Pb 1 $W(s) : 50K \dfrac{(s-1)}{(s^2+25)} \Rightarrow \tilde{W}(s) : \dfrac{-50\left(1-\frac{s}{1}\right)}{25\left(1+\frac{s^2}{25}\right)}$ $\rightarrow \omega_t = 1$ rad/s ; $\rightarrow \omega_n = 5$ rad/s, $\zeta = 0$

guadagno : $K_W = -2 \neq 0$ $\begin{cases} |K_W|_{dB} : 20\log_{10}|-2| \cong 6\ dB \\ \langle K_W \rangle = \pm\pi \end{cases}$

binomio (numeratore) $\omega_t = 1$ rad/s $\begin{cases} \text{moduli} : +20\ dB/dec & \omega \in [1, +\infty) \\ \text{fasi} : -\pi/4\ rad/dec & \omega \in [0.1, 10] \end{cases}$

trinomio (denominatore) $\omega_n = 5$ rad/s, $\zeta = 0$ $\begin{cases} \text{moduli} : -40\ dB/dec & \omega \in [5, +\infty) \\ \text{fasi} : \text{sfasamento di } -\pi \text{ in } \omega = \omega_n = 5\ \frac{rad}{s} \end{cases}$

Riassumendo:

MODULI

- $\omega \in (0, 1)$ 0 dB/dec
- $\omega \in (1, 5]$ +20 dB/dec
- $\omega \in (5, +\infty)$ −20 dB/dec
- picco di amplificazione in $\omega_n = 5$ rad/s

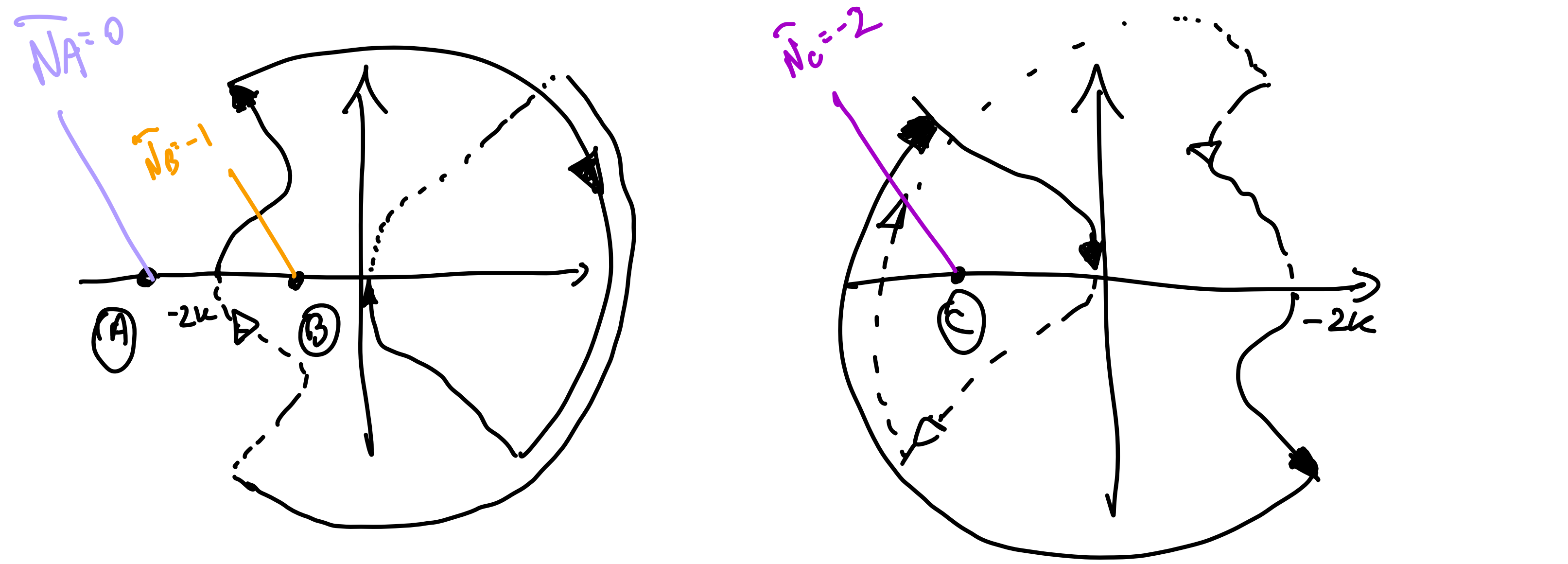
FASI

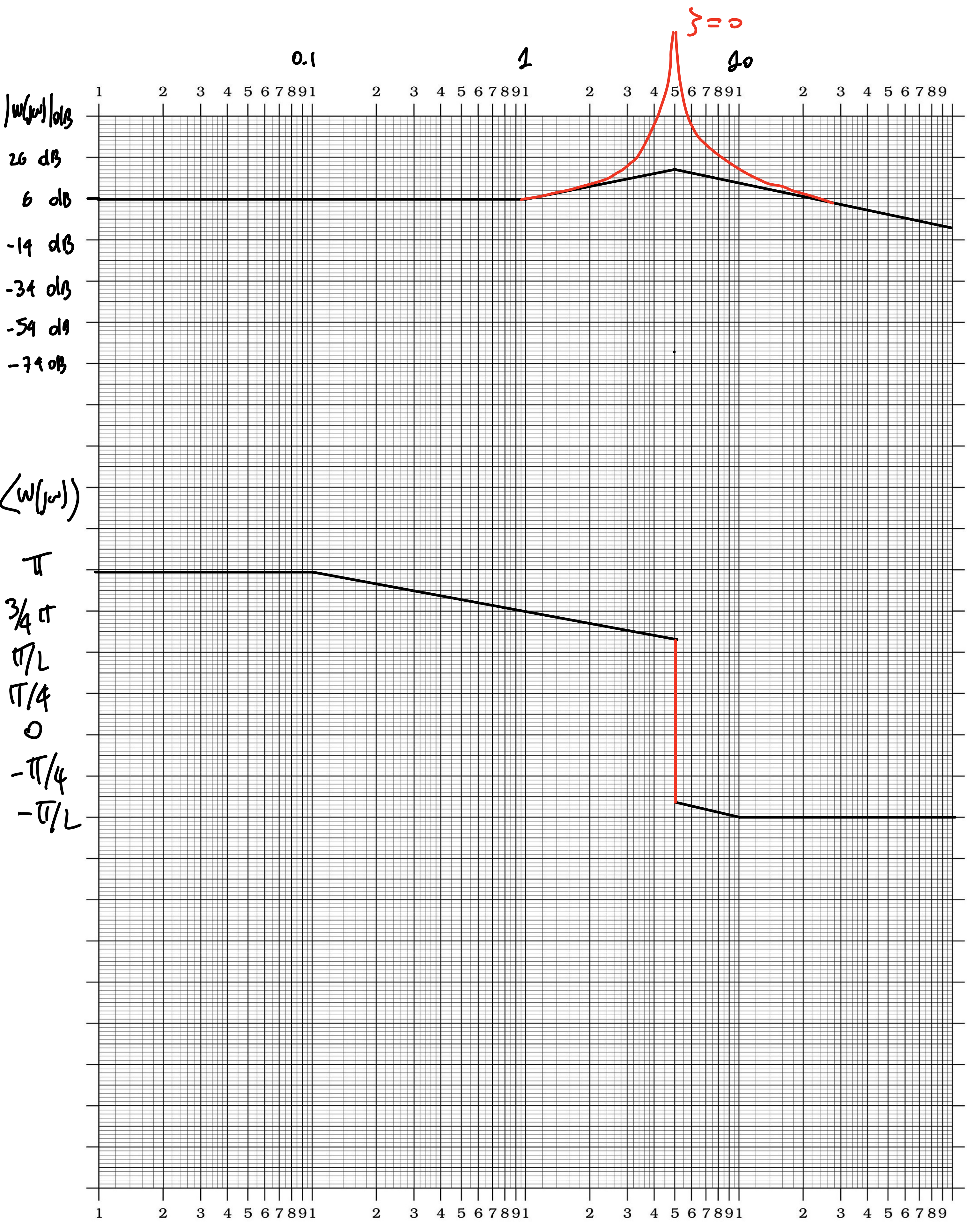
- $\omega \in (0, 0.1)$ 0 rad/dec
- $\omega \in (0.1, 10)$ $-\frac{\pi}{4}$ rad/dec
- $\omega \in (10, +\infty)$ 0 rad/dec
- sfasamento di $-\pi$ in $\omega = \omega_n = 5$ rad/s

diagrammi alla pagina successiva

Diagramma polare di $W_{AP}(s)$ al variare di $K \in \mathbb{R}$:

$K > 0$





ζ = 0
0.1
1
10
|W(jω)|dB
26 dB
6 dB
-14 dB
-34 dB
-54 dB
-74 dB
∠W(jω)
π
3/4 π
π/2
π/4
0
-π/4
-π/2

Studio della stabilità di $W_{CH}(s) = \dfrac{W_{AP}(s)}{1+W_{AP}(s)}$

CRITERIO DI NYQUIST

$N^+_{CH} = N^+_{AP} - \tilde{N}$ dove $N^+_{AP} = 0$ (W_{AP} ha due poli immaginari puri, parte reale nulla)

$\underline{K>0}$ $\quad -1$ è in (A) se $-1 < -2K \iff 0 < K < \frac{1}{2}$ $\qquad \tilde{N}_A = 0$

$\quad -1$ è in (B) se $-1 > -2K \iff K > \frac{1}{2}$ $\qquad N_B = -1$

$\underline{K<0}$ $\quad N_C = -2$

Quindi: $N^+_{CH} = \begin{cases} 0 & \text{se } K \in (0, \frac{1}{2}) \\ 1 & \text{se } K \in (\frac{1}{2}, +\infty) \\ 2 & \text{se } K \in (-\infty, 0) \end{cases}$

CRITERIO DI ROUTH:

$D_{CH}(s) = 50K(s-1) + s^2 + 25 = s^2 + 50Ks - 50K + 25$

$= s^2 + 50Ks + 25(1-2K)$

$D_{CH}(s)$ ha grado 2 FD regola di Cartesio

1	+	+	+
$50K$	−	+	+
$25(1-2K)$	+	+	−
	2V	0V	1V
	$K \in (-\infty, 0)$	$K \in (0, \frac{1}{2})$	$K \in (\frac{1}{2}, +\infty)$

FD come Nyquist.

Pb. 2

$$A = \begin{bmatrix} -1/3 & -1 \\ 1 & -1/3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\det(\lambda I - A) = 0 \iff (\lambda + 1/3)^2 + 1 = 0 \iff \lambda + 1/3 = \pm\sqrt{-1}$$

$$\iff \lambda_1 = -\frac{1}{3} + j$$

$$\lambda_2 = \lambda_1^* = -\frac{1}{3} - j$$

$$r_1 : (\lambda_1 I - A) r_1 = 0 \quad \begin{bmatrix} \lambda_1 + 1/3 & -1 \\ 1 & \lambda_1 + 1/3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \iff jx = y \quad \overset{x=1}{\iff} \quad r_1 = \begin{pmatrix} 1 \\ j \end{pmatrix}$$

$$r_2 = r_1^* = \begin{pmatrix} 1 \\ -j \end{pmatrix} \qquad R = \begin{bmatrix} r_1 \mid r_1^* \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}$$

$$L = R^{-1} = -\frac{1}{2j} \begin{bmatrix} -j & -1 \\ -j & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2j} \\ \frac{1}{2} & -\frac{1}{2j} \end{bmatrix} \begin{matrix} \to l_1^T \\ \to l_2^T \end{matrix}$$

$$e^{At} = \cancel{2} \mathrm{Re}\left\{ e^{(-\frac{1}{3}+j)t} \begin{pmatrix} 1 \\ j \end{pmatrix} \begin{pmatrix} \frac{1}{\cancel{2}} & \frac{1}{\cancel{2}j} \end{pmatrix} \right\} = e^{-\frac{1}{3}t} \mathrm{Re}\left\{ (\cos(t) + j\sin(t)) \begin{bmatrix} 1 & -j \\ j & 1 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} e^{-\frac{1}{3}t}\cos(t) & -e^{-\frac{1}{3}t}\sin(t) \\ e^{-\frac{1}{3}t}\sin(t) & e^{-\frac{1}{3}t}\cos(t) \end{bmatrix}$$

$$w(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} & e^{At} & \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = e^{-\frac{1}{3}t}(\cos(t) - \sin(t))$$

$$W(s) = \mathcal{L}\{w(t)\} = \frac{(s+1/3)}{(s+1/3)^2+1} - \frac{1}{(s+1/3)^2+1} = \frac{s-2/3}{(s+1/3)^2+1}$$

$$y_{prod}(t) = \mathcal{L}^{-1}\left\{W(s)\cdot\frac{1}{s}\right\} = \frac{R_1}{s} + \frac{R_2}{(s+\frac{1}{3}-j)} + \frac{R_2^*}{(s+\frac{1}{3}-j)}$$

$$R_1 = \lim_{s\to 0} \frac{s-2/3}{(s+1/3)^2+1} = \frac{-2/3}{1/9+1} = \frac{-2/\cancel{3}}{10/\cancel{9}} = -\frac{6}{10} = -\frac{3}{5}$$

$$R_2 = \lim_{s\to -\frac{1}{3}+j} \frac{s-2/3}{s(s+\frac{1}{3}+j)} = \frac{-1+j}{(-\frac{1}{3}+j)(2j)} = \frac{-1+j}{-\frac{2}{3}j-2}$$

$$= \frac{(-1+j)(-2+\frac{2}{3}j)}{(-2-\frac{2}{3}j)(-2+\frac{2}{3}j)} = \frac{(2-\frac{2}{3})-2j-\frac{2}{3}j}{(4+\frac{4}{9})} = \frac{\dots}{(\frac{40}{9})}$$

$$= \frac{\frac{4}{\cancel{3}}-\frac{8}{\cancel{3}}j}{\frac{40}{\cancel{9}\,3}} = \frac{3}{10} - \frac{6}{10}j \qquad R_2^* = \frac{3}{10} + \frac{6}{10}j$$

$$y_f(t) = -\frac{3}{5}\delta_{-1}(t) + \frac{3}{10}\left(e^{(\frac{1}{3}+j)t} + e^{(-\frac{1}{3}-j)t}\right) - \frac{6}{10}j\left(e^{(-\frac{1}{3}+j)t} - e^{(-\frac{1}{3}+j)t}\right)$$

$$= -\frac{3}{5}\delta_{-1}(t) + \frac{3}{5}e^{-\frac{1}{3}t}\cos(t) + \frac{6}{5}e^{-\frac{1}{3}t}\sin(t)$$

$\underline{Pb. 3}$ $\quad w(t) = \left(\frac{1}{10}\right)^{t-1} - \left(\frac{1}{2}\right)^{t-1} \quad t > 0$

Risp. armonica esiste perché $\left|\frac{1}{10}\right| < 1$ e $\left|\frac{1}{2}\right| < 1$
(entrambi i modi sono asint. stabili)

$$W(z) = \frac{\mathcal{Z}\left\{\left(\frac{1}{10}\right)^t\right\}}{z} - \frac{\mathcal{Z}\left\{\left(\frac{1}{2}\right)^t\right\}}{z} = \frac{1}{z - 1/10} - \frac{1}{z - 1/2}$$

$$= \frac{-\frac{1}{2} + \frac{1}{10}}{(z - 1/10)(z - 1/2)} = \frac{-2/5}{(z - 1/10)(z - 1/2)}$$

$u(t) = \sqrt{505}\, \sin\left(\frac{\pi}{2} t\right) \qquad \omega = \frac{\pi}{2} \quad M = \sqrt{505}$

$$y_{om}(t) = M \left|W(e^{j\frac{\pi}{2}})\right| \sin\left(\frac{\pi}{2} t + \angle W(e^{j\frac{\pi}{2}})\right)$$

$$W(e^{j\frac{\pi}{2}}) = W(j) = \frac{-2/5}{(j - 1/10)(j - 1/2)}$$

$$\begin{cases} |W(j)| = \dfrac{2/5}{\sqrt{\frac{1}{100}+1}\sqrt{\frac{1}{4}+1}} = \dfrac{2/\cancel{5}^{\,1}}{\frac{\sqrt{101}}{2\cancel{10}}\frac{\sqrt{5}}{2}} = \dfrac{8}{\sqrt{505}} \\ \angle W(j) = \pi - (\operatorname{atan}(-10) + \pi) - (\operatorname{atan}(-2) + \pi) \approx -0{,}56 \text{ rad} \end{cases}$$

$$\Longrightarrow \quad y_{om}(t) = 8 \sin\left(\frac{\pi}{2} t - 0{,}56\right)$$

Pb. 4

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & -1 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \qquad \text{rank}(R) = 2$$

$$\mathcal{R} = \text{Im}(R) = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\} = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 2 & 0 & -2 & 0 \\ 0 & 0 & 2 & 0 \\ 2 & 0 & -2 & 0 \end{bmatrix} \qquad \text{rank}(Q) = 3$$

$$\Rightarrow \dim(\text{Ker}(Q)) = n - \text{rank}(Q) = 4 - 3 = 1$$

$$\bar{\mathcal{Q}} = \text{Ker}(Q) = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

NOTA: per determinare l'unico vettore di base di $\text{Ker}(Q)$ si può equivalentemente:

a) Risolvere il sistema $Q \cdot v = 0$ e determinare v t.c. $\text{Ker}(Q) = \text{span}\{v\}$;

b) notare che la più semplice combinazione lineare non banale (cioè coefficienti non tutti nulli) delle colonne q_1, q_2, q_3, q_4 di Q che dà come risultato il vettore nullo è $\alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3 + \alpha_4 q_4$ con $\alpha_1 = \alpha_3 = \alpha_4 = 0$ e $\alpha_2 = 1$

cioè:

$$\alpha_1 \begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ -2 \\ 2 \\ -2 \end{pmatrix} + \alpha_4 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \left[\begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 2 \\ -2 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

con: $\alpha_1 = \alpha_3 = \alpha_4 = 0$, $\alpha_2 = 1$

il vettore di coefficienti così ottenuto è il vettore di base per $\text{Ker}(Q)$ desiderato $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

$\mathcal{X}_1 = R \cap Q = Q$

$\mathcal{X}_2 : \mathcal{X}_1 \oplus \mathcal{X}_2 = R \Rightarrow \mathcal{X}_2 = \mathrm{span}\left\{\begin{pmatrix}0\\0\\-1\\1\end{pmatrix}\right\}$

$\mathcal{X}_3 : \mathcal{X}_1 \oplus \mathcal{X}_3 = Q \Rightarrow \mathcal{X}_3 = \{0\}$

$\mathcal{X}_4 : \mathcal{X}_1 \oplus \dots \oplus \mathcal{X}_4 = \mathbb{R}^4 \Rightarrow \mathcal{X}_4 = \mathrm{span}\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right\}$

$x_e = \begin{bmatrix}1\\0\\2\\0\end{bmatrix} \notin Q \Rightarrow$ OSS.
$\notin R \Rightarrow$ NON RAGG.

$\begin{bmatrix}0\\1\\-1\\1\end{bmatrix} \notin Q \Rightarrow$ OSS.
$\in R \Rightarrow$ RAGG.

Pb.5

$$\dot{x}_1 = -5x_1 - (x_2+2)^2$$

$$\dot{x}_2 = k(x_2+2) + 2x_1(x_2+2)$$

Cambio di coordinate:

$\zeta_1 = x_1$
$\zeta_2 = x_2 + 2$
$\Rightarrow$
$\dot{\zeta}_1 = -5\zeta_1 - \zeta_2^2$
$\dot{\zeta}_2 = k\zeta_2 + 2\zeta_1\zeta_2$

$$J(\zeta) = \begin{bmatrix}-5 & -2\zeta_2\\ 2\zeta_2 & k\end{bmatrix}_{(0,0)} = \begin{bmatrix}-5 & 0\\ 0 & k\end{bmatrix} \qquad \lambda(J(\zeta_e)) = \{-5, k\}$$

Dunque per $k<0$ $\quad x_e$ loc. A.S.

$k>0$ $\quad x_e$ instabile

$k=0$ $\quad$ caso critico

$\underline{k=0}$ $\Rightarrow$ $\begin{cases} \dot{\xi}_1 = -5\xi_1 - \xi_2^2 \\ \dot{\xi}_2 = 2\xi_1\xi_2 \end{cases}$

Funzione di Lyapunov quadratica $V(\xi) = \frac{1}{2}\xi_1^2 + \frac{\alpha}{2}\xi_2^2 >$ se $\alpha>0$

(centrata nel punto d'equilibrio nelle nuove coordinate!)

$$\dot{V}(\xi) = \left[\xi_1 \;\middle|\; \alpha\xi_2\right]\begin{bmatrix} -5\xi_1 - \xi_2^2 \\ 2\xi_1\xi_2 \end{bmatrix} = -5\xi_1^2 - \cancel{\xi_1\xi_2^2} + \cancel{2\alpha\xi_1\xi_2^2} \quad \text{se } \alpha = \frac{1}{2}$$

Dunque con: $V(\xi) = \frac{1}{2}\xi_1^2 + \frac{1}{4}\xi_2^2 > 0$

$\dot{V}(\xi) = -5\xi_1^2 \leq 0 \Rightarrow x_e$ semplicemente stabile per $k=0$.