Adiabatic transport of Cooper pairs in arrays of small Josephson junctions

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Abstract

We present a quantitative theory of Cooper pair pumping in gated one-dimensional arrays of Josephson junctions. The pumping accuracy is limited by quantum tunneling of Cooper pairs out of the propagating potential well and by direct supercurrent flow through the array. Both corrections decrease exponentially with the number N of junctions in the array, but give a serious limitation of accuracy for any practical array.

1 Introduction

When a potential well propagates adiabatically along an electron system which is effectively one-dimensional it carries with it additional electron density and induces a dc electric current through the system. Such a pumping effect is observed in mesoscopic systems ranging from small metallic tunnel junctions in the Coulomb blockade regime [1-4], to semiconductor quantum dots [5] and one-dimensional ballistic channels [6]. The propagation of the potential well is arranged either through the propagation of an acoustoelectric wave [2, 6] or by phase-shifted gate voltages [1, 3, 5]. Of particular interest is the pumping regime when the potential well carries a quantized number m of electrons so that the induced current I is related to the frequency f, with which the well crosses the system, by the fundamental relation $I = mef$. A well with a definite number of electrons can be created either by the Coulomb interaction, as, for instance, in the Coulomb blockade pumps [1-4], or it can be caused by the discrete nature of single-particle states inside the well [7, 8]. In the case of Coulomb blockade pumps, the precision of the pumped charge is reaching a level sufficient for metrological applications [3, 4]. Different sources of inaccuracy in the pumps have been discussed in the literature [3,9-12].

Until recently the pumping effect has been studied almost exclusively in normal systems where transport is due to individual electrons. A timely motivation for studying Cooper pair transfer comes from quantum computation, where pumping can play an important role in the dynamics of quantum logic gates [13]. This work presents a quantitative theory of Cooper pair pumping in one-dimensional arrays of superconducting tunnel junctions. In particular, we find fundamental corrections to the quantized pumping regime and show that they are unexpectedly large in arrays with a small number of junctions. These large quantum corrections explain the fact that the first experiment with pumping of Cooper pairs failed to demonstrate accurate pumping [14].

2 Cooper pair pump

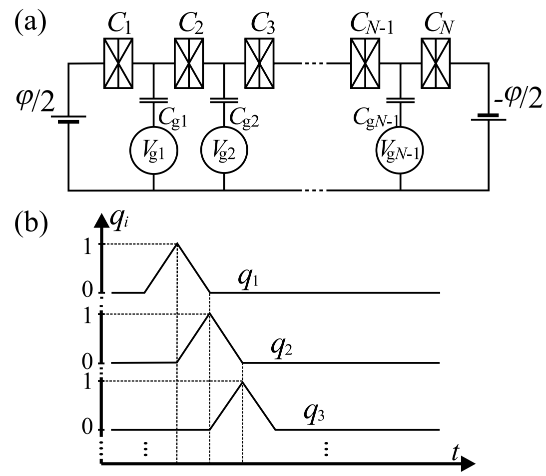

Figure 1: **(a)** A schematic drawing of a gated Josephson array of N junctions. In pumping Cooper pairs gate voltages $V_{\mathrm{g}i}$ are operated cyclically. C_i are the capacitances of the junctions, and $C_{\mathrm{g}i}$ are gate capacitances. In a uniform pump $C_i \equiv C$ for all $i = 1, 2, \ldots, N$. **(b)** A train of gate voltages to carry a charge in a pump. Here $q_i = -C_{\mathrm{g}i}V_{\mathrm{g}i}/2e$.

First we present the general expression for the charge transferred through an array of N superconducting tunnel junctions in the Coulomb blockade regime by adiabatic pumping of Cooper pairs [15]. In the standard model such arrays are characterized by two energies, the charging energy H_{C} as a system of capacitors, and the energy associated with tunneling [16]. The array is assumed uniform, so $C_1 = C_2 = \cdots = C_N \equiv C$ (fig. 1). We also assume that the characteristic energy $E_{\mathrm{C}} \equiv (2e)^2/2C$ of a Cooper pair in the array and the temperature ($k_{\mathrm{B}}T$) are both much lower than the superconducting energy gap of the electrodes. The relation for E_{C} replaces the usual condition in the normal Coulomb blockade where the junction resistances should be larger than the quantum resistance. With these conditions fulfilled quasiparticle tunneling is exponentially suppressed, while the tunneling energy of Cooper pairs in junction i reduces to a constant $E_{\mathrm{J}i}/2$. ($E_{\mathrm{J}i}$ is the Josephson coupling energy.) In this work, the bias voltage is set to zero, and thus a constant Josephson phase difference φ is fixed across the array. We can then treat the two external electrodes of the array as one, so that effectively the array forms a loop and φ plays the role of external flux threading it. Then, the Hamiltonian of the N-pump is [15]:

$$H = H_{\rm C}(n-q) - \sum_{k=1}^{N} \frac{E_{{\rm J}k}}{2} \left(|n\rangle\langle n+\delta_k| e^{i\varphi/N} + h.c. \right) . \tag{1}$$

The array $n \equiv \{n_1, n_2, \ldots, n_{N-1}\}$ represents the number n_i of Cooper pairs on the islands and $q \equiv \{q_1, q_2, \ldots, q_{N-1}\}$ the charge, normalized by $2e$, induced on each island by the gate voltage $V_{{\rm g}i}$ (fig. 2.1). The term δ_k describes the change of n due to tunneling of one Cooper pair in the kth junction. The charging energy of the homogeneous array can be written as

$$H_{\rm C} = \frac{E_{\rm C}}{N} \left[\sum_{k=1}^{N-1} k(N-k)u_k^2 + 2 \sum_{l=2}^{N-1} \sum_{k=1}^{l-1} k(N-l)u_k u_l \right] , \tag{2}$$

where $u_k \equiv n_k - q_k$. We will also need the current operator of the kth junction:

$$I_k = \frac{ieE_{{\rm J}k}}{2\hbar} \left(|n\rangle\langle n+\delta_k| e^{i\varphi/N} - h.c. \right) . \tag{3}$$

2.1 Adiabatic approximation

There are two mechanisms of Cooper pair transport in the array. One is the direct supercurrent through the whole array and the other is the pumping, i.e. the charge transfer in response to the adiabatic slow variation of the induced charges q_i. To derive the general expression for the total charge Q transferred during one pumping period we introduce the basis of instantaneous eigenstates $\{|m_{(t)}\rangle\}$ with eigenenergies $\{E_m^{(t)}\}$ of the full Hamiltonian (1) for a given $q(t)$. Assuming slowly varying gate voltages we may solve the time-dependent Schrödinger equation with the initial condition $|\psi(t_0)\rangle = |m_{(t_0)}\rangle$ to obtain [18]

$$|\psi_{(t_0+\delta t)}\rangle = e^{-iE_m^{(t_0)}\delta t/\hbar} |m_{(t_0)}\rangle + |\delta m_{(\delta t)}\rangle. \tag{4}$$

Here the term $|\delta m_{(\delta t)}\rangle$ is a correction to the state $|m_{(t_0)}\rangle$ due to the change in gate charges q. The amount of charge that passes through the junction k during a short time interval δt is then

$$\begin{aligned} \delta Q_k &= \int_{t_0}^{t_0+\delta t} \langle \psi_{(t)} | I_k | \psi_{(t)} \rangle dt \\ &= \delta t \langle I_k \rangle_{|m_{(t_0)}\rangle} - 2\hbar \sum_{l(\neq m)} {\rm Im} \left[\frac{\langle m|I_k|l\rangle\langle l|\delta m\rangle}{E_l - E_m} \right] \end{aligned} \tag{5}$$

where we have neglected the term quadratic in $|\delta m\rangle$ and oscillatory terms by assuming that the inequality $\delta t \gg \hbar/(E_l - E_m)$ holds for all l.

For a closed path γ the transferred charge must be equal for all N junctions so the total amount of charge, Q, transferred through the array over a pumping period τ is then given by $Q = Q_k = \int_0^\tau \langle I_k \rangle_{|m(t_0)\rangle} dt + Q_\mathrm{P}$. The first term gives the charge transferred via direct supercurrent. The second term, the charge transfer induced by gates, can be written as

$$\frac{Q_\mathrm{P}}{-2e} = \frac{\hbar}{e} \oint_\gamma \sum_{l(\neq m)} \mathrm{Im}\left[\frac{\langle m|I_k|l\rangle\langle l|dm\rangle}{E_l - E_m}\right], \tag{6}$$

where $|dm\rangle$ is the differential change of $|m\rangle$ due to a differential change of the gate voltages dq (see [15]).

2.2 Homogeneous pump

In the regime of accurate pumping the main contribution to Q comes from the induced charge transfer Q_P while the supercurrent gives only small corrections limiting the pumping accuracy [15]. The necessary condition for this regime to exist is $E_\mathrm{J} \ll E_\mathrm{C}$, which we assume from now on. We consider two different pumping paths around the degeneracy point, which occurs when $q_k = 1/N$ for all k. At this degeneracy point the energies of several charge states coincide as illustrated in fig. 2(a) for $N = 3$. If the pumping process is slow, the Cooper pair is transported adiabatically between the islands by the usual two-state level-crossing transitions that shift it along the array following the gate voltages. One Cooper pair is then transported through the array per cycle corresponding to a q-space trajectory circling once around the degeneracy point. One more condition necessary for accurate pumping is that the probability of the Landau-Zener transitions to the excited states is negligible and the array remains in the minimum-energy state throughout the cycle. This condition limits the rate of pumping, $1/\tau$, by the relation, $\hbar/\tau \ll E_\mathrm{J}^2/E_C$. However, even then, i.e. in the regime of the present work, the pumping is not accurate due to the nonvanishing $E_\mathrm{J}/E_\mathrm{C}$.

For the trajectory illustrated in fig. 1(b) we obtain by perturbation theory in E_J and by eq. (6) [15]:

$$\frac{Q_\mathrm{P}}{-2e} = 1 - \frac{N^{N-1}(N-1)}{(N-2)!}\left(\frac{E_\mathrm{J}}{2E_\mathrm{C}}\right)^{N-2} \cos\varphi\,. \tag{7}$$

Thus the probability of Cooper pair tunneling limiting the pumping accuracy decreases with increasing N.

For $N = 3$, the triangular pumping trajectory in the (q_1, q_2) plane shown in fig. 2(a) corresponds to triangular gate voltages. Another pumping scheme in the $N = 3$ pump [1, 14] is provided by harmonic gate voltages, corresponding to a circular trajectory around the degeneracy point $q_1 = q_2 = 1/3$. In this case it is possible to calculate the pumped charge directly from eq. (6). For $\varphi = 0$ we obtain [15]:

$$\frac{Q_\mathrm{P}}{-2e} = 1 - \frac{3}{2}\left(\frac{1}{3\sqrt{2}\delta} + \frac{1}{2 - 3\sqrt{2}\delta} + \frac{1}{\frac{3}{\sqrt{5}}\delta} + \frac{1}{1 - \frac{3}{\sqrt{5}}\delta}\right)\frac{E_\mathrm{J}}{E_\mathrm{C}}, \tag{8}$$

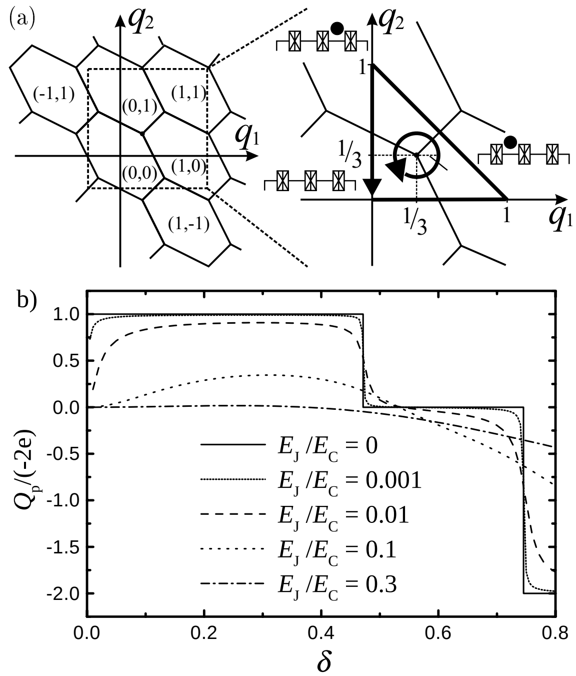

Figure 2: **(a)** The states with minimum charging energy of the uniform $N = 3$ pump on the (q_1, q_2) plane. The vector (n_1, n_2) denotes the stable configuration inside each hexagon. A circular path with radius δ, and the triangular path of fig. 1(b) are shown. Only states contributing to the inaccuracy in charge transport in the leading order are shown. **(b)** Numerically calculated quantum inaccuracies of a uniform 3-pump for different values of E_J/E_C. The analytical result of eq. (8) is exact in the limit of small E_J/E_C.

where $\delta \equiv \left[(q_1 - 1/3)^2 + (q_2 - 1/3)^2\right]^{1/2}$ is the radius of the trajectory. The results of eqs. (8) and (7) for $N = 3$ almost coincide for the optimum radius of $\delta \simeq 0.3$. It should be noted that the quantum inaccuracy in pumping is very significant: it is more than 20 % at $E_J/E_C = 0.03$ (practically, E_J is limited from below by the temperature, while the maximum E_C is limited by physical dimensions of the fabrication.) The accurate coherent pumping is thus practically impossible in the $N = 3$ pumps. Figure 2(b) shows Q_P calculated numerically from eq. (6) for $\varphi = 0$ (no direct supercurrent present) as a function of δ. For small radii the charge is quadratic in δ, $Q_P = \pi\delta^2\,(8E_C/27E_J)^2$, as can be derived from eq. (6). At large δ the pumped charge in fig. 2 starts to decrease since the trajectory approaches another degeneracy point at $q_1 = q_2 = 2/3$.

2.3 Further results

To obtain quantitatively more precise results than by the first order perturbation theory one can use the renormalisation method (see, e.g., [17]). With this method it is possible to calculate the higher order corrections to the pumping inaccuracy in case of homogeneous arrays, inhomogeneity of the array, and nonideal pumping sequences [18].

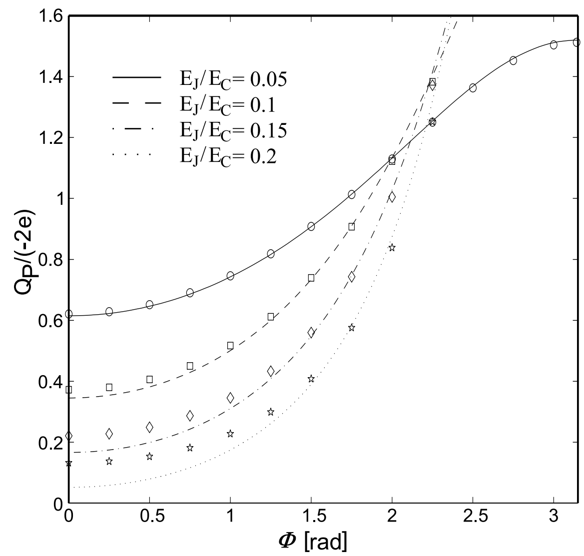

Figure 3: The pumped charge $Q_{\rm P}/(-2e)$ as a function of φ for some values of $E_{\rm J}/E_{\rm C}$ and $N=3$. Curves denote renormalised values and symbols numerical values which were obtained for a 41-state basis. The points calculated by eq.(2.7) for $\varphi=0$ and $E_{\rm J}/E_{\rm C}=0.05$ and 0.1 are 0.55 and 0.10, respectively. The pumped charge is symmetric in φ and its period is 2π.

In fig. 3 the pumped charge $Q_{\rm P}$ for $N=3$ is shown as a function of the phase difference φ. Values calculated with renormalisation and numerical results are in good agreement and they clearly indicate that the deviations from the leading order result, $[Q_{\rm P}/(-2e) = 1 - 9(E_{\rm J}/E_{\rm C})\cos\varphi]$ are important also for finite ϕ.

Also the inhomogeneity of the array can be treated with renormalisation theory. To do this we define the inhomogeneity index of the array

$$X_{\rm inh} = \sqrt{\frac{1}{N}\sum_{k=1}^{N}\left(\frac{C-C_k}{C}\right)^2}, \tag{9}$$

where the "average" capacitance C is given by $C = N/\sum_{k=1}^{N} C_k^{-1}$. To make a comparison easier we consider $W_{\rm inh}$, defined as the ratio between the inhomogeneous and homogeneous inaccuracies. The effects due to inhomogeneity can be parametrised by obtaining limits for $W_{\rm inh}$ as a function of $X_{\rm inh}$. In fig. 4 we

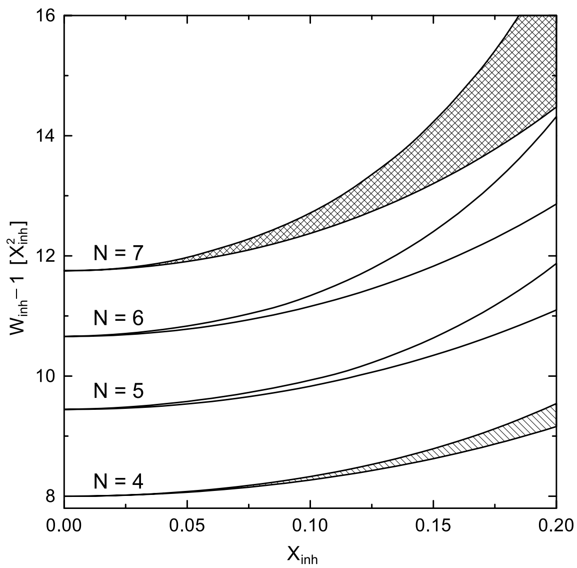

Figure 4: The limits for the ratio $W_{\rm inh}$ as a function of $X_{\rm inh}$ for array lengths $N = 4$ to $N = 7$. For small values of $X_{\rm inh}$, $W_{\rm inh} \approx 1 + a_N^{(\rm inh)} \cdot X_{\rm inh}^2$, where the N-dependent constant $a_N^{(\rm inh)}$ can be evaluated [18].

graphically present these limits for $(W_{\rm inh} - 1)/X_{\rm inh}^2$ as a function of $X_{\rm inh}$ in the cases $N = 4$ to $N = 7$.

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References

[1] H. Pothier, P. Lafarge, C. Urbina, D. Esteve and M.H. Devoret, Europhys. Lett. **17**, 249 (1992).

[2] J.P. Pekola, A.B. Zorin and M.A. Paalanen, Phys. Rev. B **50**, 11255 (1994).

[3] M.W. Keller, J.M. Martinis, N.N. Zimmerman and A.H. Steinbach, Appl. Phys. Lett. **69**, 1804 (1996); M.W. Keller, J.M. Martinis, and R.L. Kautz, Phys. Rev. Lett. **80**, 4530 (1998).

[4] S. V. Lotkhov, S. A. Bogoslovsky, A. B. Zorin and J. Niemeyer *The single electron R-pump: first experiment*, cond-mat/0001206

[5] L.P. Kouwenhoven, A.T. Johnson, N.C. Van der Vaart, C.J.P.M. Harmans and C.T. Foxon, Phys. Rev. Lett. **67**, 1626 (1991).

[6] V.I. Talyanskii, J.M. Shilton, M. Pepper, C.G. Smith, C.J.B. Ford, E.H. Linfield, D.A. Ritchie and G.A.C. Jones, Phys. Rev. B **56**, 15180 (1997).

[7] D.J. Thouless, Phys. Rev. B **27**, 6083 (1983).

[8] Q. Niu, Phys. Rev. Lett. **64**, 1812 (1990).

[9] H.D. Jensen and J.M. Martinis, Phys. Rev. B **46**, 13407 (1992).

[10] D.V. Averin, A.A. Odintsov and S.V. Vyshenskii, J. Appl. Phys. **73**, 1297 (1993).

[11] J.M. Martinis, M. Nahum and H.D. Jensen, Phys. Rev. Lett. **72**, 904 (1994).

[12] L. R. C. Fonseca, A. N. Korotkov and K. K. Likharev, Appl. Phys. Lett. **69**, 1858 (1996).

[13] D.V. Averin, Solid State Commun. **105**, 659 (1998).

[14] L.J. Geerligs, S.M. Verbrugh, P. Hadley, J.E. Mooij, H. Pothier, P. Lafarge, C. Urbina, D. Esteve and M.H. Devoret, Z. Phys. B **85**, 349 (1991).

[15] J.P. Pekola, J.J. Toppari, M. Aunola, M.T. Savolainen and D.V. Averin, Phys. Rev. B **60**, R9931 (1999).

[16] D.V. Averin and K. K. Likharev, in: *Mesoscopic Phenomena in Solids*, ed. by B. L. Altshuler, P. A. Lee, and R. A. Webb (Elsevier, Amsterdam, 1991).

[17] P.J. Ellis and E. Osnes, Rev. Mod. Phys. **49**, 777 (1978).

[18] M. Aunola, J.J. Toppari and J.P. Pekola, *Arrays of Josephson junctions in an environment with vanishing impedance*, Phys. Rev. B, *in print*.