continuous states x_0 and $x(t)$ of the *SLS* $\mathcal{G}$ are observable for almost every state trajectory if and only if every pair of different *LS* are either distinguishable for almost every state trajectory according to Proposition 18 or their distinguishability subspace $\hat{\mathcal{W}}_{ij}$ is symmetric.

V. Conclusions

A geometric characterization for the observability of continuous—time *SLS* under partially unknown inputs has been presented, separately for continuous and discrete states, for three different cases *a)* observability for every nonzero state trajectory; *b)* observability for almost every control input; *c)* observability for almost every state trajectory. These results are based on the geometric concept of (A, B)—invariant subspaces and provide testable conditions for verifying the different notions of the observability property under fully known inputs and partially or fully unknown inputs.

Future work can regard the study of systems whose output is affected by the disturbance and a possible connection between the results presented here and existing results for discrete—time *SLS* subject to unknown inputs. Also, as future work we consider to study the conditions for the observability under the assumption of this work, but when a maximum dwell time is set. In this situation the state does not need to be recovered before the first switching time, but after a finite number of switches.

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How Nonlinear Parametric Wiener System Identification is Under Gaussian Inputs?

Zhijun Cai, *Member, IEEE*, and Er-Wei Bai, *Fellow, IEEE*

***Abstract*—A number of methods exist for identifying nonlinear Wiener systems. However, there is no attempt to address the fundamental question of how nonlinear these identification problems really are? In this technical note, we try to address this question by investigating the average squared error cost function used in identification. By a proper normalization and a clever characterization of the cost function in terms of the angle between the true but unknown parameter vector and its estimate, it is shown in the technical note that under iid Gaussian inputs for parametric Wiener systems with polynomial nonlinear parts and FIR linear parts, the cost function is globally monotonic and has one and only one (local and global) minimum. The implication is that identification of such systems is nonlinear but very close to linear. Further, any local search based identification algorithms would converge globally for such systems.**

***Index Terms*—Block-oriented nonlinear system, nonlinear system identification, Wiener systems.**

I. Introduction

Wiener systems are well known nonlinear systems that are special classes of the block oriented nonlinear systems. They find applications in a variety of fields [9] and therefore, identification of Wiener systems has been an active research area for a long time [2]–[8], [12] and a number of identification algorithms have been developed with sound theoretical properties in the literature. Obviously, a difficulty in Wiener system identification is the interaction between the static nonlinear part and the dynamic linear part. Without the static nonlinearity, identification would be linear and any linear method could apply. Because of the static nonlinearity, identification of such systems becomes nonlinear and nontrivial.

Consider a common average squared error cost function for identification. The purpose of parametric Wiener system identification is to determine the unknown parameters by minimizing the cost function. In a sense, identification is a minimization problem. If a system is linear in the unknown parameters, the average squared error cost function is quadratic in the unknown parameters and thus, the problem is convex and the solution is unique which can be solved efficiently by e.g., least squares or gradient type algorithms. The problem of a general nonlinear system identification is that the underlining minimization problem is no longer convex and moreover there may exist many local minima. Thus, identification algorithms designed for linear systems or local search based algorithms fail to reach the global minimum.

It is a fact that Wiener system identification is also nonlinear. Various algorithms have been developed trying to overcome the nonlinear and non-convexity problem presented in Wiener system identification [2], [3], [8], [12]. To the best of our knowledge, no effort has been devoted to find out how nonlinear Wiener system identification really is. This is a very interesting and important issue. Though nonlinear, nonlinearities can have various degrees. If a nonlinear system identification problem is non-convex but is monotonic and has one and only one (local and global) minimum, this identification problem is not very hard. In fact, any gradient based search algorithm converges globally. For these kinds of identification problems, though nonlinear, they are very close to a linear problem and many algorithms developed for linear systems can be applied with minor modifications. To develop sophisticated nonlinear identification algorithms for these problems is an unnecessary over-kill. In this technical note, we focus on parametric Wiener systems with polynomial nonlinear parts, FIR linear parts and Gaussian inputs. The problem to be investigated is how nonlinear identification of such systems really is. By a proper normalization and a clever characterization of the cost function in terms of the angle between the true but unknown parameter vector and its estimate, we are able to show under iid Gaussian inputs that the average squared error cost function is globally monotonic in the angle, and has one and only one (local and global) minimum. Therefore, any local search based identification algorithms and in fact, many other identification algorithms converge globally. The derived results in the current technical note are also useful in explaining why some existing methods work for Wiener system identification.

II. Systems and Assumptions

In this technical note, we study identification of parametric Wiener systems with polynomial nonlinear parts and FIR linear parts given, respectively, by

$$y(t) = \sum_{j=0}^{n} b_j^* x(t)^j + v(t), \quad x(t) = \sum_{i=0}^{m} a_i^* u(t-i). \tag{1}$$

In (1), $u(t)$, $x(t)$, $y(t)$ and $v(t)$ are the system input, not measured internal signal, output and noise respectively.

Assumption II.1: The input $u(t)$ is assumed to be iid, zero mean and unit variance Gaussian. Noise is assumed to be iid, zero mean, finite

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The authors are with Department of Electrical and Computer Engineering, University of Iowa, Iowa City 52242 USA (e-mail: zhijun.cai@gmail.com; erwei@engineering.uiowa.edu).

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