Long Josephson junctions in a field-induced deterministic ratchet

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We address the problem of a relativistic particle in a periodic asymmetric potential of the ratchet type. As a solid state realization of such a particle, we consider a single flux quantum in a long annular Josephson junction embedded in an inhomogeneous magnetic field. Deterministic (non thermal) regime is numerically investigated and compared with theoretical results. The ratchet velocity of the relativistic fluxon has been found qualitatively different from the one known for a nonrelativistic particle.

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Motion in a periodic potential lacking reflection symmetry, known as a ratchet potential [1], has attracted a considerable interest in the last years. Though the main interest was initially in the *thermal ratchet* [2–8], recently the *deterministic ratchet* [8,9] has also been addressed. The net unidirectional motion exhibited in ratchet potentials is the key feature potentially interesting for applications. Magnetic flux cleaning [10] in superconducting films, fluxon diodes [11] or voltage rectifier are an example of proposed applications of the ratchet effect. In Josephson junctions systems, a voltage-rectifier [12] based on a three-juntions SQUID and a fluxons ratchet [13] based on especially engineered arrays have been theoretically and experimentally [14] investigated.

Until now, principally nonrelativistic regime of particles in ratchet potentials has been addressed. Here we address an experimentally controllable way to study both nonrelativistic and relativistic regime of a particle in a ratchet potential. To this purpose, we consider a well known [15] solid state example of a relativistic particle: a single flux quantum in a long Josephson junction. To apply an effective ratchet potential to the fluxon in the junction we consider the inhomogeneous field generated by a control current passing in a properly shaped control line deposited on the top of the long junction. Main topics presented here concern the equation of motion, the depinning currents, the velocity-force relation, and the ratchet velocity of the fluxon forced by a square wave drive in the deterministic (non thermal), principally overdamped regime.

Field-induced sawtooth ratchet— The model for a long, unidimensional junction in an inhomogeneous field $h(x)$ is [16,17]

$$\varphi_{xx} - \varphi_{tt} - \sin\varphi - \alpha\varphi_t = \frac{\partial h}{\partial x} - \eta - \eta_{ac}(t) \tag{1}$$

where φ is the quanto-mechanical phase difference, x is the length normalized to the Josephson penetration length λ_J, and t is the time normalized to the inverse of the plasma frequency $\omega_J = \overline{c}/\lambda_J$, with $\overline{c}$ the maximum velocity of electromagnetic waves in the junction (Swihart velocity). The α term accounts for the quasiparticle current, the $\sin\varphi$ term accounts for the Josephson current, φ_t is proportional to the instantaneous voltage, η is the dc bias current I [see Fig. 1(a)] normalized to the critical current I_0, η_{ac} is an ac normalized bias current, and h is the magnetic field normalized to the critical field $B_0 = 4\pi\lambda_J J_0/c$ of the junction. For the annular geometry shown in Fig. 1(a) the x coordinate in Eq. (1) means a curvilinear coordinate and h means the radial component of the external field. Moreover, for this geometry the boundary conditions of the perturbed sine-Gordon equation Eq. (1) are

$$\varphi_x(0) = \varphi_x(l), \tag{2a}$$
$$\varphi(l) = \varphi(0) + m2\pi, \tag{2b}$$

where the integer m is the number of flux quanta trapped in the junction.

To generate an inhomogeneous magnetic field we can feed with a control current Ic

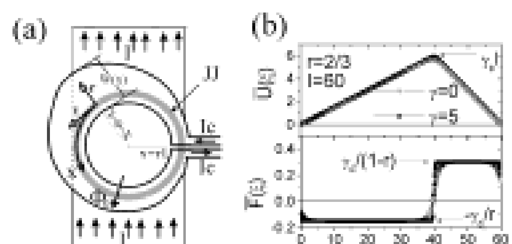

FIG. 1. (a) A way to apply a sawtooth magnetic field to a long annular Josephson junction. (b) The effective potential and force experienced by a fluxon when the sawtooth field is turned on.

a control line of variable width $W(x)$ deposited on the top of (and insulated from) the junction, as it is shown in Fig. 1(a). In such a case, we generate a field

$$h(x) = \frac{\overline{\gamma_c} l}{2\overline{w}(x)} \tag{3}$$

where $\overline{\gamma_c}$ is the control current normalized to the critical current of the junction, l is the length of the junction normalized to λ_J, and $\overline{w}(x)$ is the width of the control line normalized to the physical width of the junction.

From relation (3) a control line with a width [see Fig. 1(a)] varying as

$$W(x) = \begin{cases} \frac{W_{\max} W_{\min} rL}{W_{\min} rL + (W_{\max} - W_{\min})x} & 0 < x < rL \\ \frac{W_{\max} W_{\min}(1-r)L}{W_{\max}(1-r)L - (W_{\max} - W_{\min})(x - rL)} & rL < x < L \end{cases},$$

where $W_{\max} = \max_x \{W(x)\}$ and $W_{\min} = \min_x \{W(x)\}$, gives us the sawtooth normalized field

$$h(x) = \begin{cases} h_0 + \frac{\gamma_c}{r} x & 0 < x < rl \\ h_0 + \gamma_c l - \frac{\gamma_c}{(1-r)}(x - rl) & rl < x < l \end{cases}, \tag{4}$$

where

$$h_0 = \frac{\overline{\gamma_c} l}{2} \frac{1}{\overline{w}_{\max}}, \quad \gamma_c = \frac{\overline{\gamma_c}(\overline{w}_{\max} - \overline{w}_{\min})}{2(\overline{w}_{\max} \overline{w}_{\min})}. \tag{5}$$

The corresponding forcing term in the Eq. (1) is then

$$\frac{\partial h}{\partial x} = f(x) = \begin{cases} \frac{\gamma_c}{r} & 0 < x < rl \\ -\frac{\gamma_c}{(1-r)} & rl < x < l \end{cases}. \tag{6}$$

We remark that though here we focus on the so-called "rocking ratchet" [2], potentially all known ratchets can be easily realized in our physical system. For example a "flashing ratchet" [4,6] could be realized if an ac control current were used. Moreover, the thermal problem could also be indagated if the thermal noise of the dc bias current were considered or an artificially colored current noise were added on purpose.

Equation of motion—A fluxon with center of mass $\xi(t)$ travelling in the junction with velocity $\dot{\xi} = u$ is described by

$$\varphi = 4 \arctan e^{\gamma(x-\xi)} \tag{7}$$

where $\gamma = 1/\sqrt{1-u^2}$ is the relativistic factor. Following the classical energetic approach [18], the equation of motion for $\xi(t)$ is obtained inserting solution (7) in the power-balance equation

$$\frac{dH_{SG}}{dt} = \eta \int_0^l \varphi_t dx - \int_0^l f(x) dx - \alpha \int_0^l \varphi_t^2 dx, \tag{8}$$

where

$$H_{SG} = \int_0^l \left(\frac{1}{2}\varphi_x^2 + \frac{1}{2}\varphi_t^2 + 1 - \cos\varphi \right) dx.$$

The resulting equation is

$$\frac{4}{\pi}\left(1-u^2\right)^{-\frac{3}{2}} \frac{du}{dt} + \frac{4\alpha}{\pi} \frac{u}{\sqrt{1-u^2}} = \overline{F_0}(\xi) + \overline{F}(\xi), \tag{9}$$

with

$$\overline{F_0}(\xi) = \eta \frac{4}{2\pi} \arctan\left[\frac{\sinh(\gamma l/2)}{\cosh(\gamma(\xi - l/2))}\right], \tag{10a}$$

$$\overline{F}(\xi) = -\frac{\gamma_c}{r}\frac{4}{2\pi} \arctan\left[\frac{\sinh(\gamma r l/2)}{\cosh(\gamma(\xi - rl/2))}\right] + \frac{\gamma_c}{1-r}\frac{4}{2\pi} \arctan\left[\frac{\sinh(\gamma(1-r)l/2)}{\cosh[\gamma(\xi - (1+r)l/2)]}\right]. \tag{10b}$$

For very long junctions the forcing terms simplifie in

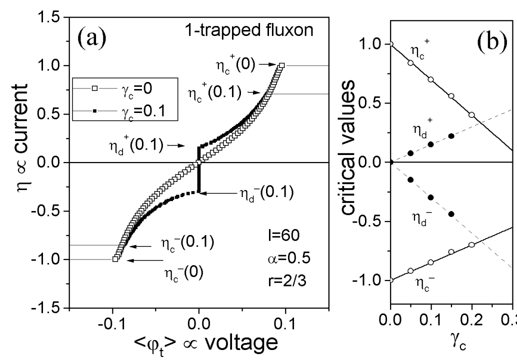

FIG. 2. (a) Current-voltage (force-velocity) characteristic of the junction with one trapped fluxon when the ratchet potential is off (open circles) or on (solid circles). (b) The critical and depinning currents versus the normalized control current. The points are numerical results, the lines are analytical results.

$$\overline{F_0}(\xi, l \to \infty) = \eta \qquad 0 < x < l, \tag{11a}$$

$$\overline{F}(\xi, l \to \infty) \equiv F(\xi) = \begin{cases} -\frac{\gamma_c}{r} & 0 < x < rl \\ +\frac{\gamma_c}{1-r} & rl < x < l \end{cases}. \tag{11b}$$

A plot of the effective force $\overline{F}(\xi)$ and the corresponding potential acting on the fluxon is given in Fig. 1(b) both for static ($\gamma = 1$) and dynamical case.

Restricting ourself to the very long junctions regime, we approximate the forcing terms in Eq. (9) with Eqs. (11). Moreover, considering quite overdamped (high α values) junctions to neglect inertial effects [du/dt in Eq. (9)], we can reduce our equation of motion to

$$\frac{4\alpha}{\pi} \frac{\dot{\xi}}{\sqrt{1-\dot{\xi}^2}} = -\frac{\partial}{\partial \xi}\left(-\eta\xi + U(\xi)\right) = \eta + F(\xi). \tag{12}$$

Single fluxon depinning currents and ratchet velocity—In Fig. 2(a) it is shown the effect of the sawtooth potential on the current-voltage [$\eta - < \varphi_t >$, with $< ... >$ spatio-temporal mean] characteristic of a junction with one trapped fluxon. Noticing that the mean voltage generated by a fluxon moving with velocity u is given by $V = < \varphi_t > = 2\pi u/l$, and that, from Eq. (12), η means a force, we can think of the plots in Fig. 2(a) also as force-velocity characteristic. Results refer to a junction with $l = 60$, $r = 2/3$, and are calculated integrating Eq. (1)

with forcing term Eq. (6), $\eta_{ac} = 0$, and $m = 1$ in boundary conditions Eqs. (2). As it is seen, when the ratchet potential is turned on ($\gamma_c \neq 0$), the current extension of the step is reduced, and a current range with zero mean voltage (zero velocity) appears. The relevant critical currents $\eta_c^{\pm}$ and depinning currents $\eta_d^{\pm}$ indicated in Fig. 2(a) are found using a solution $\varphi = \varphi_0$ in Eq. (1) and $\dot{\xi} = 0$ in Eq. (12), respectively.

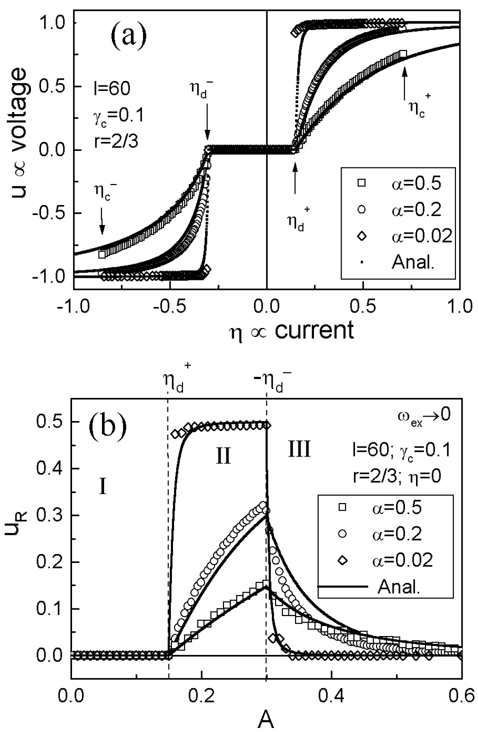

FIG. 3. (a) The fluxon velocity as a function of the bias current for different values of dissipation α. (b) Ratchet velocity of the fluxon induced by a square wave ac forcing in the adiabatic limit. The constant force is $\eta = 0$.

The result is

$$\eta_c^{+} = 1 - \frac{\gamma_c}{1-r}, \quad \eta_c^{-} = -1 + \frac{\gamma_c}{r}, \tag{13a}$$

$$\eta_d^{+} = \frac{\gamma_c}{r}, \quad \eta_d^{-} = -\frac{\gamma_c}{1-r}. \tag{13b}$$

A plot of these critical values versus the control current is given in Fig. 2(b).

The mean velocity of the fluxon in the inhomogeneous forcing is

$$\overline{u} \equiv \frac{1}{T}\int_0^T \dot{\xi}\, dt = \frac{\xi(T)-\xi(0)}{T} = \frac{l}{T},$$

where the revolution period T is

$$T = \int_0^T dt = \int_0^l \frac{d\xi}{\dot{\xi}}.$$

From Eq. (12) we have

$$\overline{u} = \frac{1}{r\frac{\sqrt{\left(\frac{4\alpha}{\pi}\right)^2+\left(\eta-\eta_d^+\right)^2}}{\left(\eta-\eta_d^+\right)} + (1-r)\frac{\sqrt{\left(\frac{4\alpha}{\pi}\right)^2+\left(\eta-\eta_d^-\right)^2}}{\left(\eta-\eta_d^-\right)}}. \tag{14}$$

So, the full step extension will be analytically described by the pieces

$$u(\eta,\gamma_c,r) = \begin{cases} \overline{u}(\eta) & -1+\frac{\gamma_c}{r} < \eta < -\frac{\gamma_c}{1-r} \\ 0 & -\frac{\gamma_c}{1-r} \leq \eta \leq \frac{\gamma_c}{r} \\ \overline{u}(\eta) & \frac{\gamma_c}{r} < \eta < 1-\frac{\gamma_c}{1-r} \end{cases}. \tag{15}$$

Numerically calculated velocity-current curves are compared with the analytical description Eq. (15) in Fig. 3(a).

The ratchet velocity [2,10] u_R is the velocity of the particle mediated over the period T_{ex} of an ac drive $\eta_{ac}(t)$. If the external drive is a square wave of amplitude A and period T_{ex} , the ratchet velocity of our fluxon in the "adiabatic" limit ($T_{ex} \to \infty$, $\omega_{ex} \to 0$) can be deduced by Eq. (14) and Eq. (15) as

$$u_R(A) = \begin{cases} 0 & 0 < A < \eta_d^+ \\ \overline{u}(A)/2 & \eta_d^+ < A < -\eta_d^- \\ \left[\overline{u}(A)+\overline{u}(-A)\right]/2 & A > -\eta_d^- \end{cases} \tag{16}$$

Analytical prediction Eq. (16) is compared with numerical results in Fig. 3(b). The static (I), active (II), and overdriven (III) regions typical [2] of the ratchet effect can be recovered in the plot. As a qualitative difference with respect to non-relativistic particles, the active region for our fluxon shows a deviation from the quasi-linear trend [2], [8], [10] exhibited in non-relativistic motion. The relativistic nature (i.e., the approaching of a limit ratchet velocity due to the existence of a limit velocity in our system) is more and more pronounced

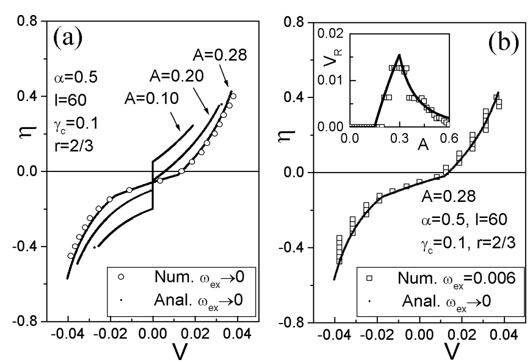

FIG. 4. (a) Modification of the current-voltage characteristic of the fluxon induced by a square wave drive in the adiabatic limit. (b) Same as in (a) but here we are not in the adiabatic limit. The ratchet voltage (velocity) at $\eta = 0$ for this drive is shown in the inset.

as α values are lowered. As it is seen, when full relativistic regime is achieved [$\alpha = 0.02$ in Fig. 3(b)] the ratchet velocity versus forcing drive amplitude takes the peculiar form of a window centered in the active region. In other words, the ratchet velocity of a particle in full relativistic regime is almost independent on the forcing amplitude in the active region and vanishing small otherwise. As a further consideration, inspection of Fig. 3(b) shows that the efficience of this kind of Josephson fluxon diode is maximized in the relativistic regime.

In the adiabatic limit, the ratchet velocity as a function of η, in other words the modification of the current-voltage characteristic due to a square wave forcing, can be constructed piecewise using Eq. (14). For $A < A_{cr} = (\eta_d^+ - \eta_d^-)/2$ we have

$$u_R(\eta, A) = \begin{cases} \frac{[\overline{u}(\eta+A)+\overline{u}(\eta-A)]}{2} & \eta < \eta_d^- - A \\ \frac{\overline{u}(\eta+A)}{2} & \eta_d^- - A < \eta < \eta_d^- + A \\ 0 & \eta_d^- + A < \eta < \eta_d^+ - A \\ \frac{\overline{u}(\eta-A)}{2} & \eta_d^+ - A < \eta < \eta_d^+ + A \\ \frac{[\overline{u}(\eta+A)+\overline{u}(\eta-A)]}{2} & \eta > \eta_d^+ + A \end{cases} \tag{17}$$

while for $A > A_{cr}$ we have

$$u_R(\eta, A) = \begin{cases} \frac{[\overline{u}(\eta+A)+\overline{u}(\eta-A)]}{2} & \eta < \eta_d^- - A \\ \frac{\overline{u}(\eta+A)}{2} & \eta_d^- - A < \eta < \eta_d^+ - A \\ \frac{[\overline{u}(\eta+A)+\overline{u}(\eta-A)]}{2} & \eta_d^+ - A < \eta < \eta_d^- + A \\ \frac{\overline{u}(\eta-A)}{2} & \eta_d^- + A < \eta < \eta_d^+ + A \\ \frac{[\overline{u}(\eta+A)+\overline{u}(\eta-A)]}{2} & \eta > \eta_d^+ + A \end{cases} . \tag{18}$$

The modification of current-voltage curves induced by a square wave forcing of increasing amplitude described by Eqs. (17) and (18) is shown in Fig. 4(a). As it is expected, a crossing of the zero current axis at a ratchet voltage $V_R = 2\pi u_R/l$ is found. In Fig. 4(b) we report the numerical result obtained for a faster square wave forcing: synchronized

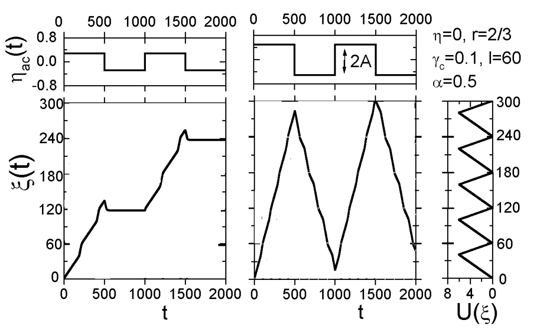

FIG. 5. Trajectory of the center of mass of the fluxon under the effect of the ac drive for two different amplitudes of the square wave. The effective potential seen by the fluxon is included as a guide for eyes. The angular frequency of the drive is $\omega_{ex} = 0.001 \cdot 2\pi$, and the bias current is $\eta = 0$.

current steps appear at voltages $V_{m,n} = (m/n)\omega_{ex}$, with m, n integers. This voltage quantization phenomenon accounts for the synchronization of the fluxon motion with the $m-th$ harmonic or the $n-th$ subharmonic of the ac drive. Obviously also the ratchet velocity (voltage) at $\eta = 0$ is now quantized [8], [13]. Same general trend shown in Fig. 4 was found in numerical simulations performed with a sinusoidal ac drive.

Finally, in Fig. 5 we show the time evolution of $\xi(t)$ at two values of the amplitude A of the square wave forcing. In the left panel the amplitude A falls in the active ratchet region, while in the right panel A falls in the overdriven region. As it is seen, the resulting net motion is unidirectional in both cases, but the advance in one period T_{ex} is only marginal when we are in the overdriven region.

In summary, we considered the problem of a relativistic particle in a ratchet potential and we individuated a physical realization for such a particle in a fluxon trapped in a long annular Josephson junction embedded in a saw-tooth magnetic field. For very long junctions simple analytical results have been found, concerning the depinning currents, the I-V curve and the ratchet velocity of the fluxon under an adiabatic square wave forcing. The ratchet velocity of our relativistic particle is found qualitatively different from the one known for non relativistic particles. Only simple deterministic, noninertial effects have been discussed here, but the proposed physical system could deserve further attention because it allows to indagate in an experimentally controllable way the physics of relativistic particles in deterministic or thermal ratchets.

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