TEORIA DEI SISTEMI (Systems Theory)

Prof. C. Manes, Prof. A. Germani

Some solutions of the written exam of November 18th, 2013

Problem 1. Consider a feedback control system with unit feedback gain, with the following transfer function in open-loop

$$W(s) = K\frac{s-1}{s(s^2+64)}.$$

1. Draw the amplitude and phase Bode diagrams, and the polar diagram for $K = 1$;
2. Compute the denominator of the closed-loop transfer function;
3. Compute the number of poles with negative real part of the closed loop transfer function as a function of the gain $K \in (-\infty, \infty)$, using both the Nyquist criterion and the Routh criterion.

We consider the Bode plots and polar plot of $\widetilde{W}(s)$

$$\widetilde{W}(s) = \frac{s-1}{s(s^2+64)} \qquad \left(\widetilde{W}(s) = W(s) \text{ for } K = 1\right).$$

The low frequency gain is

$$K_0 = \lim_{s\to 0} |\widetilde{W}(s)s| = \frac{1}{64} = 2^{-6} \quad \Rightarrow \quad |K_0|_{dB} = 20\log_{10}(2^{-6}) = -6\big(20\,\log_{10}(2)\big) \approx -36\ dB.$$

(recall that $20\,\log_{10}(2) \approx 6$). Thus, the term $\dfrac{K_0}{s}$ is a straight line with -20 dB/decade slope, passing at -36 dB for $\omega = 1$.

The Bode plots and the polar plot of the open loop transfer function are in the enclosed file.

The closed-loop transfer function is

$$W_{ch}(s) = \frac{W(s)}{1+W(s)} = \frac{K\frac{s-1}{s(s^2+64)}}{1+K\frac{s-1}{s(s^2+64)}} = \frac{K(s-1)}{s(s^2+64)+K(s-1)}$$

The characteristic polynomial of the closed-loop system is the denominator of $W_{ch}(s)$:

$$d_{ch}(s) = s(s^2+64) + K(s-1) = s^3 + (64+K)s - K$$

The first two rows (rows 3 and 2) of the Routh table are:

$$\begin{array}{c|cc} 3 & 1 & 64+K \\ 2 & 0 & -K \end{array}$$

There is a 0 in the first column. To continue the table construction, replace the 0 with $\varepsilon > 0$

$$\begin{array}{c|cc} 3 & 1 & 64+K \\ 2 & \varepsilon & -K \\ 1 & \frac{-K-\varepsilon(64+K)}{-\varepsilon} & \\ 0 & -K & \end{array}$$

The term in the row 1 is rewritten as

$$\frac{K}{\varepsilon} + 64 + K.$$

By multiplying the row 1 by $\varepsilon > 0$ we have

$$\begin{array}{c|cc} 3 & 1 & 64+K \\ 2 & \varepsilon & -K \\ 1 & K-\varepsilon(64+K) & \\ 0 & -K & \end{array}$$

Thus, for ε sufficiently small the sign of the term coincides with the sign of K. Thus, the signs of the first column are the signs of

$$1 \qquad \varepsilon \qquad K \qquad -K.$$

For $K > 0$ we have only one sign variation, while for $K < 0$ we have two sign variations.

Thus, for $K > 0$ we have one pole with positive real part in the closed loop transfer function, while for $K < 0$ we have two poles with positive real part. In both cases we have the instabilty of the closed loop system.

Problem 5. Given the system

$$\begin{cases} \dot{x}_1(t) = -k\left(1 + x_2^2(t)\right)x_1(t) + x_2(t) + 1 \\ \dot{x}_2(t) = -\left(1 + x_2^2(t)\right)\left(1 + x_2(t)\right) - x_1^3 \end{cases}$$

study the stability of the equilibrium point $x_e = (0, -1)$ for all the values of the parameter $k \in (-\infty, \infty)$, using the method of linear approximation at the equilibrium point, and the Lyapunov method, if necessary. (*Suggestion for the Lyapunov function:* $V(x) = (x_1 - x_{e,1})^4 + \beta(x_2 - x_{e,2})^2$, with suitable $\beta > 0$.)

Solution of problem 5.
The system considered is of the form $\dot{x}(t) = f\big(x(t); k\big)$, where $x(t)$ is the state and k is a constant parameter. The vector function $f(x; k) = [f_1(x; k)\ f_2(x)]^T$ is as follows

$$\begin{cases} f_1(x; k) = -k\,(1 + x_2^2)\,x_1 + x_2 + 1 \\ f_2(x) = -(1 + x_2^2)(1 + x_2) - x_1^3 \end{cases}$$

The Jacobian is

$$J(x) = \begin{bmatrix} \partial_{x_1} f_1 & \partial_{x_2} f_1 \\ \partial_{x_1} f_2 & \partial_{x_2} f_2 \end{bmatrix} = \begin{bmatrix} -k\,(1 + x_2^2) & -2k\,x_2\,x_1 + 1 \\ -3\,x_1^2 & -2\,x_2(x_2 + 1) - (1 + x_2^2) \end{bmatrix}$$

The value of the Jacobian at the equilibrium point $x_e = (0, -1)$ and the characteristic polynomial are

$$J(x_e) = \begin{bmatrix} -2k & 1 \\ 0 & -2 \end{bmatrix}, \qquad |\lambda I_2 - J(x_e)| = \left|\begin{bmatrix} \lambda + 2k & -1 \\ 0 & \lambda + 2 \end{bmatrix}\right| = (\lambda + 2k)(\lambda + 2).$$

The eigenvalues are $\lambda_1 = -2k$ and $\lambda_2 = -2$. Thus, both eigenvalues of $J(x_e)$ have strictly negative real part *if and only if* $k > 0$. As a consequence, if $k > 0$ then x_e is an asymptotically stable (A.S.) equilibrium point, while if $k < 0$, then λ_1 is positive, and then x_e is an unstable equilibrium point.

When $k = 0$ the two eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = -2$, and the origin is a simply stable equilibrium point *of the linear approximation* of the nonlinear system.

However, this **does not** imply the simple stability of the original nonlinear system (only asymptotic stability of the linear approximation implies the asymptotic stability of the nonlinear system). Thus, we must study the stability of the point $x_e = (0, -1)$ when $k = 0$ by using a suitable Lyapunov function.

Following the suggestion, for the system $\dot{x}(t) = f\big(x(t); 0\big)$ we consider the Lyapunov function

$$V(x) = x_1^4 + \beta(x_2 + 1)^2,$$

which is positive definite for any $\beta > 0$. According to the Lyapunov theorem, if the derivative $\dot{V}(x) = (dV/dx)f(x; \alpha)$ is semidefinite negative, then the equilibrium x_e is (simply) stable, while if $\dot{V}(x)$ is definite negative, then the equilibrium is asymptotically stable. for $k = 0$ we have

$$\begin{aligned} f_1(x; 0) &= x_2 + 1, \\ f_2(x) &= -(1 + x_2^2)(1 + x_2) - x_1^3. \end{aligned}$$

The computation of $\dot{V}(x)$ gives

$$\begin{aligned} \dot{V}(x) &= \frac{dV}{dx} f(x; 0) = (\partial_{x_1} V) f_1(x; 0) + (\partial_{x_2} V) f_2(x) \\ &= 4\,x_1^3(x_2 + 1) + 2\beta\,(x_2 + 1)\big(-(1 + x_2^2)(1 + x_2) - x_1^3\big) \\ &= 4\,x_1^3(x_2 + 1) - 2\beta\,(1 + x_2^2)(1 + x_2)^2 - 2\beta\,(x_2 + 1)x_1^3. \end{aligned}$$

Note that the term $-2\beta\,(1 + x_2^2)(1 + x_2)$ is never positive for any $\beta > 0$, and is strictly negative when $x_2 \neq -1$, while the first and the last terms have indefinite sign. If we choose $\beta = 2$ we can cancel the two terms with indefinite sign, and get the following derivative of the Lyapunov function

$$\dot{V}(x) = -4\,(1 + x_2^2)(1 + x_2)^2.$$

This function is negative semidefinite (i.e. $\dot{V}(x) \leq 0$) and therefore the equilibrium $x_e = (0, -1)$ is stable. Note that the chosen Lyapunov function does not prove asymptotic stability, because $V(x)$ is not negative definite (it is easy to see that $V(x) = 0$ for $x = (x_1, -1)$, for any $x_1 \in \mathbb{R}$).

In conclusion, for $k \in (-\infty, 0)$ the equilibrium $x_e = (0, -1)$ is unstable, for $k = 0$ is stable, and for $k \in (0, \infty)$ is asymptotically stable.