State space representation of a class of MIMO Systems via positive systems

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Abstract—In many technological frameworks the only choice to implement the designed signal processing scheme (filter or control law) is to provide a positive state-space realization. On the other hand, by setting an a priori positivity constraint to the processing algorithm could be a heavy restriction to its performances. It is worthwhile, then, to look for a state-space realization through combination of positive systems. This paper a state-space representation for a class of MIMO systems is proposed in the discrete-time framework. The approach followed provides an easy implementation by means of combinations of positive systems, whose total order is fixed regardless of the poles location in the complex plane. The stability of the positive realization of a stable system is also investigated.

Index Terms—Positive systems, State-space realization, Linear systems.

I. INTRODUCTION

A positive system is a system whose state and output evolutions are always nonnegative provided that the initial state and the input sequence are nonnegative [10, 12]. The design of positive realizations plays a crucial role when coping with the necessity of providing a realizable control system according to the available technology. Indeed there are many cases for which a positive realization design is the only implementable choice; it happens, for instance, for charge coupled device technology such as Charge Routing Networks (CRN) [9], or for fiber-optic filters [4], because nonnegativity is a consequence of the underlying physical mechanism. In these cases the constraint of positive system may well impair the properties of the designed processing algorithm (filter or control law). For instance, the design of positive filters heavily restrict the performances w.r.t. other filters such as Butterworth or Chebyshev, which have no sign limitation on their impulse response, [2]. It has been also shown in the literature that approximating a given system by means of a positive realization does not lead to satisfactory results [3]. These drawbacks are even enhanced when the system implements an algorithm that solves an unconstrained theoretical problem (e.g. optimal control, optimal filtering, etc.).

A way to overcome such a drawback is to design the signal processing scheme without any positive constraints, and to realize it by means of combination of positive systems. Such an approach has been successfully applied for SISO (Single Input/Single Output) systems in [6, 4] in a CRN and in a fiber-optic framework, respectively. The MIMO (Multi Input/Multi Output) case is not a straightforward extension of the SISO case and, according to our knowledge, is still an open problem.

This paper proposes a different methodology to solve the previously mentioned problem of system representation by means of combinations of positive realizations. The proposed approach may well be applied both to SISO and MIMO systems, and does not require numerical computations to achieve the dynamic matrices of the positive systems. As in [6, 4] an upper bound for the total order of the system is given: it simply doubles the dimension of the system to be realized regardless to the I/O transfer function poles placement.

When applying the proposed methodology to a stable system, it may happen the resulting positive realization to be unstable. Such a limitation is investigated in the paper, and a necessary and sufficient condition is given which ensures the stability of the positive realization.

The paper is organized as follows. Basic results on state-space realization through combination of positive systems are briefly reported in Section two. The new realization algorithm is presented in Section three, while Section four deals with the stability analysis of the proposed state space representation. Section five reports a numerical example. Conclusions follow.

II PRELIMINARY RESULTS

A positive system is a system whose state and output evolutions are always nonnegative provided that the initial state and the input sequence are nonnegative [10, 12]. When dealing with LTI (Linear Time-Invariant) systems in the discrete-time framework, we speak of positive systems iff the system matrices are all nonnegative.
positive). The problem of positive realizations for LTI systems from a strictly proper rational transfer function is a well stated problem, and it is resumed in the following Theorems (see the tutorial [5] and references therein for more details).

Theorem 1: [11] Let \( H(z) \in \mathbb{C}^{n\times p} \) be a strictly proper rational transfer function of order \( n \) and let \( \{A,B,C\}, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times p}, C \in \mathbb{R}^{1\times n} \) be a minimal realization of \( H(z) \). Then, \( H(z) \) has a positive realization iff there exists a polyhedral proper cone \( K \subset \mathbb{R}^{n} \) such that:

i) \( K \) is \( A \)-invariant;

ii) \( K \subset \mathcal{O}_j = \{x \in \mathbb{R}^n : c_j A^{k-1} x \geq 0, \; k = 1,2,\ldots\}, \forall j = 1,\ldots,q, \) with \( c_j \) rows of \( C \).

iii) \( b_i \in K, \forall i = 1,\ldots,p, \) with \( b_i \) columns of \( B \).

Once the cone \( K \) is found, one has to solve an overdimensioned linear system to obtain the positive state-space realization, namely \( \{A_+, B_+, C_+\} \). According to the SISO case, if \( K \) is the matrix whose columns are the extremal vectors of the cone \( K \), the triple \( \{A_+, B_+, C_+\} \) is given by the solutions of the following systems [7]:

\[
AK = KA_+, \quad B = KB_+, \quad C_+ = CK. \quad (1)
\]

The number of the columns of \( K \) gives back the dimension of the positive realization.

It clearly comes from the definition of positive systems that a necessary condition for the existence of a positive state space realization is that the impulse response of the LTI system has to be nonnegative. Below follows a sufficient condition ensuring the existence of a positive realization for a given scalar transfer function.

Theorem 2: [1] Given a scalar strictly proper transfer function, with a nonnegative impulse response, it has a unique (possibly multiple) dominant pole it admits a positive realization.

When dealing with a generic I/O transfer function which does not admit a positive state-space realization, it is worthwhile to look for a realization through the combination of positive systems. In [6, 4] an algorithm is proposed for the SISO case, which gives the possibility of realizing a general I/O strictly proper transfer function \( H(z) \) with \( n \) asymptotically stable simple poles by means of the difference of an \( N \)-dimensional positive system and a one-dimensional positive system. An upper bound for \( N \) is given, which depends on the position of the poles of \( H(z) \) in the complex plane. By naming \( \mathcal{P}_i, \; i = 3,4,\ldots, \) the set of points in the complex plane that lie in the interior of the regular polygon with \( i \) edges having one vertex in point \( 1 + j0 \) and inscribed in the unit disk centered at the origin, the upper bound for the integer \( N \) is given by:

\[
n + N_2 + \sum_{i \geq 3} (i - 2) N_i, \quad (2)
\]

where:

- \( N_2 \) is the sum of the number of negative real poles and the number of nonnegative real poles with negative residues of \( H(z) \);
- \( N_3 \) is the number of pairs of complex poles of \( H(z) \) belonging to \( \mathcal{P}_3 \);
- \( N_i, \; i > 3, \) is the number of pairs of complex poles of \( H(z) \) belonging to \( \mathcal{P}_i \setminus \bigcup_{j=3}^{i-1} \mathcal{P}_j \).

Such an approach is based on the decomposition of the original transfer function into the difference of two nonnegative transfer functions:

\[
H(z) = H^1_+(z) - H^2_+(z), \quad (3)
\]

each admitting a positive realization. In [6, 4] an algorithm is also given which provides the polyhedral proper cone from which is numerically computed the positive state-space realization of \( H^1_+(z) \) (the other positive realization is readily obtained, being \( H^2_+(z) \) a first order transfer function).

III. MAIN RESULT

Consider the linear time-invariant MIMO system:

\[
\begin{align*}
x(t+1) &= Ax(t) + Bu(t), & x(0) &= x_0, \\
y(t) &= Cx(t) + Du(t),
\end{align*} \quad (4)
\]

with \( x(t) \in \mathbb{R}^n, \; u(t) \in \mathbb{R}^p, \; y(t) \in \mathbb{R}^q \).

Theorem 3: System (4) can be realized as the difference of the outputs of a \( 2n \)-dimensional positive system.

Proof. Define the matrix \( A^+ \in \mathbb{R}^{n\times n} \) as follows:

\[
A^+(i,j) = \begin{cases} A(i,j), & \text{if } A(i,j) \geq 0, \\
0, & \text{if } A(i,j) < 0, \end{cases}
\quad (5)
\]

and \( A^- = A^+ - A \). By construction both \( A^+ \) and \( A^- \) are nonnegative matrices. The matrix \( |A| = A^+ + A^- \) will be referred to as the absolute value of \( A \). Analogously, the nonnegative matrices \( B^+, \; B^-, \; C^+, \; C^-, \; D^+, \; D^- \) are defined. Consider also the positive/negative part of the initial state and of the input sequence, according to the following positions:

\[
\begin{align*}
x_0^+ &= \begin{cases} x_0, & \text{if } x_0 \geq 0, \\
0, & \text{if } x_0 < 0, \end{cases} \\
u_0^+(t) &= \begin{cases} u(t), & \text{if } u(t) \geq 0, \\
0, & \text{if } u(t) < 0, \end{cases}
\text{ for } t = 0,1,\ldots
\end{align*}
\]

and \( x_0^- = x_0^+ - x_0, \; u_0^-(t) = u_0^+(t) - u(t) \).
Then, the positive/negative part of the state $x(t)$ are defined according to the following positive recursive equations:
\[
x^+(t+1) = A^+ x^+(t) + A^- x^-(t)
+ B^+ u^+(t) + B^- u^-(t), \quad x^+(0) = x_0^+,
\]
\[
x^-(t+1) = A^+ x^+(t) + A^- x^-(t)
+ B^+ u^+(t) + B^- u^-(t), \quad x^-(0) = x_0^-.
\]
(8)

from which it readily comes that $x(t) = x^+(t) - x^-(t)$. By setting the nonnegative extended state $X = (x^+T \ x^-T)^T \in \mathbb{R}^{2n}$, and the nonnegative extended input $U = (u^+T \ u^-T)^T \in \mathbb{R}^{2p}$ the equations in (8) may be written in the following more compact form:
\[
X(t+1) = AX(t) + BU(t), \quad X(0) = \left( \begin{array}{c} x_0^+ \\ x_0^- \end{array} \right) \tag{9}
\]
with $A = \begin{bmatrix} A^+ & A^- \\ A^- & A^+ \end{bmatrix}$, $B = \begin{bmatrix} B^+ & B^- \\ B^- & B^+ \end{bmatrix}$.

(10)
The output $y(t)$ is obtained as the difference $y(t) = y^+(t) - y^-(t)$ of the nonnegative outputs of system (9) defined as:
\[
y^+(t) = \begin{bmatrix} C^+ & C^- \end{bmatrix} x(t) + \begin{bmatrix} D^+ & D^- \end{bmatrix} u(t),
\]
\[
y^-(t) = \begin{bmatrix} C^+ & C^- \end{bmatrix} x(t) + \begin{bmatrix} D^+ & D^- \end{bmatrix} u(t). \tag{11}
\]

Fig. 1 - Block diagram of the positive realization.

The order of system (9) may be further reduced in presence of nonnegative initial state and input sequence of the system. Such a framework is common when the signal processing scheme to be realized is synthesized in the frequency domain. Before getting into the details, consider the following notation. Let $\lambda_i$ be an eigenvalue associated to a matrix $M$, and $\bar{\eta}_i = \{\eta_1, \ldots, \eta_{m_i}\}$ be the multi-index associated to $\lambda_i$, with $m_i$ the number of chains of generalized eigenvectors associated to $\lambda_i$. $\eta_{ij}$ is the length of the $j$-th chain, $1 \leq j \leq m_i$, related to $\lambda_i$. The modulus $|\bar{\eta}_i|$ of the multi-index $\bar{\eta}_i$, defined as $|\bar{\eta}_i| = \sum_{j=1}^{m_i} \eta_{ij}$, gives back the algebraic multiplicity of $\lambda_i$. Moreover, we denote with $J_{\eta_i}(\lambda_i)$ the Jordan block of order $\eta_{ij}$ related to $\lambda_i$, and with:
\[
J_{\bar{\eta}_i}(\lambda_i) = \text{diag} \{J_{\eta_1}(\lambda_i), \ldots, J_{\eta_{m_i}}(\lambda_i)\}, \tag{13}
\]
the Jordan block-diagonal matrix of order $|\bar{\eta}_i|$ related to $\lambda_i$. $I_n$ will denote the identity matrix of order $n$.

The following Lemma provides a coordinate transformation for a class of matrices, which will be useful in the sequel. The proof is omitted for brevity.

**Lemma 5:** Let $M$ be a square matrix of order $n$ with a pair of complex eigenvalues, $\lambda_1 = \alpha + j \beta$ and $\lambda_2 = \lambda_1^*$, each related to the multi-index $\bar{\eta}_1 = \{\eta_1, \ldots, \eta_m\}$, with $n = 2|\bar{\eta}_1|$. Then $M$ is similar to the following Jordan block-diagonal form, \cite{13}:
\[
M = \text{diag} \{J_{\bar{\eta}_1}(\lambda_1), J_{\bar{\eta}_1}(\lambda_2)\}, \tag{14}
\]
with $J_{\bar{\eta}_j}(\lambda_j)$ the Jordan block-diagonal matrix associated to $\lambda_j$, $j = 1, 2$. Then, there exists a coordinate transformation $T$ such that the Jordan block-diagonal matrix $M$ is transformed into the following block matrix:
\[
J_{\bar{\eta}_j}(\alpha, \beta) = TMT^{-1} = \begin{bmatrix} J_{\bar{\eta}_1}(\alpha) & \beta I_{|\bar{\eta}_1|} \\ -\beta I_{|\bar{\eta}_1|} & J_{\bar{\eta}_1}(\alpha) \end{bmatrix}. \tag{15}
\]

Let $\nu$ be the number of real eigenvalues and $\mu$ be the number of pairs of complex eigenvalues of $A$. According to Lemma 5, assume that matrix $A$ is in the following block-diagonal form:
\[
A = \text{diag} \{J_{\eta_1}, \ldots, J_{\eta_v}, J_{\bar{\eta}_1}, \ldots, J_{\bar{\eta}_\mu}\}, \tag{16}
\]
where $J_{\tilde{\eta}_i}$ is the Jordan block-diagonal matrix associated to the real eigenvalue $\lambda_i$ with the multi-index $\tilde{\eta}_i = \{\eta_{i1}, \ldots, \eta_{i\nu}\}$, $i = 1, \ldots, \nu$, and $J_{\tilde{\sigma}_j}$ is a block matrix of order $2|\tilde{\sigma}_j|$ of the type of (15), related to the complex pair of eigenvalues $\alpha_j \pm j\beta_j$ with the multi-index $\tilde{\sigma}_j = \{\sigma_{j1}, \ldots, \sigma_{j\mu}\}$, $j = 1, \ldots, \mu$. Then, the whole system (4) can be rewritten in the following form:

\[
\begin{align*}
x_i(t + 1) &= J_{\tilde{\eta}_i}x_i(t) + B_iu(t), & i = 1, \ldots, \nu, \\
x_{\nu+j}(t + 1) &= J_{\tilde{\sigma}_j}x_{\nu+j}(t) + B_{\nu+j}u(t), & j = 1, \ldots, \mu, \\
y(t) &= \sum_{i=1}^{\nu} C_i x_i(t) + \sum_{j=1}^{\mu} C_{\nu+j}x_{\nu+j}(t) + Du(t),
\end{align*}
\]

with $x_i(t) \in \mathbb{R}^{|\tilde{\eta}_i|}$, $i = 1, \ldots, \nu$ and $x_{\nu+j}(t) \in \mathbb{R}^{2|\tilde{\sigma}_j|}$, $j = 1, \ldots, \mu$. If $J_{\tilde{\eta}_i}$ is related to nonnegative real eigenvalues, and the corresponding input/state and state/output matrices $B_i$ and $C_i$ are nonnegative, there is no reason to make the positive/negative decomposition for the quadruple $(J_{\tilde{\eta}_i}, B_i, C_i, D)$. Assume the first $k$ quadruples $(J_{\tilde{\eta}_i}, B_i, C_i, D)$, $i = 1, \ldots, k$, satisfy such a property. Then, the whole state may be decomposed as:

\[
\begin{align*}
x(t) = \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix}, & \xi_1 = \begin{bmatrix} x_1^T & \cdots & x_k^T \end{bmatrix}^T \in \mathbb{R}^{n_1}, \\
& \xi_2 = \begin{bmatrix} x_{k+1}^T & \cdots & x_{\nu}^T & x_{\nu+1}^T & \cdots & x_{\nu+\mu}^T \end{bmatrix}^T, \\
& \xi_2 \in \mathbb{R}^{n-n_1}, \text{ where } n_1 = \sum_{i=1}^{k} |\tilde{\eta}_i| \text{ and:}
\end{align*}
\]

\[
\begin{align*}
\xi_1(t + 1) &= \tilde{A}_1\xi_1(t) + \tilde{B}_1u(t), \\
\xi_2(t + 1) &= \tilde{A}_2\xi_2(t) + \tilde{B}_2u(t), \\
y(t) &= \tilde{C}_1\xi_1(t) + \tilde{C}_2\xi_2(t) + Du(t),
\end{align*}
\]

with:

\[
\begin{align*}
\tilde{A}_1 &= \text{diag}\{J_{\tilde{\eta}_1}, \ldots, J_{\tilde{\eta}_k}\}, \\
\tilde{B}_1 &= \begin{bmatrix} B_1^T & \cdots & B_k^T \end{bmatrix}^T, \\
\tilde{A}_2 &= \text{diag}\{J_{\tilde{\eta}_{k+1}}, \ldots, J_{\tilde{\eta}_\nu}, J_{\tilde{\sigma}_1}, \ldots, J_{\tilde{\sigma}_\mu}\}, \\
\tilde{B}_2 &= \begin{bmatrix} B_{k+1}^T & \cdots & B_\nu^T & B_{\nu+1}^T & \cdots & B_{\nu+\mu}^T \end{bmatrix}^T, \\
\tilde{C}_1 &= [C_1 \cdots C_k], \\
\tilde{C}_2 &= [C_{k+1} \cdots C_\nu \ C_{\nu+1} \cdots C_{\nu+\mu}].
\end{align*}
\]

By applying the positive/negative decomposition only to the dynamical equations of $\xi_2(t)$, it comes that the output is achieved through the following combination:

\[
y(t) = y_1(t) + y_2^+(t) - y_2^-(t),
\]

where $y_1(t) = \tilde{C}_1\xi_1(t)$ and $y_2^+(t)$, $y_2^-(t)$ are the outputs of a $2(n-n_1)$ positive system realized according to Theorem 3 (fig.2). Then, the order of the whole system becomes $n_1 + 2(n-n_1) = 2n - n_1$.

\[\text{Fig. 2 - Reduced order positive realization.}\]

## IV. STABILITY ANALYSIS

According to Theorem 3, the proposed approach provides a positive realization on a state space whose dimension is twice the dimension of the original (minimal) realization. From an applicative point of view, it is important to investigate whether a positive realization scheme coming from an originally stable system maintains the stability property. In this Section a necessary and sufficient condition is given which ensures the stability of the positive realization described in Theorem 3, starting from a state-space minimal realization of the original system with matrix $A$ in the block-diagonal form of the type in (16).

**Lemma 6:** The spectrum of $A$, eq.(10), namely $\sigma(A)$, is given by:

\[\sigma(A) = \sigma(A) \cup \sigma(|A|).\]  

(22)

The proof is omitted for brevity.

**Definition 7:** A positive $J$-class realization is the quadruple $(A, B, C, D)$ given by equations (10–12) when the original (minimal) system from which it is computed is in the block-diagonal form of (16).

**Theorem 8:** Assume to have a stable minimal realization of the type (16). Then, the positive $J$-class realization provided by Theorem 3 is stable if and only if the eigenvalues of matrix $A$ are in $\mathcal{P}_4$, the regular square inscribed in the unit circle, with one vertex in the point $1 + j0$ in the complex plane.

**Proof:** According to the general block diagonal decomposition of (16), it comes that:

\[\sigma(|A|) = \left( \bigcup_{i=1}^{\nu} \sigma(|J_{\tilde{\eta}_i}|) \right) \cup \left( \bigcup_{j=1}^{\mu} \sigma(|J_{\tilde{\sigma}_j}|) \right).\]  

(23)

Let us name $\lambda_i$, $i = 1, \ldots, \nu$ the real eigenvalues and $\alpha_j \pm j\beta_j$, $j = 1, \ldots, \mu$ the pairs of complex conjugate eigenvalues.

Consider, first, the real eigenvalues, which obviously belong to $\mathcal{P}_4$. If $\lambda_i \geq 0$, then $|J_{\tilde{\eta}_i}| = J_{\tilde{\eta}_i}^T = J_{\tilde{\eta}_i}$, from
which follows the stability of $|J_{\tilde{H}}|$. If $\lambda_i < 0$, then:

$$
|J_{\tilde{H}}| = \begin{bmatrix}
|\lambda_i| & \varphi_{i,1} & 0 & \cdots & 0 \\
0 & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & |\lambda_i|^{-1}
\end{bmatrix}, \quad (24)
$$

with $\varphi_{i,j} \in \{0,1\}$, $j = 1, \ldots, |\tilde{H}_i| - 1$, from which it follows that $\sigma(|J_{\tilde{H}}|) = \{|\lambda_i|\}$, with $|\lambda_i| \leq 1$.

Consider, then, the case of complex eigenvalues. According to (15):

$$
|J_{\sigma_j}(\alpha, \beta)| = \begin{bmatrix}
|J_{\sigma_j}(\alpha_j)| & |\beta_j| I_{|\sigma_j|} \\
|\beta_j| I_{|\sigma_j|} & \sigma_j(\alpha_j)
\end{bmatrix}. \quad (25)
$$

In order to compute the eigenvalues of $|J_{\sigma_j}|$, we apply the following formula involving the determinant of block matrices [8]:

$$
\det \begin{bmatrix}
U & V \\
W & Z
\end{bmatrix} = \det U \cdot \det (Z - W U^{-1} V). \quad (26)
$$

Recalling that, in our case, it is:

$$
U = Z = \lambda I_{|\sigma_j|} - |J_{\sigma_j}(\alpha_j)|, \quad V = W = -|\beta_j| I_{|\sigma_j|}, \quad (27)
$$

it comes:

$$
\det (\lambda I_{|\sigma_j|} - |J_{\sigma_j}|) = \det ((\lambda I_{|\sigma_j|} - |J_{\sigma_j}(\alpha_j)|)^2 - |\beta_j|^2 I_{|\sigma_j|}) = ((\lambda - |\alpha_j|)^2 - |\beta_j|^2)^{|\sigma_j|}, \quad (28)
$$

that means, the eigenvalues of $|J_{\sigma_j}|$ are $|\alpha_j| \pm |\beta_j|$, each with $|\sigma_j|$ algebraic multiplicity. A necessary and sufficient condition for $|\alpha_j| \pm |\beta_j|$ to be in the unit circle is that $\alpha_j \pm j \beta_j \in P_4$.

It can be proven that all the unstable modes of a positive $J$-class realization that occur when the original system has stable eigenvalues outside $P_4$, are observable. The following example reports the proof for the case of a matrix $A$ of order $n = 2$, with a pair of complex poles. (the whole theorem is not reported for the lack of space).

**Example 9:** Assume matrix $A \in \mathbb{R}^{2 \times 2}$ is given by:

$$
A = \begin{bmatrix}
\alpha & \beta \\
-\beta & \alpha
\end{bmatrix}, \quad \alpha, \beta > 0. \quad (29)
$$

Then, $\alpha + \beta$, the dominant eigenvalue of $A$, is related to an observable mode.

**Proof.** Consider the explicit form of matrix $A$, (10):

$$
A = \begin{bmatrix}
\alpha & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\alpha & 0 & 0 & \beta \\
\beta & 0 & 0 & \alpha
\end{bmatrix}, \quad (30)
$$

from which it follows that:

$$
(\alpha + \beta) I_{4} - A = \begin{bmatrix}
\beta & -\beta & 0 & 0 \\
0 & \beta & -\beta & 0 \\
0 & 0 & \beta & -\beta \\
-\beta & 0 & 0 & \beta
\end{bmatrix}. \quad (31)
$$

Then, the eigenvectors $u$ associated to $(\alpha + \beta)$ are in the span of $[1 \; 1 \; 1]^T$, so that:

$$
[C^+ \; C^-] u \not= 0, \quad [C^- \; C^+] u \not= 0. \quad (32)
$$

\[ \square \]

**Remark 10:** It has to be stressed that also the approach followed in [6] may produce unobservable unstable modes. Nevertheless, assuming that the original I/O transfer function $H(z)$ is asymptotically stable, there always exists a reduced order positive realization with the dominant eigenvalue given by a pole of $H(z)$, [1]. Unfortunately, such a theorem cannot be applied to our case, because we build the positive systems directly starting from a minimal realization.

**V. NUMERICAL EXAMPLES**

In this Section the proposed approach is applied in a numerical example in comparison with the existing methodology [6, 4]. Of course the comparison is done only for SISO systems, and only eigenvalues in $P_4$ will be considered. The following numerical example is reported to show how easy is the algorithm. The example is taken from [6]; it is a fourth-order low-pass digital Chebyshev filter with 0.5dB of ripple in the passband and with cut-off frequency 0.5-times half the sample rate:

$$
H(z) = \frac{0.06728(z+1)^4}{(z^2 + 0.0526z + 0.7095)(z^2 - 0.5843z + 0.2314)}. \quad (33)
$$

As a preliminary step, the direct transmission term $D = 0.06728$ is eliminated, so that we obtain $H(z) = H'(z) + D$, with $H'(z)$ given by the following ratio:

$$
\frac{0.30489z^3 + 0.34244z^2 + 0.29619z + 5.6234 \cdot 10^{-2}}{(z^2 + 0.0526z + 0.7095)(z^2 - 0.5843z + 0.2314)}. \quad (34)
$$

As it can be verified, all the poles lie in $P_4$. More in details we have $\lambda_{1/2} = 0.29215 \pm 0.38216$ and $\lambda_{3/4} = -0.0263 \pm 0.84191 \in P_4 \cap P_3$. Then $H'(z)$ is decomposed as follows:

$$
H'(z) = \frac{-0.2831z + 0.0992}{z^2 + 0.0526z + 0.7095} + \frac{0.5880z + 0.0469}{z^2 - 0.5843z + 0.2314}. \quad (35)
$$
from which comes the block-diagonal matrix state-space minimal decomposition (16):

$$\begin{bmatrix}
-0.0263 & 0.84191 & 0 & 0 \\
-0.84191 & -0.0263 & 0 & 0 \\
0 & 0 & 0.29215 & 0.38216 \\
0 & 0 & -0.38216 & 0.29215
\end{bmatrix}. \tag{36}$$

According to the positive/negative decomposition, Theorem 3, the positive state-space realization readily comes, without any further computation (and in case of a system directly available in a state space form, no computations at all are required):

$$A^+ = \begin{bmatrix}
0 & 0.84191 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0.29215 & 0.38216 \\
0 & 0 & 0 & 0.29215
\end{bmatrix}, \quad A^- = \begin{bmatrix}
0.0263 & 0 & 0 & 0 \\
0.84191 & 0.0263 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.38216
\end{bmatrix}. \tag{38}$$

$$B^+ = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0.2049 & 0.0782 & 0 & 0 \\
0.0079 & 0.5810 & 0 & 0
\end{bmatrix}, \quad C^+ = [1 \ 1 \ 1], \quad C^- = [0 \ 0 \ 0]. \tag{39}$$

**Remark 11:** It is easy to verify that, in case of $\nu$ real poles with $n_1$ nonnegative real poles with nonnegative residues, $\mu_1$ pairs of conjugate complex poles in $P_3 \cap P_4$ and $\mu_2$ pairs of conjugate complex poles in $P_4 \setminus P_3$ (that means: $n = \nu + 2\mu_1 + 2\mu_2$), the total dimension of our positive realizations is:

$$n + (n - n_1) = n + \nu - n_1 + 2\mu_1 + 2\mu_2. \tag{40}$$

while, following the existing methodology [6, 4], the total dimension of the two positive systems is, (2):

$$n + \nu - n_1 + \mu_1 + 2\mu_2 + 1. \tag{41}$$

It is trivial to verify that the total order of the positive realization is the same for the two approaches in case of $\mu_1 = 1$. If $\mu_1 = 0$ (the case of only real eigenvalues), our approach provides a total order lower (even if of one dimension only). If $\mu_1 \geq 2$ the approach in [6, 4] has a lower total order.

**Remark 12:** In case of complex poles in $P \setminus \bigcup_{j=3}^{i-1} P_j$, $i > 4$, we do have an effective lower total order w.r.t. the existing methodology. However, being involved poles outside $P_4$, the stability properties are not maintained.

VI. CONCLUSIONS

In this paper a new methodology is proposed to realize a given signal processing scheme by means of the output combinations of positive systems. The approach is very easy to implement, in that it does not require further numerical computations once a minimal realization of the system is obtained. Moreover, it allows to be applied in the same way both to SISO and MIMO systems. An upper bound is also given for the total order of the system, which consists of twice the dimension of the system to be realized, regardless to the I/O transfer function poles allocation. A necessary and sufficient condition that ensures the stability of the positive $J$-class realization of a stable system is provided.

REFERENCES


