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RESEARCH ARTICLE

Digital Self Triggered Robust Control of Nonlinear Systems

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In this paper we develop novel results on self triggered control of nonlinear systems, subject to perturbations, and sensing/computation/actuation delays. First, considering an unperturbed nonlinear system with bounded delays, we provide conditions that guarantee the existence of a self triggered control strategy stabilizing the closed–loop system. Then, considering parameter uncertainties, disturbances, and bounded delays, we provide conditions guaranteeing the existence of a self triggered strategy, that keeps the state arbitrarily close to the equilibrium point. In both cases, we provide a methodology for the computation of the next execution time. We show on an example the relevant benefits obtained with this approach, in terms of energy consumption, with respect to control algorithms based on a constant sampling, with a sensible reduction of the average sampling time.

Keywords: Self triggered control, non-linear systems, robust control.

1 Introduction

Wireless networked control systems are spatially distributed control systems where the communication between sensors, actuators, and computational units is supported by a shared wireless communication network [9]. The use of wireless networked control systems in industrial automation results in flexible architectures and generally reduces installation, debugging, diagnostic and maintenance costs with respect to wired networks. The main motivation for studying such systems is the emerging use of wireless technologies in control systems, see e.g. [1], [17] and references therein. Although wireless networks offer many advantages, communication nodes generally consist of battery powered devices. For this reason, when designing a control scheme closed on a wireless sensor network, it is fundamental to adopt power aware control algorithms to reduce energy consumption. An example is notably given by the so–called intelligent (or smart) tires, equipped with sensors embedded in the tread, giving information on pressure, road–tire friction, etc. [16], [13], [4]. The sensors are only supplied with the energy provided by the tire motion, and it is fundamental to trigger wireless transmission only when necessary, to prevent energy shortage and, possibly, to reduce the probability of information packet losses during the transmission.

With the aim of addressing the above issues in the controller design phase, self triggered control strategies have been introduced in [20], where a heuristic rule is provided to self–trigger the next execution time of a control task on the basis of the last measurement of the state. In [11], [12], a robust self triggered strategy is proposed, which guarantees that the $\mathscr{L}_2$ gain of a linear time invariant system is kept under a given threshold. In [14] a self triggered strategy distributed over a wireless sensor network is proposed for linear time invariant systems.

In [18] sufficient conditions for the existence of a stabilizing event–triggered control strategy are given for nonlinear systems. In [3] the authors propose a self–triggered emulation of the event–triggered control strategy proposed in [18]. In particular a methodology for the computation of the next execution time as a function of the last sample is presented, under a homogeneity condition. We extend the previous results in two directions. These results are extended in [22] to smooth unperturbed nonlinear systems.

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In this paper we consider nonlinear systems perturbed by norm–bounded parameter uncertainties and disturbances, and affected by bounded delays. Under weaker conditions than those used in [18] we prove the existence of a self triggered strategy keeping the state in a *safe set* arbitrarily close to the equilibrium point, and provide a methodology for computing the next execution time. Given a δ boundary of the equilibrium point, a sensing/computation/actuation delay bounded by a positive real Δ_{max} and a disturbance upper bounded in norm by a class $\mathscr{K}$ function $\nu(\delta)$, we provide a self-triggering rule to compute the next execution time as a function of δ and of the bounds $\Delta_{max}, \nu(\delta)$ on delays and disturbances. This is the main contribution of the paper: indeed, to the best of the authors' knowledge, this is the first work that provides results on self–triggered control for nonlinear systems with uncertainties, disturbances and delays. Our technique is based on polynomial approximations of Lyapunov functions, and therefore differs from the one recently developed in [22]. As a consequence an additional contribution of this paper is providing a novel technique for the computation of the next execution time, which represents an alternative to [22] for guaranteeing asymptotic stability of unperturbed non-linear systems affected by bounded sensing/computation/actuation delays.

The paper is organized as follows. In Section 2, we illustrate the mathematical setting and the problem formulation. In Section 3 we derive results for the asymptotic stability of unperturbed systems. In Section 4 we consider the safety problem of perturbed systems and provide the main results of the paper. In Section 5, on a significant example, we show that the results obtained introduce strong benefits in terms of energy consumption, with respect to digital controls based on a constant sampling time, by reducing the average sampling time. A preliminary version of the results provided in this paper can be found in [6].

2 Problem Formulation

Consider a generic nonlinear system

$$\dot{x} = f(x, u, \mu, d) \tag{1}$$

where $x \in \mathscr{D}_x \subset \mathbb{R}^n$, $\mathscr{D}_x$ a domain containing the origin, $u \in \mathscr{D}_u \subset \mathbb{R}^p$, μ is a parameter uncertainty vector varying in a compact set $\mathscr{D}_\mu \subset \mathbb{R}^r$, with $0 \in \mathscr{D}_\mu$, d is an external bounded disturbance vector taking values in a compact set $\mathscr{D}_d \subset \mathbb{R}^s$, with $0 \in \mathscr{D}_d$. In the following, we may refer to (1) as the perturbed system. Furthermore, we define the nominal (or "unperturbed") system associated to the "perturbed" system (1) as

$$\dot{x} = f_0(x, u) \doteq f(x, u, 0, 0). \tag{2}$$

Given a state feedback control law $\kappa\colon \mathscr{D}_x \to \mathscr{D}_u$, the closed loop perturbed system is

$$\dot{x} = f(x, \kappa(x), \mu, d) \tag{3}$$

and the closed loop nominal system is

$$\dot{x} = f_0(x, \kappa(x)). \tag{4}$$

We will denote by $x(t), t \geq t_0$, the solution of the closed loop system (3) (or (4), according to the context), with initial condition $x_0 = x(t_0)$. Given a state feedback control law κ it is well–known that, if κ locally stabilizes the origin of system (4) and if $f_0(x, \kappa(x)) \in C^\ell(\mathscr{D}_x)$, $\ell > 1$ integer, then there exists a Lyapunov function $V(x)$ of class $C^1(\mathscr{D}_x)$ such that

$$\begin{aligned} \alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \\ \frac{\partial V(x)}{\partial x} f_0(x, \kappa(x)) \leq -\alpha_3(\|x\|) \\ \left\| \frac{\partial V(x)}{\partial x} \right\| \leq \alpha_4(\|x\|) \end{aligned} \tag{5}$$

with $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathscr{K}$ [8], [10], [21].

Moreover, given a state feedback control law κ, we say that system (3) is safe with respect to the set $\mathscr{S} \subseteq \mathscr{D}_x$ for the time interval $\mathscr{T} \subseteq \mathbb{R}_0^+$, if $x(t) \in \mathscr{S}$, $\forall t \in \mathscr{T}$.

The feedback control signal $u(t) = \kappa(x(t))$ requires continuous measurements of the state of the system. We assume that state measurements are available at sampling times t_k, which define a sequence $\mathscr{I} = \{t_k\}_{k\geq 0}$, and that the applied control is

$$u_{\mathscr{I}}(t) = \begin{cases} 0 & \forall t \in [t_0, t_0 + \Delta_0) \\ \kappa(x_k) & \forall t \in [t_k + \Delta_k, t_{k+1} + \Delta_{k+1}),\ k \geq 0 \end{cases} \tag{6}$$

where $\{\Delta_k\}_{k\geq 0}$ is a sequence of delays, due to the transmission time from the sensor to the controller, the computation time, and the transmission time from the controller to the actuator. On the basis of this assumption, we address the following problems.

Problem 2.1 Given a system (2), and a state feedback control law κ such that the origin of (4) is asymptotically stable with region of attraction $\Omega \subset \mathscr{D}_x$ containing the origin, determine a function $\tau_s : \mathscr{D}_x \to [\tau_{\min}, \infty)$, $\tau_{\min} > 0$ and a maximum allowed delay $\Delta_{\max} \in [0, \tau_{\min}]$ such that if the sequence of sampling instants $\mathscr{I}$ is inductively defined by

$$t_{k+1} = t_k + \tau_s(x_k) \tag{7}$$

and if the delays are such that

$$\Delta_k \in [0, \Delta_{\max}), \quad \forall k \geq 0, \tag{8}$$

then the origin of the closed loop system (4) with control input signal $u_{\mathscr{I}}(t)$ as in (6) is asymptotically stable with region of attraction Ω. $\diamond$

Problem 2.2 Given a system (1) (resp. (2)), and a state feedback control law κ such that the origin of (3) (resp. (4)) is asymptotically stable with region of attraction $\Omega \subset \mathscr{D}_x$ containing the origin, and an arbitrary safe set $\mathscr{B}_\delta = \{x \in \mathbb{R}^n \mid \|x\| < \delta\} \subset \Omega$, $\delta > 0$, determine τ_s and $\Delta_{\max}$ as defined in Problem 2.1 such that if $\mathscr{I}$ is inductively defined by (7) and if Δ_k satisfies (8), then the closed loop system (3) with control input signal $u_{\mathscr{I}}(t)$ as in (6) is safe with respect to $\mathscr{B}_\delta$ for the time interval $[t_0, \infty)$. $\diamond$

In Problems 2.1 and 2.2, the function τ_s is used to determine the next sampling instant as a function of the current measurement of the system. The purpose is to obtain a self triggered control system that is robust with respect to delays bounded by a design parameter $\Delta_{\max}$. By choosing the next sampling instant t_{k+1} as a function of the current measurement at time t_k, we perform sampling only when needed for guaranteeing asymptotic stability or safety. The aim is to determine a sampling instant sequence $\mathscr{I}$ such that the intersampling time $t_{k+1} - t_k$ is as large as possible, in order to reduce transmission power of the sensing and actuation data transmissions, and to reduce the CPU effort due to the computation of the control.

As a comparison of the above definitions with the concepts of Maximally Allowable Transmission Interval (MATI) and Maximally Allowable Delay (MAD) introduced in [7], we can interpret $\Delta_{\max}$ as the

MAD, and $t_{k+1} - t_k = \tau_s(x_k) - t_k$ as the *local* MATI of the system in the time interval $[t_k, t_{k+1}]$ on the basis of the measurement $x_k = x(t_k)$ of the state $x(t)$ at $t = t_k$.

3 Self Triggered Stabilizing Control

The results developed in this section address Problem 2.1 for system (2), and are based on the following assumptions to compute the next sampling time as a function τ_s of the current state of the system.

Assumption 3.1 Assume that

1. $f_0 \in C^\ell(\mathscr{D}_x \times \mathscr{D}_u)$, with ℓ a positive integer sufficiently large;
2. There exists a nonempty set $\mathscr{U}$ of state feedback laws $\kappa\colon \mathscr{D}_x \to \mathscr{D}_u$, such that $\kappa \in C^\ell(\mathscr{D}_x)$ and the origin of (4) is asymptotically stable, with region of attraction a certain compact $\Omega \subset \mathscr{D}_x$ containing the origin;
3. The functions $\alpha_3, \alpha_4 \in \mathscr{K}$ in (5) are such that α_3^{-1}, α_4 are Lipschitz. $\diamond$

The assumption of existence of a stabilizing control (i.e. non–emptiness of the set $\mathscr{U}$) is not restrictive, since if the nominal system cannot be stabilized using continuous time measurement and actuation, then it is clear that the nominal system cannot be stabilized using a digital control with zero–order holders. The main limitation of Assumption 3.1 is the Lipschitz condition on $\alpha_3^{-1}(\cdot)$ and $\alpha_4(\cdot)$. We will show how to weaken this assumption in Section 4, which will be devoted to safety control.

Theorem 3.2 : *Under Assumption 3.1, Problem 2.1 is solvable for system (2), and the function τ_s can be iteratively computed as a function of the current state of the system and the maximum allowable delay $\Delta_{\max}$.* $\diamond$

Proof: We first prove the result for $\Delta_k = 0$. Since $\mathscr{U}$ is not empty, by Assumption 3.1, we pick a state feedback control law $\kappa \in \mathscr{U}$. Since $f_0(x, \kappa(x)) \in C^\ell(\mathscr{D}_x)$ with $\ell > 1$, there exists a Lyapunov candidate $V(x)$ that satisfies (5). Choose $r > 0$ such that the ball $B_r = \{x \in \Omega : \|x\| \le r\} \subset \Omega$. For $x_k \in B_r$,

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial x} f_0(x, \kappa(x_k)) = \frac{\partial V}{\partial x} f_0(x, \kappa(x)) + \frac{\partial V}{\partial x}\Big(f_0(x, \kappa(x_k)) - f_0(x, \kappa(x))\Big) \\ &\le -\alpha_3(\|x\|) + \alpha_4(\|x\|)\|d_h\|\end{aligned} \tag{9}$$

where

$$d_h = f_0(x(t), \kappa(x_k)) - f_0(x(t), \kappa(x(t)))$$

can be considered as a perturbation due to the holding.

Under Assumption 3.1, there exists a $\zeta_k > 0$ such that $\dot{x} = f(x, \kappa(x_k))$ has a unique solution over $[t_k, t_k + \zeta_k]$. Hence, we can expand the components $d_{h,i}$ of d_h in Taylor series. Consider the i^{th} component $d_{h,i}$, $i = 1, \cdots, n$, of the n–dimensional vector d_h. One can expand each component in Taylor series with respect to $t \in [t_k, t_k + \zeta_k]$, on the right of t_k, up to the 2^{nd} term, with Lagrange remainder of the 3^{rd} term [19]. According to Taylor theorem with Lagrange remainder, there exists $\bar{t}_i \in [t_k, t]$, with $\bar{x}_i = x(\bar{t}_i)$, $i = 1, \cdots, n$, such that

$$d_{h,i} = \varphi_{1,i}(x_k)(t - t_k) + \varphi_{2,i}(\bar{x}_i, x_k)(t - t_k)^2, \tag{10}$$

with

$$\varphi_{1,i}(x_k) = \left.\frac{\mathrm{d}_+ d_{h,i}}{\mathrm{d}t}\right|_{x(t)=x_k}, \quad \varphi_{2,i}(\bar{x}_i, x_k) = \frac{1}{2}\left.\frac{\mathrm{d}_+^2 d_{h,i}}{\mathrm{d}t^2}\right|_{x(t)=\bar{x}_i}$$

and where $\frac{\mathrm{d}_+^n(\cdot)}{\mathrm{d}t^n}$ denotes the n–th right derivative. Hence,

$$\|d_h\| \leq \|\varphi_1(x_k)\|(t-t_k) + \|\varphi_2(\bar{x},x_k)\|(t-t_k)^2$$

where $\bar{x} \doteq (\bar{x}_1,\cdots,\bar{x}_n)$ and

$$\varphi_1(x_k) \doteq \Big(\varphi_{1,1}(x_k),\cdots,\varphi_{1,n}(x_k)\Big)^T$$
$$\varphi_2(\bar{x},x_k) \doteq \Big(\varphi_{2,1}(\bar{x}_1,x_k),\cdots,\varphi_{2,n}(\bar{x}_n,x_k)\Big)^T.$$

Consider the set $\Omega_{V(x_k)} \doteq \{x \in \Omega : V(x) \leq V(x_k)\}$, and define

$$M_1(x_k) \doteq \|\varphi_1(x_k)\|, \qquad M_2(x_k) \doteq \max_{\bar{x}\in\Omega_{V(x_k)}} \|\varphi_2(\bar{x},x_k)\|.$$

Since $f,\kappa \in C^\ell$ and $\Omega_{V(x_k)}$ is compact, then $M_1(x_k)$ is finite for any $x_k \in \Omega_{V(x_k)}$, and $M_2(x_k) \in \mathbb{R}^+$ exists and is finite for any $x_k \in \Omega_{V(x_k)}$.

Note that there exists a time interval $[t_k,t_{k+1}]$ such that $t_{k+1} < t_k + \zeta_k$ and

$$\alpha_4(\|x\|)\|d_h\| \leq \vartheta\alpha_3(\|x\|) \tag{11}$$

is satisfied for a fixed $\vartheta \in (0,1)$. In fact, (11) is satisfied if

$$\alpha_3^{-1}\left(\frac{1}{\vartheta}\alpha_4(\|x\|)\left(M_1(x_k)(t-t_k)+M_2(x_k)(t-t_k)^2\right)\right) \leq \|x\|.$$

Since α_3^{-1} and α_4 are Lipschitz, then equation (11) is satisfied if

$$\frac{1}{\vartheta}L_{\alpha_3^{-1}}L_{\alpha_4}\|x\|\Big(M_1(x_k)(t-t_k)+M_2(x_k)(t-t_k)^2\Big) \leq \|x\|$$

where $L_{\alpha_3^{-1}}, L_{\alpha_4} > 0$ are the Lipschitz constants of α_3^{-1}, α_4, respectively. The above equation directly implies that (11) is satisfied if

$$M_1(x_k)(t-t_k)+M_2(x_k)(t-t_k)^2 \leq \frac{\vartheta}{L_{\alpha_3^{-1}}L_{\alpha_4}}. \tag{12}$$

Hence, if we define

$$\tau_s(x_k) \doteq \max\left\{t-t_k:\ (12) \text{ is satisfied for each } t-t_k \in [0,\tau_s(x_k)]\right\}$$
$$\tau_{\min} \doteq \min_{x_k\in\Omega_{V(x_k)}} \tau_s(x_k)$$

and we choose $t_{k+1} = t_k + \tau_s(x_k)$, then $\dot{V}(t) \leq -(1-\vartheta)\alpha_3(\|x\|) < 0$ for all $t \in [t_k,t_{k+1}]$ and for all $k \geq 0$. This implies that the origin is asymptotically stable. Equation (12) is a second degree inequality in the form $ay^2+by \leq c$, where a,b are non–negative and upper bounded for each $x_k \in \mathscr{D}_x$, and c is strictly positive and upper bounded. This trivially implies that $\tau_s(x_k)$ is strictly positive for each $x_k \in \Omega_{V(x_k)}$, and thus $\tau_{\min}$ is strictly positive as well. In this way, $\tau_s(\cdot)$ remains defined iteratively for each $k \geq 0$. This completes the proof for $\Delta_k = 0$.

For the case of $\Delta_k > 0$, following the same reasoning

$$\begin{aligned}\dot{V}(t) &= \frac{\partial V}{\partial x} f_0(x(t), \kappa(x_k)) = \frac{\partial V}{\partial x} f_0(x, \kappa(x)) + \frac{\partial V}{\partial x}(d_h + d_{\Delta_k}) \\ &\leq -\alpha_3(\|x\|) + \alpha_4(\|x\|)\|d_h\| + \alpha_4(\|x\|)\|d_{\Delta_k}\|\end{aligned}$$

for $t \geq t_k + \Delta_k$ where

$$\begin{aligned}d_h &= f_0(x(t), \kappa(x(t_k + \Delta_k))) - f_0(x, \kappa(x)) \\ d_{\Delta_k} &= f_0(x(t), \kappa(x_k)) - f_0(x(t), \kappa(x(t_k + \Delta_k)))\end{aligned}$$

can be considered as perturbations due to the holding and to the sensing/computation/actuation delay. Since also the solution $x(t)$ is Lipschitz, as well as f_0 and κ, then

$$\|d_{\Delta_k}\| \leq M_3 \Delta_k, \quad M_3 = L_{f_0} L_\kappa L_x$$

where L_{f_0}, L_κ, L_x are the Lipschitz constants of f_0, κ, x. Proceeding for d_h as in the previous case, we conclude that (11) is satisfied if

$$M_1(x_k)(t - t_k) + M_2(x_k)(t - t_k)^2 + M_3 \Delta_k \leq \frac{\vartheta}{L_{\alpha_3^{-1}} L_{\alpha_4}}. \tag{13}$$

Setting $\vartheta = \vartheta_1 + \vartheta_2$, with $\vartheta_1, \vartheta_2 \in (0,1)$, equation (13) implies that the stability condition (11) holds if

$$M_1(x_k)(t - t_k) + M_2(x_k)(t - t_k)^2 \leq \frac{\vartheta_1}{L_{\alpha_3^{-1}} L_{\alpha_4}}, \tag{14}$$

and

$$\Delta_k \leq \Delta_{\max} \doteq \frac{\vartheta_2}{M_3 L_{\alpha_3^{-1}} L_{\alpha_4}}. \tag{15}$$

Defining

$$\begin{aligned}\tau_s(x_k) &\doteq \max\left\{t - t_k :\ \text{(14) is satisfied for each } t - t_k \in [0, \tau_s(x_k)]\right\} - \Delta_{\max} \\ \tau_{\min} &\doteq \min_{x_k \in \Omega_{V(x_k)}} \tau_s(x_k)\end{aligned}$$

and if we choose $t_{k+1} = t_k + \tau_s(x_k)$, then $\dot{V}(t) \leq -(1-\vartheta)\alpha_3(\|x\|) < 0$ for all $t \in [t_k + \Delta_k, t_{k+1} + \Delta_{\max}]$ and for all $k > 0$. This ensures the asymptotic stability of the origin. $\Delta_{\max}$ is non-negative, and for ϑ_2 sufficiently small $t_{k+1} - t_k = \tau_s(x_k) > \Delta_{\max} \geq 0$ for each $x_k \in \Omega_{V(x_k)}$. Therefore, $\tau_{\min}$ is strictly positive as well. This completes the proof. ∎

Remark 1 : It is worth noting that $\tau_{\min} > 0$, as shown in the proof, implies that the time interval between two sampling instants is lower bounded by the minimum sampling time $\tau_{\min} > 0$. As a consequence, the self-triggering rule is implementable and undesired Zeno behaviors are avoided. ◇

It should not only be said that the Zeno phenomenon is avoided but simply that the inter-execution times is strictly positive which is what we (also) want to know before implementing the triggering rule

Remark 2 : The choice of $\vartheta_1 \in (0,1)$ corresponds to a simple tradeoff between larger intersampling times $\tau_s(x_k)$ and robustness with respect to larger delays $\Delta_{\max}$. As ϑ_1 decreases, $\tau_s(x_k)$ decreases and

$\Delta_{\max}$ increases. This implies that we improve robustness vs delays, paid by stronger sampling requirements. $\diamond$

Remark 3: When applying the self triggered rule defined in the above theorem in a real scenario, it is necessary to compute on–line the next sampling time for each time instant t_k. This computation corresponds to solving a second degree equality, which is reasonable in an embedded system. On the contrary, the functions $M_1(\cdot)$ and $M_2(\cdot)$ can be determined off–line, and then (numerically) computed on–line in x_k. However, $M_2(\cdot)$ might be still difficult to determined in closed form. In this case, one can define

$$M_2 \doteq \max_{\bar{x},x_k \in \Omega} \|\varphi_2(\bar{x},x_k)\|$$

and use it in equation (12) to compute the next sampling time. This new definition clearly implies shorter sampling times. $\diamond$

The above remarks also apply to Theorems 4.3 and 4.5 in the following Sections.

4 Self Triggered Safety Control

The main limitation of the results developed in Section 3 is the Lipschitz continuity assumption of $\alpha_3^{-1}(\cdot)$ and $\alpha_4(\cdot)$. The following example shows that even exponentially stabilizable systems do not always satisfy this assumption.

Example 4.1 Consider the system $\dot{x} = Ax + Bu + f(x,u) = f_0(x,u)$ with

$$f_0(x,u) = \begin{pmatrix} -x_1 + x_2 + x_1^2 \\ (1+x_1)u \end{pmatrix}.$$

Let $u = \kappa(x) = -x_2 \in \mathscr{U}$. Consider the Lyapunov candidate $V(x) = x^T Px$, with P solution of the Lyapunov equation $PA_c + A_c^T P = -Q$, with $Q = 2I$, I the identity matrix, and $A_c = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$. Since $P = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$, then $\lambda_{\min}^P \cong 1.382$ and $\lambda_{\max}^P \cong 3.618$ denote respectively the minimum and the maximum eigenvalue of P. For $\|x\| \leq 2/3$, the time derivative of V satisfies

$$\dot{V} = -\|x\|_Q^2 + 2|x_1|^3 + 3|x_1|x_2^2 \leq -2x_1^2 - 2x_2^2 + 2(2/3)x_1^2 + 3(2/3)x_2^2 \leq -\frac{1}{2}\|x\|^2$$

thus the origin is locally exponentially stable, with $\alpha_1(\|x\|) = \lambda_{\min}^P \|x\|^2$, $\alpha_2(\|x\|) = \lambda_{\max}^P \|x\|^2$, $\alpha_3(\|x\|) = \|x\|^2/2$, $\alpha_4(\|x\|) = \lambda_{\max}^P \|x\|$.

It is clear that, for the chosen Lyapunov candidate, Assumption 3.1 is not satisfied since $\alpha_3^{-1}(\cdot)$ is not Lipschitz. For this reason, on the basis of the previous results, one can not ensure the existence of a stabilizing self triggered strategy. $\diamond$

The main technical problem is that, if $\alpha_3^{-1}(\cdot)$ is not Lipschitz, the next sampling time $\tau_s(x_k)$ goes to zero as x_k approaches the equilibrium point, and this might generate Zeno behaviors. The results developed in this section address Problem 2.2, both for the nominal system (2) and the generic system (1), and are based on the following assumption, that does not require $\alpha_3^{-1}(\cdot)$ to be Lipschitz.

Assumption 4.2 Assume that $f_0 \in C^\ell(\mathscr{D}_x \times \mathscr{D}_u)$, with ℓ a positive integer sufficiently large. Assume that there exists a nonempty set $\mathscr{U}$ of state feedback laws $\kappa\colon \mathscr{D}_x \to \mathscr{D}_u$, such that $\kappa \in C^\ell(\mathscr{D}_x)$ and the origin of the system (4) is asymptotically stable, with region of attraction a certain compact $\Omega \subset \mathscr{D}_x$ containing the origin. $\diamond$

For system (2) (unperturbed case) we determine a function τ_s to compute the next sampling time as a function of the current state of the system and the maximum allowable delay $\Delta_{\max}$, such that the closed loop system applying a self triggered strategy is safe. On the basis of Assumption 4.2, in Theorem 4.3 we provide a different computation of τ_s providing less conservative (less frequent) sampling instants.

For system (1) (perturbed case), given a δ boundary of the equilibrium point and a disturbance that is upper bounded in norm by a class $\mathscr{K}$ function $\nu(\delta)$, we determine a function τ_s to compute the next sampling time as a function of the current state of the system and the maximum allowable delay $\Delta_{\max}$, such that the closed loop system applying a self triggered strategy is safe with respect to δ. We remark that, according to well known results in [8], a locally stable system subject to a bounded disturbance always satisfies a safety property with respect to δ sufficiently small. Nevertheless neither the computation of the function τ_s nor the relation among the safe boundary δ and the disturbance upper bound $\nu(\delta)$ are straightforward from the results in [8].

4.1 *Unperturbed Systems*

The following theorem states that, if a system (2) is asymptotically stabilizable using a continuous time state feedback control law, then it is always possible to keep the state arbitrarily close to the equilibrium point by applying a digital self triggered strategy. Note that, in order to guarantee that the state is *arbitrarily* close to the equilibrium point, we need the stabilizability assumption.

Theorem 4.3: *Under Assumption 4.2, Problem 2.2 is solvable for system (2), and the function τ_s can be iteratively computed as a function of the current state of the system and the maximum allowable delay $\Delta_{\max}$.* ◇

Proof: Using the same reasoning of Theorem 3.2 proof, and directly considering the case $\Delta_k > 0$, we conclude that the following inequality

$$\dot{V} \leq -(1-\vartheta)\alpha_3(\|x\|) + \alpha_4(\|x\|)(\|d_h\| + \|d_\Delta\|) - \vartheta\alpha_3(\|x\|) \ \leq -(1-\vartheta)\alpha_3(\|x\|)$$

holds when

$$\alpha_4(\|x\|)\Big(M_1(x_k)(t-t_k) + M_2(x_k)(t-t_k)^2 + M_3\Delta_k\Big) \leq \vartheta\alpha_3(\|x\|)$$

with $\vartheta \in (0,1)$, and d_h, d_Δ, $M_1(x_k)$, $M_2(x_k)$, M_3 defined as in Theorem 3.2. The above inequality holds if

$$\|x\| \geq \eta \doteq \alpha_3^{-1}\left(\frac{\alpha_4(\delta)}{\vartheta}\Big(M_1(x_k)(t-t_k) + M_2(x_k)(t-t_k)^2 + M_3\Delta_k\Big)\right).$$

This implies, by [8], that there exists $b := \alpha_1^{-1}(\alpha_2(\eta)) > 0$ such that $\|x(\tau)\| \leq b$, $\forall \tau \in [t_k, t]$ if $x_k \in \mathscr{B}_b$ and the following holds

$$\alpha_4(\delta)\Big(M_1(x_k)(t-t_k) + M_2(x_k)(t-t_k)^2 + M_3\Delta_k\Big) \leq \vartheta\alpha_3\Big(\alpha_2^{-1}(\alpha_1(\delta))\Big) \tag{16}$$

where we imposed the constraint $b = \delta$. Equation (16) holds if the following inequalities hold

$$\begin{aligned} &\alpha_4(\delta)\Big(M_1(x_k)(t-t_k) + M_2(x_k)(t-t_k)^2\Big) \leq \vartheta_1\alpha_3\Big(\alpha_2^{-1}(\alpha_1(\delta))\Big) \\ &\alpha_4(\delta)M_3\Delta_k \leq \vartheta_2\alpha_3\Big(\alpha_2^{-1}(\alpha_1(\delta))\Big) \end{aligned} \tag{17}$$

where we have set $\vartheta = \vartheta_1 + \vartheta_2$, with $\vartheta_1, \vartheta_2 \in (0,1)$ and $\vartheta_1 + \vartheta_2 < 1$. Defining

$$\begin{aligned}
\Delta_{\max} &\doteq \vartheta_2 \frac{\alpha_3\left(\alpha_2^{-1}(\alpha_1(\delta))\right)}{\alpha_4(\delta) M_3} \\
\tau_s(x_k) &\doteq \max\{t - t_k : \text{ (17) is satisfied for each } t - t_k \in [0, \tau_s(x_k)]\} - \Delta_{\max} \\
\tau_{\min} &\doteq \min_{x_k \in \mathscr{B}_\delta} \tau_s(x_k)
\end{aligned}$$

and if we choose $t_{k+1} = t_k + \tau_s(x_k)$, then (17) holds for all $t \in [t_k + \Delta_k, t_{k+1} + \Delta_{\max}]$ and for all $k \geq 0$, with $\Delta_{\max}$ non-negative. Since $M_1(x_k)$, $M_2(x_k)$ and M_3 are non–negative and upper bounded for each $x_k \in \mathscr{B}_\delta$, and since α_4, $\alpha_3 \circ \alpha_2^{-1} \circ \alpha_1 \in \mathscr{K}$, then the first of (17) is a second degree inequality in the form $ay^2 + by - c \leq 0$, where a, b are non–negative and bounded and c is strictly positive and bounded. Therefore, for ϑ_2 sufficiently small, $t_{k+1} - t_k = \tau_s(x_k) > \Delta_{\max} \geq 0$ for each $x_k \in \mathscr{B}_\delta$, and thus $\tau_{\min}$ is strictly positive as well. This completes the proof. ■

4.2 *Perturbed Systems*

A generic system (1), subject to disturbances and parameter variations, can be seen as the nominal system (2), perturbed by the term

$$d_g \doteq g(x, u, \mu, d) = f(x, u, \mu, d) - f_0(x, u) \doteq d_g. \tag{18}$$

Hence, (1) can be rewritten as follows

$$\dot{x} = f_0(x, u) + g(x, u, \mu, d). \tag{19}$$

Definition 4.4: Under Assumption 4.2, and given the perturbed system (1) and a safe set $\mathscr{B}_\delta$, $\delta > 0$, we say that the perturbation (18) is δ–admissible if there exists a state feedback control law $\kappa \in \mathscr{U}$ and a constant $\vartheta_g \in (0,1)$ such that the function $g(x, \kappa(x_0), \mu, d)$ satisfies

$$\max_{\substack{x, x_k \in \mathscr{B}_\delta \\ d \in \mathscr{D}_d \\ \mu \in \mathscr{D}_\mu}} \|g(x, \kappa(x_k), \mu, d)\| \leq \nu(\delta) \doteq \vartheta_g \frac{\alpha_3\left(\alpha_2^{-1}(\alpha_1(\delta))\right)}{\alpha_4(\delta)} \tag{20}$$

with $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ as in (5). ◇

The δ–admissible perturbations are those for which the safety problem with respect to a ball $\mathscr{B}_\delta$ can be solved using continuous time measurement and actuation, namely it is a necessary condition to achieve safety with respect to $\mathscr{B}_\delta$ using sampled measurements and actuations. Note that in condition (4.4) the expression of $\nu(\delta)$ can be explicitly computed.

The following theorem states that, if a system is asymptotically stabilizable using a continuous time state feedback control law and the perturbation is δ–admissible, then it is possible to keep the state in a boundary $\mathscr{B}_\delta$ of the equilibrium point by applying a digital self triggered strategy.

Theorem 4.5: *Under Assumption 4.2, Problem 2.2 is solvable for system (1) for any δ–admissible perturbation (18), and the function τ_s can be iteratively computed as a function of the current state of the system and the maximum allowable delay $\Delta_{\max}$.* ◇

Proof: Using the same reasoning as in the proof of Theorem 4.3, and since the perturbation is assumed

δ–admissible, we conclude that the following inequality

$$\begin{aligned}\dot{V} &\leq -(1-\vartheta)\alpha_3(\|x\|) - \vartheta\alpha_3(\|x\|) + \alpha_4(\|x\|)(\|d_h\| + \|d_\Delta\| + \|d_g\|)\\ &\leq -(1-\vartheta)\alpha_3(\|x\|)\end{aligned}$$

with d_g defined in (18), and d_h, d_Δ defined as in Theorem 3.2, holds when

$$\begin{aligned}&\alpha_4(\delta)\left(M_1(x_k)(t-t_k) + M_2(x_k)(t-t_k)^2\right) \leq \vartheta_1\alpha_3\left(\alpha_2^{-1}(\alpha_1(\delta))\right)\\ &\alpha_4(\delta)M_3\Delta_k \leq \vartheta_2\alpha_3\left(\alpha_2^{-1}(\alpha_1(\delta))\right)\end{aligned} \tag{21}$$

where $\vartheta = \vartheta_1 + \vartheta_2 + \vartheta_g$, with $\vartheta_1, \vartheta_2, \vartheta_g \in (0,1)$, and $\vartheta_1 + \vartheta_2 < 1 - \vartheta_g$, and $M_1(x_k)$, $M_2(x_k)$, M_3 are as in Theorem 3.2. Defining

$$\begin{aligned}\Delta_{\max} &\doteq \vartheta_2 \frac{\alpha_3\left(\alpha_2^{-1}(\alpha_1(\delta))\right)}{\alpha_4(\delta)M_3}\\ \tau_s(x_k) &\doteq \max\left\{t - t_k : \text{ (21) is satisfied for each } t - t_k \in [0, \tau_s(x_k)]\right\} - \Delta_{\max}\\ \tau_{\min} &\doteq \min_{x_k \in \mathscr{B}_\delta} \tau_s(x_k)\end{aligned}$$

and if we choose $t_{k+1} = t_k + \tau_s(x_k)$, then (21) holds for all $t \in [t_k + \Delta_k, t_{k+1} + \Delta_{\max}]$ and for all $k \geq 0$, with $\Delta_{\max}$ non-negative. Arguing as in the proof of Theorem 4.3, for ϑ_2 sufficiently small, $t_{k+1} - t_k = \tau_s(x_k) > \Delta_{\max} \geq 0$ for each $x_k \in \mathscr{B}_\delta$, and thus $\tau_{\min}$ is strictly positive as well. This completes the proof.■

As discussed in Section 3, the choice of ϑ_1, ϑ_2 and ϑ_g corresponds to a simple tradeoff between larger intersampling times (ϑ_1), and robustness with respect to larger delays (ϑ_2) and perturbations (ϑ_g).

Remark 4: Theorems 4.3 and 4.5 prove the existence of a self triggered strategy characterized by the time sequence $\mathscr{I} = \{t_k\}_{k\geq 0}$, with $t_k \geq \tau_{\min} > 0$ for each $k \geq 0$, such that the closed loop system satisfies a given safety specification. Moreover, they provide a formula to explicitly compute the next sampling time t_{k+1} as a function of the state x_k at time t_k.

Although the simulation results, illustrated in Section 5, show strong benefits of the proposed self triggered strategy with respect to controllers based on constant sampling, the sequence $\mathscr{I}$ might be conservative, in the sense that longer sampling times might be determined, because of the approximations used in the proof. A trivial way to obtain a less conservative sequence $\mathscr{I}$ without introducing more restricting assumptions is the use of Taylor expansions of order higher than 2.

5 An Example of Application of the Digital Self Triggered Robust Control

Consider the system defined in Example 4.1. As already shown, we can not imply the existence of a stabilizing self triggered strategy. However, since Assumption 4.2 holds, Theorem 4.3 implies the existence of a self triggered strategy that guarantees safety for an arbitrary small neighborhood of the equilibrium point. In particular, since the origin of the system is locally exponentially stabilizable for $\|x\| \leq 2/3$, we define the safe set $\mathscr{B}_\delta$ with $\delta = 10^{-4} < 2/3$. We performed simulations using Matlab, with initial condition $x_0 = (10^{-5}, 10^{-5})^T \in \mathscr{B}_\delta$.

When a discrete time control law with constant sampling time greater than 2.1 s is used, the closed loop system is unstable.

In Figure 1, the closed loop behavior is illustrated when the proposed self triggered control algorithm is used, with $\vartheta_1 = 0.99$ and $\vartheta_2 = 0.009$. The closed loop system is not asymptotically stable, but is safe with respect to $\mathscr{B}_\delta$ for the time interval $[t_0, \infty)$. It is interesting to remark that the average sampling time is 6.2 s, i.e. more than 295% longer than the constant sampling time of 2.1 s (which yields an unstable

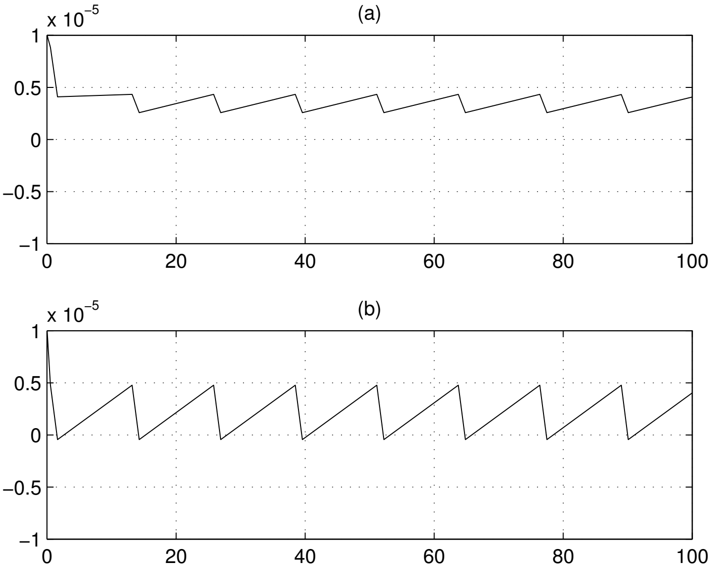

Figure 1. Self triggered control with $\vartheta_1 = 0.99$ and $\vartheta_2 = 0.009$: (a) x_1; (b) x_2 vs time

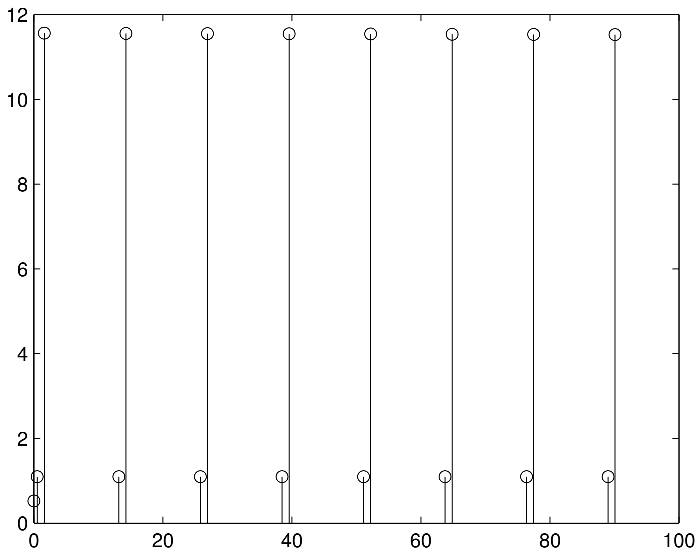

Figure 2. Sequence of sampling instants $\mathscr{I} = \{t_k\}_{k\geq 0}$ [s] with $\vartheta_1 = 0.99$ and $\vartheta_2 = 0.009$

control loop). Thus, using the proposed self triggered control algorithm, we achieve safety reducing of more than 295% the battery energy consumption, with respect to an unstable control strategy with constant sampling. However, since we have chosen $\vartheta_2 = 0.009$, we can only guarantee robustness with respect to delays bounded by $\Delta_{\max} = 0.17$ ms.

In Figure 3, the closed loop behavior is illustrated when the proposed self triggered control algorithm law is used, with $\vartheta_1 = 0.5$ and $\vartheta_2 = 0.499$. The closed loop system is not asymptotically stable, but is still safe with respect to $\mathscr{B}_\delta$ for the time interval $[t_0, \infty)$. However, since we have chosen $\vartheta_1 = 0.5$ in order to be robust with respect to delays, the average sampling time 3 s is more conservative with respect to the case $\vartheta_1 = 0.99$. Nevertheless, the average sampling time is almost 50% longer than the constant sampling time of 2.1 s (which yields an unstable control loop). Since we have chosen $\vartheta_2 = 0.009$, we can guarantee robustness with respect to delays bounded by $\Delta_{\max} = 9$ ms. Thus, using the proposed self triggered control algorithm, we achieve safety reducing of almost 50% the battery energy consumption,

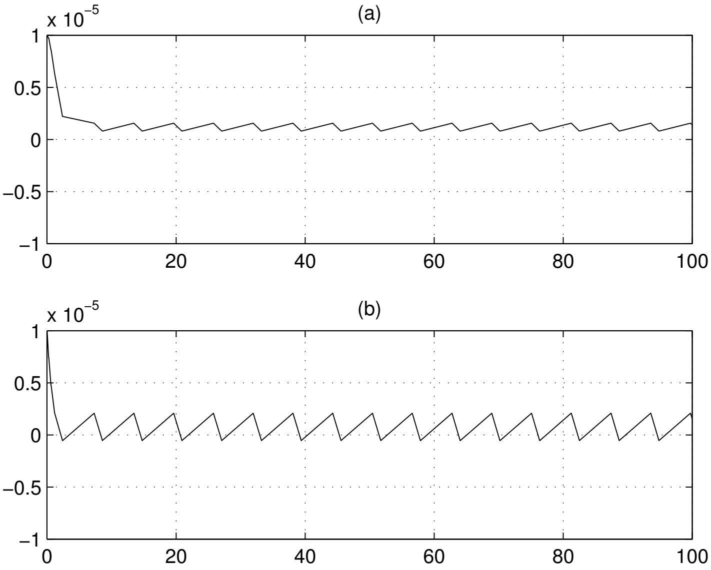

Figure 3. Self triggered control with $\vartheta_1 = 0.5$ and $\vartheta_2 = 0.499$: (a) x_1; (b) x_2 vs time

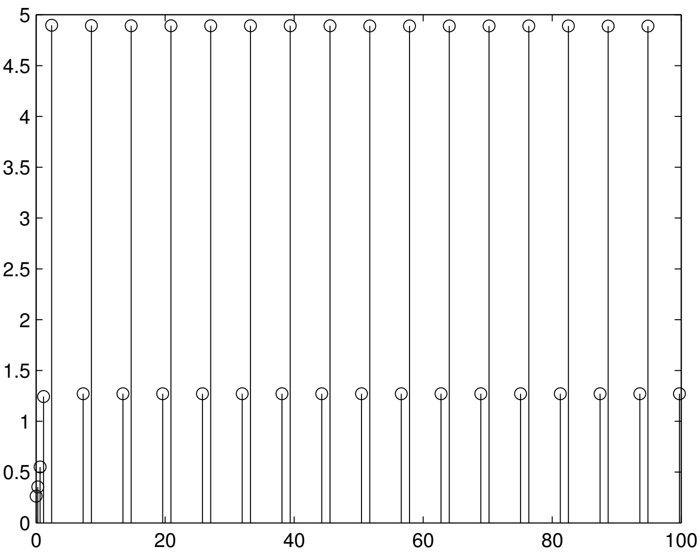

Figure 4. Sequence of sampling instants $\mathscr{I} = \{t_k\}_{k\geq 0}$ [s] with $\vartheta_1 = 0.5$ and $\vartheta_2 = 0.499$

with respect to an unstable control strategy with constant sampling, while guaranteeing robustness with respect to delays bounded by $\Delta_{\max} = 9$ ms.

6 Conclusions

We have developed novel results on self triggered control for guaranteeing safety of nonlinear systems perturbed by norm–bounded parameter uncertainties and disturbances and affected by bounded delays. Our self triggered algorithm can also be exploited as an alternative to existing techniques for guaranteeing asymptotic stability of unperturbed non-linear systems affected by bounded delays. We have showed on a simple case study that the proposed results provide strong benefits in terms of energy consumption, with respect to digital controls based on constant samplings, by reducing the average sampling times. As a next step of this research line, we aim to tackle more complex case studies, and obtain results for less conservative sampling time sequences.

References

[1] I. F. Akyildiz and I. H. Kasimoglu, WirelessHART: Wireless sensor and actor networks: research challenges, *Ad hoc networks*, VOL. 2, NO. 4, pp. 351–367, 2004.
[2] A. Anta, and P. Tabuada, Self–Triggered Stabilization of Homogeneous Control Systems, *Proceedings of the 2008 American Control Conference – ACC 2008*, pp. 4129–4134, 2008.
[3] A. Anta, and P. Tabuada, To Sample or not to Sample: Self–Triggered Control for Nonlinear Systems, *IEEE Transactions on Automatic Control*, Vol. 55, No. 9, 2010.
[4] APOLLO Final Report, Intelligent Tyre for Accident–Free Traffic, *IST-2001–34372*, Deliverable 22/23, 2001.
[5] D. Bianchi, A. Borri, G. Burgio, M. D. Di Benedetto, and S. Di Gennaro, Adaptive Integrated Vehicle Control using Active Front Steering and Rear Torque Vectoring, *International Journal of Vehicle Autonomous Systems*, Special Issue on: “Autonomous and Semi–Autonomous Control for Safe Driving of Ground Vehicles”, Vol. 8, No. 2/3/4, pp. 85–105, 2010.
[6] M. D. Di Benedetto, S. Di Gennaro, and A. D’Innocenzo, Digital Self Triggered Robust Control of Nonlinear Systems, *Proceeding of the* 50^{th} *Conference on Decision and Control and European Control Conference*, Orlando, FL, USA, pp. 1674–1679, 2011.
[7] W.P.M.H. Heemels, A.R. Teel, N. van de Wouw and D. Neic. Networked Control Systems with Communication Constraints: Tradeoffs between Transmission Intervals, Delays and Performance. *IEEE Transactions on Automatic Control*, Vol. 55, No. 8, pp. 1781-1796, 2010.
[8] H. K. Khalil, *Nonlinear Systems*, Third Edition, Prentice Hall, Upper Saddle River, New Jersey, U.S.A., 2002.
[9] H. Karl and A. Willig, *Protocols and Architectures for Wireless Sensor Networks*, Wiley, 2005.
[10] J. Kurzweil, On the Inversion of Liapunov’s Second Theorem on Stability of Motion, *Translation of American Mathematical Society*, Vol. 24, pp. 19–77, 1963. Originally appeared on *Czechoslovak Mathematica Journal*, Vol. 81, pp. 217–259, 1956.
[11] M. Lemmon, T. Chantem, X. Hu, and M. Zyskowski, On Self–Triggered Full Information H–infinity Controllers, in *Hybrid Hybrid Systems: Computation and Control*, 2007.
[12] X. Wang, and M. Lemmon, Self–Triggered Feedback Control Systems With Finite–Gain $\mathscr{L}_2$ Stability, *IEEE Transactions on Automatic Control*, No. 3, Vol. 54, pp. 452-467, 2009.
[13] R. Matsuzakia, A. Todoroki, Wireless Strain Monitoring of Tires Using Electrical Capacitance Changes with an Oscillating Circuit, *Sensors and Actuators A*, No. 119, pp. 323–331, 2005.
[14] M. Mazo, and P. Tabuada, On Event–Triggered and Self–Triggered Control over Sensor/Actuator Networks, *Proceedings of the* 47^{th} *Conference on Decision and Control*, Cancun, Mexico, pp. 435–440, 2008.
[15] H. B. Pacejka, Tyre and Vehicle Dynamics, Elsevier Butterworth–Hein, 2005.
[16] M. E. Palmer, C. C. Boyd, J. McManus, and S. Meller, Wireless Smart–Tire for Road Friction Measurement and Self State Determination, *Proceedings of the* 43^{rd} *AIAA/ASME/ASCE/AHS Structure, Structural Dynamics, and Materials Conference*, p. 1548, Denver, CO, USA, 22-25 April, 2002.
[17] J. Song, S. Han, A. Mok, D. Chen, M. Lucas, M. Nixon, and W. Pratt, WirelessHART: Applying Wireless Technology in Real-Time Industrial Process Control, *In Proceedings of the* 14^{th} *IEEE Real-Time and Embedded Technology and Applications Symposium (RTAS)*, 2008.
[18] P. Tabuada, Event-Triggered Real-Time Scheduling of Stabilizing Control Tasks, *IEEE Transactions on Automatic Control*, Vol. 52, No. 9, pp. 1680–1685, 2007.
[19] Brook Taylor, Methodus Incrementorum Directa et Inversa [Direct and Reverse Methods of Incrementation] (London, 1715), pp. 21–23 (Proposition VII, Theorem 3, Corollary 2). Translated into English in D. J. Struik, A Source Book in Mathematics 1200–1800 (Cambridge, Massachusetts: Harvard University Press, 1969), pp. 329–332.

[20] M. Velasco, P. Marti, and J. Fuertes, The Self Triggered Task Model for Real–Time Control Systems, *Work–in–Progress Session of the* 24^{th} *IEEE Real–Time Systems Symposium – RTSS03*, pp. –, 2003.

[21] T. Yoshizawa, *On the Stability of Solutions of a System of Differential Equations*, Memoirs of the College of Sciences, University of Kyoto, Ser. A, Vol. 29, pp. 27–33, 1955.

[22] A. Anta and P. Tabuada, Exploiting Isochrony in Self–Triggered Control. IEEE Trans. on Automatic Control, 57(4):950-962, 2012.