**TEORIA DEI SISTEMI (Systems Theory)**

Prof. C. Manes, Prof. A. Germani

Some solutions of the written exam of January 13th, 2014

**Problem 1.** Consider a feedback control system with unit feedback gain, with the following transfer function in open-loop

$$W(s) = K\frac{10-s}{(s-1)^2(s+1)}.$$

1. Draw the amplitude and phase Bode diagrams, and the polar diagram for $K = 1$;
2. Compute the denominator of the closed-loop transfer function;
3. Compute the number of poles with negative real part of the closed loop transfer function as a function of the gain $K \in (-\infty, \infty)$, using both the Nyquist criterion and the Routh criterion.

Let $\widetilde{W}(s)$ denote $W(s)$ for $K = 1$:

$$\widetilde{W}(s) = \frac{10-s}{(s-1)^2(s+1)}$$

We consider the Bode plots and the polar plot of $\widetilde{W}(s)$. In Bode form we have

$$W(s) = K\,\widetilde{W}(s) = K\,10\frac{(1-s/10)}{(1-s)^2(1+s)}$$

The low frequency gain is

$$K_0 = \lim_{s\to 0}\widetilde{W}(s) = 10 \quad \Rightarrow \quad |K_0|_{dB} = 20\log_{10}10 = 20 \text{ dB}.$$

Thus

$$|\widetilde{W}(j\omega)| = \frac{10\,|1-j\omega/10|}{|1-j\omega|^2\,|1+j\omega|}$$
$$\langle\widetilde{W}(j\omega)\rangle = \langle 1-j\omega/10\rangle - 2\langle 1-j\omega\rangle - \langle 1+j\omega\rangle.$$

Recalling that $|j\omega-1| = |j\omega+1|$ and $\langle 1-j\omega\rangle = -\langle 1+j\omega\rangle$, we have

$$|\widetilde{W}(j\omega)| = \frac{10\,|1-j\omega/10|}{|1+j\omega|^3}$$
$$\langle\widetilde{W}(j\omega)\rangle = \langle 1-j\omega/10\rangle - \langle 1-j\omega\rangle$$

(the Bode plots and the Nyquist plot of the open loop transfer function are in the enclosed file).

The closed-loop transfer function is

$$W_{CL}(s) = \frac{K\,\widetilde{W}(s)}{1+K\,\widetilde{W}(s)} = \frac{K\frac{10-s}{(s-1)^2(s+1)}}{1+K\frac{10-s}{(s-1)^2(s+1)}} = \frac{K(10-s)}{(s-1)^2(s+1)+K(10-s)}$$

the denominator of $W_{CL}(s)$:

$$\begin{aligned} d_{CL}(s) &= (s-1)^2(s+1) + K(10-s) = s^3 - s^2 - s + 1 + K\,10 - K\,s \\ &= s^3 - s^2 - (K+1)s + 10K + 1 \end{aligned}$$

*NYQUIST ANALYSIS*

Let $N$ be the number of times that the Nyquist plot of $W(j\omega)$ encircles the $-1$ point in the counterclockwise direction. From the plot it is clear that

- For $K > 0$ we have $N = 0$ (the Nyquist plot does not encircle the point $-1$);
- For $K \in (-0.1, 0)$ we have $N = 0$ (the Nyquist plot does not encircle the point $-1$);
- For $K < -0.1$ we have $N = 1$ (the Nyquist plot encircles one time the point $-1$ in the counterclockwise direction);

(Note that for $K=-0.1$ the denominator of $W_{CL}(s)$ is 0 for $s=0$, and thus $s=0$ is a pole of $W_{CL}(s)$. Thus, when $K$ moves from the interval $(-0.1, 0)$, where $p_{CL}=2$, to the interval $(-\infty,-0.1)$, where $p_{CL}=1$, a pole with positive real part becomes a pole with negative real part. Thus, we have that for $K=-0.1$ $W_{CL}(s)$ has one positive pole and one in zero (the third one is negative).)

Recall the Nyquist formula in the form

$$p_{CL} = p_{OL} - N$$

where, $p_{CL}$ is the number of poles with positive real part of the Closed Loop (CL) system, and $p_{OL}$ is the number of poles with positive real part of the Open Loop (OL) system. Since for the given $W(s)$ we have $p_{OL}=2$ (the term $(s-1)^2$ in the denominator of $W(s)$ represent a double pole in 1), we have $p_{CL}=2-N$, and therefore

- For $K \in (-0.1,+\infty)$ we have $p_{CL}=2$;
- For $K \in (-\infty,-0.1]$ we have $p_{CL}=1$.

In both cases, the closed loop system is unstable.

*ROUTH ANALYSIS*

The characteristic polynomial of the closed-loop system is the denominator of $W_{CL}(s)$:

$$d_{CL}(s) = s^3 - s^2 - (K+1)s + 10K + 1.$$

The first two rows (rows 3 and 2) of the Routh table are:

$$\begin{array}{c|cc} 3 & 1 & -(K+1) \\ 2 & \text{-1} & 10K+1 \end{array}$$

The computation of the element in the third row (row 1) is

$$a_{1,1} = \frac{1}{-(-1)} \begin{vmatrix} 1 & -(K+1) \\ -1 & 10K+1 \end{vmatrix} = 10K + 1 - (K+1) = 9K$$

Thus we have

$$\begin{array}{c|cc} 3 & 1 & -(K+1) \\ 2 & \text{-1} & 10K+1 \\ 1 & 9K & \\ 0 & 10K+1 & \end{array}$$

Analyzing the signs of the first column we have:

- For $K \in (0,+\infty)$ we have two sign variations ($3 \to 2$ and $2 \to 1$) so that $p_{CL}=2$;
- For $K \in (-0.1,0)$ we have two sign variations ($3 \to 2$ and $1 \to 0$) so that $p_{CL}=2$;
- For $K \in (-\infty,-0.1)$ we have one sign variation ($3 \to 2$ and $1 \to 0$) so that $p_{CL}=1$.

In all cases we have the instabilty of the closed loop system.

These results coincide with those obtained using the Nyquist analysis.

**Problem 5.** Given the system

$$\begin{cases} \dot{x}_1(t) = \big(x_1(t)-1\big)\big(-\alpha^2 + 2\big(x_2(t)-1\big)\big) \\ \dot{x}_2(t) = (\alpha-1)\big(x_2(t)-1\big) - \big(x_1(t)-1\big)^2 \end{cases}$$

study the stability of the equilibrium point $x_e=(1,1)$ for all the values of the parameter $\alpha \in (-\infty,\infty)$, using the method of linear approximation at the equilibrium point, and the Lyapunov method (*Suggestion for the Lyapunov function:* $V(x)=(x_1-x_{e,1})^2+\beta(x_2-x_{e,2})^2$, with suitable $\beta>0$.)

**Solution of problem 5.**

The system considered is of the form $\dot{x}(t) = f\big(x(t);\alpha\big)$, where $x(t)$ is the state and $\alpha$ is a constant parameter. The vector function $f(x;\alpha)$ is as follows

$$\begin{cases} f_1(x;\alpha) = (x_1-1)\big(-\alpha^2+2(x_2-1)\big) \\ f_2(x;\alpha) = (\alpha-1)(x_2-1) - (x_1-1)^2 \end{cases}$$

The Jacobian computed at $x_e = (1,1)$ is

$$J(x_e) = \begin{bmatrix} \partial_{x_1} f_1 & \partial_{x_2} f_1 \\ \partial_{x_1} f_2 & \partial_{x_2} f_2 \end{bmatrix}_{(1,1)} = \begin{bmatrix} -\alpha^2 & 0 \\ 0 & \alpha - 1 \end{bmatrix}$$

Since $J(x_e)$ is diagonal, the two eigenvalues are the diagonal terms:

$$\lambda_1 = -\alpha^2, \qquad \lambda_2 = \alpha - 1$$

From this we conclude:

- $x_e$ is asymptotically stable when *both* eigenvalues are strictly negative, i.e. when $\alpha < 1$ and $\alpha \neq 0$;
- $x_e$ is unstable when *at least one* eigenvalue is strictily positive i.e. when $\alpha > 1$.

There are two cases where we can not conclude anything about the stability of $x_e$: when $\alpha = 0$ and when $\alpha = 1$ (in these cases the origin is a simply stable equilibrium point *of the linear approximation* of the nonlinear system, but nothing can be concluded about the nonlinear system).

In these case we must study the stability of the point $x_e = (1,1)$ by using a suitable Lyapunov function.

Following the suggestion we consider the Lyapunov function

$$V(x) = (x_1 - 1)^2 + \beta(x_2 + 1)^2,$$

which is positive definite for any $\beta > 0$. According to the Lyapunov theorem, if the derivative $\dot{V}(x) = (dV/dx) f(x;\alpha)$ is semidefinite negative, then the equilibrium $x_e$ is (simply) stable, while if $\dot{V}(x)$ is definite negative, then the equilibrium is asymptotically stable. The computation of $\dot{V}(x)$ gives

$$\begin{aligned} \dot{V}(x) &= \frac{dV}{dx} f(x;\alpha) = (\partial_{x_1} V) f_1(x;\alpha) + (\partial_{x_2} V) f_2(x;\alpha) = 2\,(x_1 - 1) f_1(x;\alpha) + 2\beta\,(x_2 - 1) f_2(x;\alpha) \\ &= 2\,(x_1 - 1)(x_1 - 1)\big(-\alpha^2 + 2(x_2 - 1)\big) + 2\beta\,(x_2 - 1)\big((\alpha - 1)(x_2 - 1) - (x_1 - 1)^2\big) \\ &= -2\alpha^2 (x_1 - 1)^2 + 4(x_1 - 1)^2 (x_2 - 1) + 2\beta\,(\alpha - 1)(x_2 - 1)^2 - 2\beta\,(x_2 - 1)(x_1 - 1)^2 \end{aligned}$$

Note that $\dot{V}(x_e) = 0$, as expected. We must check the sign of $\dot{V}(x)$ in the two situations of $\alpha = 0$ and $\alpha = 1$.

Note that in both cases, in the expression of $\dot{V}(x)$ we have two terms that are sign-indefinite:

$$4(x_1 - 1)^2 (x_2 - 1) \qquad \text{and} \qquad -2\beta\,(x_2 - 1)(x_1 - 1)^2$$

By choosing $\beta = 2$ these terms can be cancelled, thus we have

$$\dot{V}(x) = -2\alpha^2 (x_1 - 1)^2 + 2\beta\,(\alpha - 1)(x_2 - 1)^2.$$

Thus, for $\alpha = 0$ we have

$$\dot{V}(x) = -2\beta\,(x_2 - 1)^2 \leq 0,$$

and for $\alpha = 1$ we have

$$\dot{V}(x) = -2\,(x_1 - 1)^2 \leq 0.$$

In both cases the derivative of the Lyapunov function is semidefinite negative (it is never positive, but is zero not only at the equilibrium point $x_e = (1,1)$). As a consequence, when $\alpha = 0$ and $\alpha = 1$ the equilibrium point is *simply* stable (stable, but not asympotically stable).