SCRITTO TEORIA DEI SISTEMI 1/2/2022

Problema 1

$$W(s) = 10K\,\frac{(s-10)}{s^2+1}$$

Problema 2

T.C. $\lambda_1 = -i$, $u_1 = \begin{pmatrix} -1 \\ 2i \end{pmatrix}$, $v_1^T = \begin{bmatrix} -\frac{1}{2} & -\frac{i}{4} \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$

1) Proprietà modi naturali
2) Calcolare $\Phi(t) = e^{At}$ e $w(t)$
3) Calcolare $W(s)$ e $Y_{gradino}(t)$.

Problema 3

Dato il sistema a tempo discreto caratterizzato da $w(t) = \begin{cases} \left(\frac{1}{4}\right)^{t-1} & t>0 \\ 0 & t=0 \end{cases}$
calcolare la risposta forzata e la risposta armonica a $(-1)^t$

Problema 4

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}$$

1) Base per $\mathcal{R}$ e $\mathcal{I}$
2) Quattro sottospazi dec. Kalman
3) Fornire un esempio di stato non raggiungibile e non osservabile.

Problema 5

$$\begin{cases} \dot{x}_1 = (1-K)x_1^3 + 2x_2 \\ \dot{x}_2 = -2x_1 + (1-K)x_2 \end{cases} \qquad x_e = (0,0)$$

Soluzione

Problema 1 :

$$\tilde{W}(s) = -100 \frac{\left(1 - \frac{s}{10}\right)}{s^2 + 1}$$

$\omega_f = 10\ rad/s$

$\omega_n = 1\ rad/s$

guadagno : $K_{\tilde{W}} = -100$ <
- $20\log_{10}(10^2) = 40\ dB$ di traslazione globale
- $\pm\pi$ di sfasamento globale

INTERVALLI :

MODULO		FASE	
$\omega < 1$	$0\ dB/dec$	$\omega < 1$	$0\ rad/dec$
$1 < \omega < 10$	$-40\ dB/dec$	$1 < \omega < 100$	$-\pi/4\ rad/dec$
$10 < \omega$	$-20\ dB/dec$	$100 < \omega$	$0\ rad/dec$

(picco di antirisonanza in $\omega_n = 1$)

sfasamento globale di $-\pi$ in ω_n.

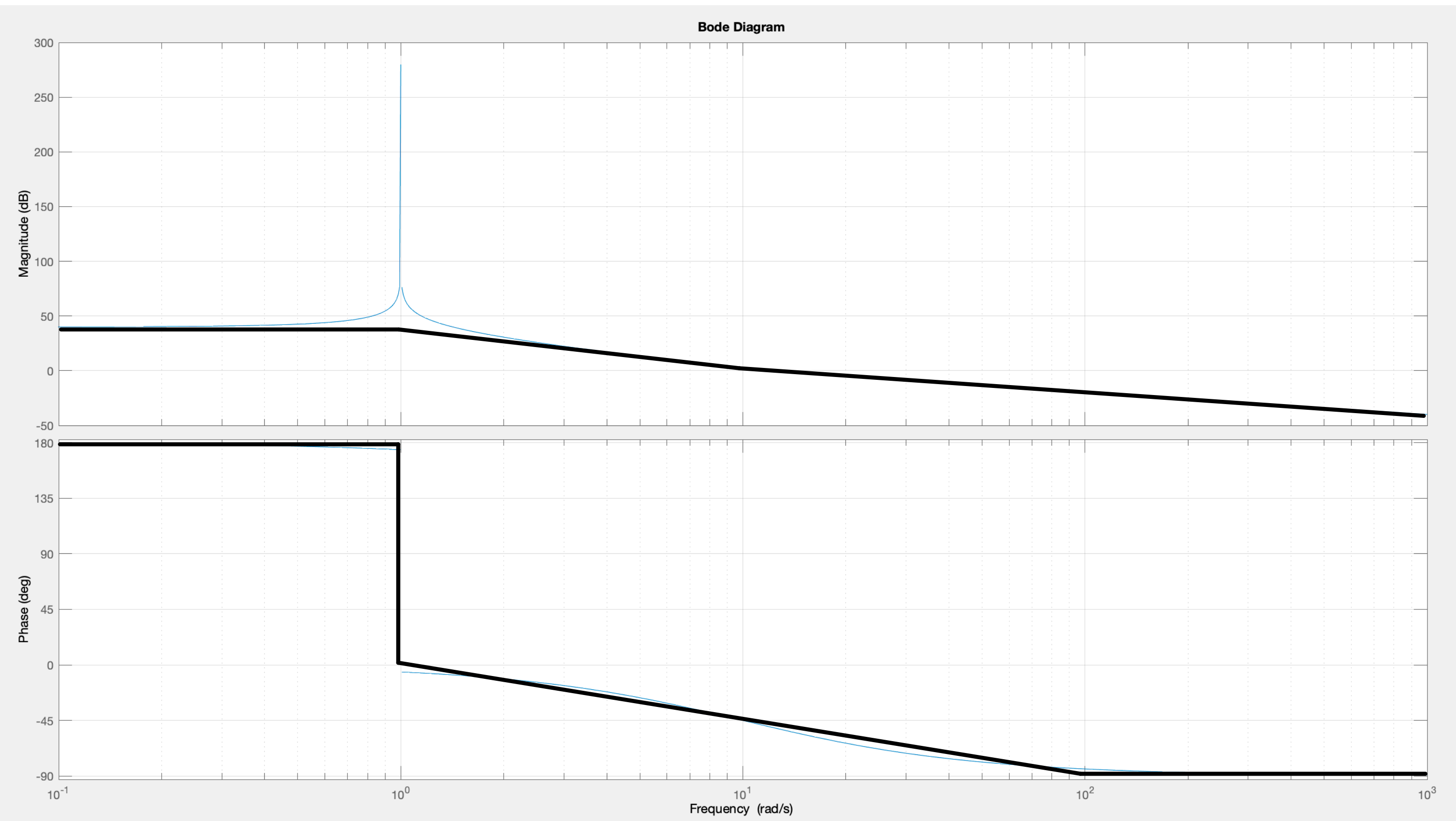


Diagramma polare (K=1)

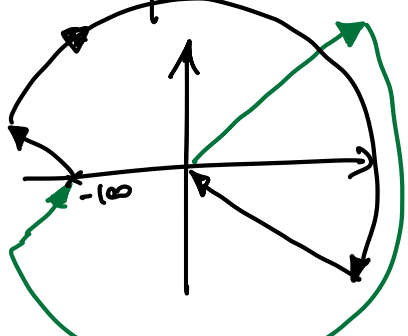
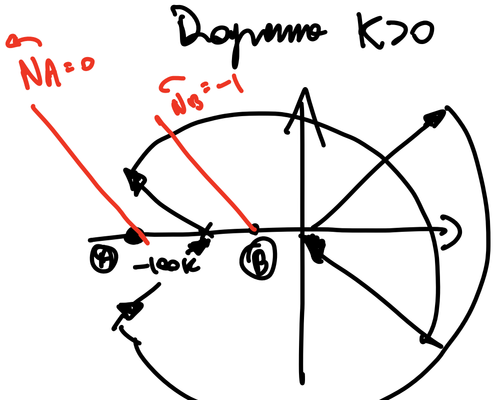
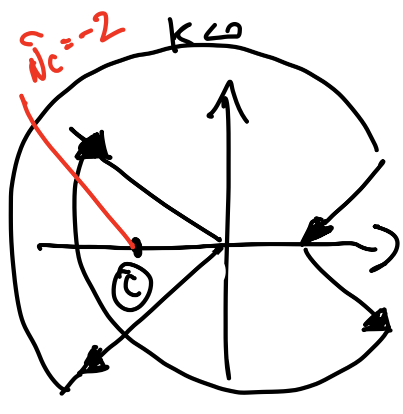


Diagramma K>0



K<0



Stabilità a ciclo chiuso:

$D_{CL}(s)$: $s^2+1+10K(s-10) = s^2+10Ks+1-100K$

se $\begin{cases} 10K>0 \\ 1-100K>0 \end{cases}$ $D_{CL}(s)$ ha solo radici $Re(\cdot)<0$

ovvero se $\begin{cases} K>0 \\ K<\frac{1}{100} \end{cases}$ $\left(K\in\left(0,\frac{1}{100}\right)\right)$ ⇒ stabilità asintotica a ciclo chiuso $P_{CL}=0$ (A)

se $K>\frac{1}{100}$ allora $D_{CL}(s)$ ha una radice $Re(\cdot)>0$ (B)

⇒ $P_{CL}=1$, instabilità a ciclo chiuso

se $K<0$ $D_{CL}(s)$ ha due radici $Re(\cdot)>0$ (C)

$P_{CL}(s)=2$, instabilità a ciclo chiuso.

<u>Problema 2</u> $R=[r_1, r_2] = \begin{bmatrix}-1 & -1\\ 2i & -2i\end{bmatrix}$ $L = R^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{i}{4}\\ -\frac{1}{2} & \frac{i}{4}\end{bmatrix} = \begin{bmatrix} l_1^T \\ l_2^T\end{bmatrix}$

$$\Lambda = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} = \begin{bmatrix}-i & 0\\ 0 & i\end{bmatrix}$$

⇒ $A = 2Re(\lambda_1 r_1 l_1^T) = 2Re\left(-i \begin{pmatrix}-1\\2i\end{pmatrix}\left(-\frac{1}{2}\ \ -\frac{i}{4}\right)\right) = \begin{bmatrix}0 & 1/2\\ -2 & 0\end{bmatrix}$

entrambi i modi sono semplicemente stabili $Re(\lambda_1)=Re(\lambda_2)=0$.

$l_1^T B = \left(-\frac{1}{2}\ \ -\frac{i}{4}\right)\begin{bmatrix}0\\1\end{bmatrix} = -\frac{i}{4} \neq 0$ $C\cdot r_1 = [0\ \ 1]\begin{bmatrix}-1\\2i\end{bmatrix} = 2i \neq 0$

$l_2^T B = +\frac{i}{4} \neq 0$ $C\cdot r_2 = -2i \neq 0$

entrambi i modi sono osservabili ed eccitabili.

$$e^{At} = 2\mathrm{Re}\left\{ e^{-it} \begin{pmatrix} -1 \\ 2i \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & -\frac{i}{4} \end{pmatrix} \right\} = 2\mathrm{Re}\left\{ (\cos(-t) + i\sin(-t)) \begin{bmatrix} \frac{1}{2} & \frac{i}{4} \\ -i & \frac{1}{2} \end{bmatrix} \right\}$$

$$= \mathrm{Re}\left\{ (\underset{\cos(t)}{\cos(-t)} + \underset{-i\sin(t)}{i\sin(-t)}) \begin{bmatrix} 1 & i/2 \\ -2i & 1 \end{bmatrix} \right\} = \begin{bmatrix} \cos(t) & \sin(t)/2 \\ -2\sin(t) & \cos(t) \end{bmatrix}$$

$$W(t) = Ce^{At}B = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \cdot \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \cos(t)$$

$$W(s) = \frac{s}{s^2+1} \qquad Y_{grad}(s) = \frac{1}{s} \cdot \frac{s}{s^2+1} = \frac{1}{s^2+1}$$

$$y_{grad}(t) = \mathcal{L}^{-1}\{ Y_{grad}(s) \} = \frac{R_1}{s+i} + \frac{R_2}{s-i}$$

$$R_1 = \lim_{s \to -i} \frac{1}{s-i} = -\frac{1}{2i} \qquad R_2 = R_1^* = \frac{1}{2i}$$

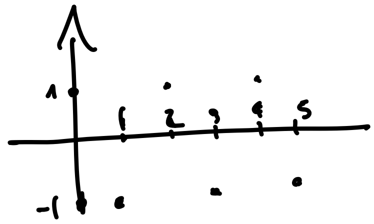
$$\Rightarrow y_{grad}(t) = \mathcal{L}^{-1}\left\{ -\frac{1}{2i} \frac{1}{s+i} + \frac{1}{2i} \frac{1}{s-i} \right\} = -\frac{1}{2i} e^{-it} + \frac{1}{2i} e^{it}$$

$$= \frac{1}{2i} \left(e^{it} - e^{-it} \right) = \sin(t).$$

Problema 3

$$W(t) = \begin{cases} \left(\frac{1}{4}\right)^{t-1} & t > 0 \\ 0 & t = 0 \end{cases} \qquad W(z) = \frac{\mathcal{Z}\left\{ \frac{1}{4}^t \right\}}{z} = \frac{1}{z - 1/4}$$

$$u(t) = (-1)^t = \cos\left(\frac{2\pi}{T}t\right) = \cos(\pi t)$$



segnale armonico di periodo $T=2$

$$U(z) = \mathcal{Z}\{(-1)^t\} = \frac{z}{z+1} = (*)$$

ovviamente: $$\mathcal{Z}\{\cos(\pi t)\} = \frac{z(z-\cos(\pi))}{z^2-2z\cos(\pi)+1} = \frac{z(z+1)}{z^2+2z+1} = \frac{z\cancel{(z+1)}}{(z+1)^{\cancel{2}}} = (*)$$

$$Y_f(z) = W(z)U(z) = \frac{1}{z-1/4}\,\frac{z}{z+1} = \frac{z}{(z-1/4)(z+1)}$$

$$\frac{Y_f(z)}{z} = \frac{R_1}{z-1/4} + \frac{R_2}{z+1}$$

$$R_1 = \lim_{z\to 1/4} \frac{1}{z+1} = \frac{4}{5}$$

$$R_2 = \lim_{z\to -1} \frac{1}{z-1/4} = -\frac{4}{5}$$

$$\Rightarrow y_{for}(t) = \frac{4}{5}\left(\frac{1}{4}\right)^t - \frac{4}{5}(-1)^t$$

$$y_{orm}(t) = M\,|W(e^{j\omega})|\cos(\omega t + \varphi + \angle W(e^{j\omega}))$$

$u(t) = (-1)^t = \cos(\pi t)$

$M = 1,\ \varphi = 0,\ \omega = \pi$

$$= |W(-1)|\cos(\pi t + \angle W(-1))$$

$$W(-1) = \frac{1}{-1-\frac{1}{4}} = -\frac{4}{5} \quad \begin{cases} |W(-1)| = \frac{4}{5} \\ \angle W(-1) = \pm\pi \end{cases}$$

$$\Rightarrow y_{orm}(t) = \frac{4}{5}\cos(\pi t + \pi)$$

$$\frac{4}{5}\cos(\pi t+\pi) = \frac{4}{5}\cos(\pi t)\cos(\pi) - \frac{4}{5}\sin(\pi t)\sin(\pi)$$

$$= -\frac{4}{5}(-1)^t$$ coincide con la risposta forzata all'esaurimento del transitorio

Problema 4

$$R = [B \;\; AB \;\; A^2B \;\; A^3B] = \begin{bmatrix} -1 & 1 & -2 & 3 \\ 0 & -1 & 2 & -3 \\ 1 & -2 & 3 & -5 \\ 0 & 1 & -1 & 2 \end{bmatrix}$$ rango 3

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ -1 & 0 & 2 & 0 \\ 2 & 0 & -3 & 0 \\ -3 & 0 & 5 & 0 \end{bmatrix}$$ rango 2

$$\mathcal{R} = \operatorname{Im}(R) = \operatorname{span}\left\{ \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} 1 \\ -1 \\ -2 \\ 1 \end{pmatrix}}_{v_2}, \underbrace{\begin{pmatrix} -2 \\ 2 \\ 3 \\ -1 \end{pmatrix}}_{v_3} \right\} = \operatorname{span}\left\{ \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{v_2+v_3-v_1}, \underbrace{\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}}_{v_3+2v_2} \right\}$$

$$\mathcal{I} = \ker(Q) = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\mathcal{X}_1 = \mathcal{R} \cap \mathcal{I} = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\mathcal{X}_2 : \mathcal{X}_1 \oplus \mathcal{X}_2 = \mathcal{R} \Rightarrow \mathcal{X}_2 = \operatorname{span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

$$\mathcal{X}_3 : \mathcal{X}_1 \oplus \mathcal{X}_3 = \mathcal{I} \Rightarrow \mathcal{X}_3 = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\mathcal{X}_4 : \mathcal{X}_1 \oplus \dots \oplus \mathcal{X}_4 = \mathbb{R}^4 \Rightarrow \mathcal{X}_4 = \{0\}.$$

Esempio di vettore non raggiungibile e non osservabile: $v = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \in \mathcal{X}_3$.

Problema 5

$$\begin{cases} \dot{x}_1 = (1-k)x_1^3 + 2x_2 \\ \dot{x}_2 = -2x_1 + (1-k)x_2 \end{cases}$$

punto d'equilibrio $x_e = (0,0)$

è t.c. $f_1(x_e) = 0$

$f_2(x_e) = 0$

$$J(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 3(1-k)x_1^2 & 2 \\ -2 & (1-k) \end{bmatrix} \qquad J(x_e) = \begin{bmatrix} 0 & 2 \\ -2 & 1-k \end{bmatrix}$$

$$p(\lambda) = \det(\lambda I - J(x_e)) = \left\| \begin{matrix} \lambda & -2 \\ 2 & \lambda + (k-1) \end{matrix} \right\| = \lambda^2 + (k-1)\lambda + 4$$

- se $k - 1 > 0$ $(k > 1)$ $J(x_e)$ ha tutti autovalori $Re(\cdot) < 0$
  $\Rightarrow$ x_e loc. asint. stabile
- se $k - 1 = 0$ $(k = 1)$ $J(x_e)$ ha due autovalori $Re(\cdot) = 0$
  $(\lambda_1 = 2i, \lambda_2 = \lambda_1^*)$ CASO CRITICO
- se $k - 1 < 0$ $(k < 1)$ $J(x_e)$ ha due autovalori $Re(\cdot) > 0$
  $\Rightarrow$ x_e INSTABILE

Discussione del caso critico $k = 1$

$$\begin{cases} \dot{x}_1 = \cancel{(1-k)x_1^3} + 2x_2 \\ \dot{x}_2 = -2x_1 + \cancel{(1-k)x_2} \end{cases} \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\sigma(A) = \{2i, -2i\} \Rightarrow$ x_e semplicemente stabile.