1. INTRODUCTION

Many results are available in the literature for the input/output exact linearization and for the stabilization of retarded (with time-delay in the state) nonlinear systems (see, for instance, Jankovic, 2001, Germani, Manes & Pepe, 2001, 2003, Oguchi, Watanabe & Nakamizo, 2002, Hua, Guan & Shi, 2004, Lien, 2004, Marquez-Martinez & Moog, 2004, Zhang & Cheng, 2005). All the above papers assume the full knowledge of the system state. On the other hand, few works dealing with the observer problem of retarded nonlinear systems are available in the literature. Contributions can be found in Germani, Manes & Pepe, 2001, Germani & Pepe, 2005. Observers for nonlinear systems with delayed output have been built up in Germani, Manes & Pepe, 2002, Kazantzis & Wright, 2005, Koshkouei & Burnham, 2009, and an output feedback stabilizing control law is found in Zhang, Zhang & Cheng, 2006. An observer based control law for the human glucose-insulin system, described by retarded nonlinear equations (see Palumbo, Panunzi & De Gaetano, 2007), is investigated in Palumbo, Pepe, Panunzi & De Gaetano, 2009. The input/output linearization method and the nonlinear observer, developed in Germani, Manes & Pepe, 2000, 2003, and in Germani & Pepe, 2005, respectively, are used. The local convergence of the glucose evolution to the reference signal is proved theoretically and shown by simulations.

In this paper we investigate an observer based control law for the class of retarded systems which admit full relative degree and time-delay matched with the control input (see Pepe, 2003, Germani & Pepe, 2005). Many practical systems belong to this class, such as the above human glucose-insulin system or the two interacting species systems, with control acting on predators (see Kolmanovskii & Myshkis, 1999). Any finite number of discrete time-delays of arbitrary size is allowed. It is assumed that the time-delays are constant and known. The input/output exact linearization method and the observer developed in Germani, Manes & Pepe, 2000, 2003, and in Germani & Pepe, 2005, respectively, are used. The state feedback control law found by the exact input/output linearization method is applied to the system using the estimated state instead of the true state. The asymptotic stability of the trivial solution of the closed-loop system is proved. In the case that the functionals describing the equations are globally Lipschitz, it is proved that the trivial solution is globally uniformly asymptotically stable. In the case that the functionals describing the equations are only locally Lipschitz, it is proved that the trivial solution of the closed loop system is locally asymptotically stable. In the global case, distributed delay terms are also allowed. A numerical example, concerning the global case, is reported with simulations.

Notations

$R$ denotes the set of real numbers, $R^+$ denotes the set of non negative reals $[0, \infty)$. The symbol $\cdot$ stands for the Euclidean norm of a real vector, or the induced Euclidean norm of a matrix. The essential supremum norm of an essentially bounded function is indicated with $\sup_{t \geq 0} |v(t)| < \infty$. For given times $0 \leq T_1 < T_2$, we indicate with $v_{[T_1,T_2]} : R^+ \rightarrow R^n$ the function given by $v_{[T_1,T_2]}(t) = v(t)$ for all $t \in [T_1,T_2)$ and $= 0$ elsewhere. An input $u$ is said to be locally essentially bounded if, for any $T > 0$, $v_{[0,T]}$ is essentially bounded. For a positive integer $n$, for a positive real $\tau$ (maximum involved time-delay), $C$ denotes the space of the continuous functions mapping $[-\tau, 0]$ into $R^n$. For a function $x : [-\tau, c) \rightarrow R^n$, with $0 < c \leq \infty$, for any positive real $\tau$, $x_\tau(t) = x(t + \tau)$, $\tau \in [-\tau, 0)$. For a positive real $H$, $C_H \subset C$ denotes the subset of the functions $x \in C$ such that $\|x\|_H < H$, $B_H(0) \subset R^n$ denotes the subset of the vectors $x \in R^n$ such that $|x| < H$. For given smooth functions $f, g : R^m \rightarrow R^n$, $h : R^n \rightarrow R$, $L_i^0 h(x) = \frac{\partial L_i^{j-1} h(x)}{\partial x} f(x)$, $i = 1, 2, \ldots$.
A control affine single-input single-output nonlinear system described by functions \( f, g, h \) admits (uniform) full relative degree in a ball \( B_{\gamma}(0) \) in \( R^n \) if, for all \( x \in B_{\gamma}(0) \),
\[
L_g L_f^j h(x) = \frac{\partial L_f^j h(x)}{\partial x} - g(x), \quad j = 0, 1, \ldots
\]  
(1)

Let \( x(t) = f(x(t)) + g(x(t)) (p_1(x,t) u(t) + p_2(x,t)), \quad t \geq 0 \)
\[
y(t) = h(x(t)) \quad t \geq -\Delta
\]  
(2)

where: \( x(t) \in R^n \); \( n \) is a positive integer; \( u(t), y(t) \in R \) are the control input and measured output, respectively; \( f : R^n \to R^n \); \( g : R^n \to R^n \); \( h : R^n \to R \) are smooth functions (admit continuous partial derivatives of any order); \( p_1 : \mathcal{C} \to R \); \( p_2 : \mathcal{C} \to R \) are continuously Frechet differentiable functionals; \( f(0) = 0, p_2(0) = 0 \).

2. RETARDED SYSTEMS WITH MATCHED TIME-DELAY AND CONTROL INPUT

In this paper we consider the following retarded nonlinear system (see Pepe, 2003, Germani & Pepe, 2005)
\[
\begin{align*}
\dot{x}(t) &= f(x(t)) + g(x(t)) (p_1(x,t) u(t) + p_2(x,t)), \quad t \geq 0 \\
y(t) &= h(x(t)) \quad t \geq -\Delta
\end{align*}
\]  
(3)

where: \( x(t) \in R^n \); \( n \) is a positive integer; \( u(t), y(t) \in R \) are the control input and measured output, respectively; \( f : R^n \to R^n \); \( g : R^n \to R^n \); \( h : R^n \to R \) are smooth functions (admit continuous partial derivatives of any order); \( p_1 : \mathcal{C} \to R \); \( p_2 : \mathcal{C} \to R \) are continuously Frechet differentiable functionals; \( f(0) = 0, p_2(0) = 0 \).

3. GLOBAL SEPARATION THEOREM

Let us introduce in this section the following hypotheses for (3):

H1) the triple \( f, g, h \) has full uniform relative degree;
H2) the nonlinear delay-free system described by the triple \((f, g, h)\) is Globally Uniformly Lipschitz Drift Observable (GULDO), see Dalla Mora, Germani, Manes, 2000), i.e., the function \( \Phi : R^n \to R^n \), defined, for \( x \in R^n \), by
\[
\Phi(x) = \begin{bmatrix} h(x) & L_f h(x) & \cdots & L_f^{n-1} h(x) \end{bmatrix}^T,
\]
is a diffeomorphism in \( R^n \), and there exist positive reals \( \gamma_0 \) and \( \gamma_0 - 1 \) such that, \( \forall v_1, v_2 \in R^n \),
\[
|\Phi(v_1) - \Phi(v_2)| \leq \gamma_0 |v_1 - v_2|,
\]
(4)

H3) there exist a positive real \( \gamma_1 \) such that, \( \forall v_1, v_2 \in R^n \),
\[
|L_f^j h(\Phi^{-1}(v_1)) - L_f^j h(\Phi^{-1}(v_2))| \leq \gamma_1 |v_1 - v_2|.
\]
(5)

H4) there exist a positive real \( \gamma_2 \) such that, \( \forall v_1, v_2 \in \mathbb{C}, \)
\[
|L_g L_f^j h(\Phi^{-1}(v_1)) p_2(\Phi^{-1}(v_1)) - L_g L_f^j h(\Phi^{-1}(v_2)) p_2(\Phi^{-1}(v_2))| \
\leq \gamma_2 \|v_1 - v_2\|_\infty,
\]
(6)

where \( \Psi^{-1}(v_i(\tau)) = \Phi^{-1}(v_i(\tau)), \tau \in [-\Delta, 0], i = 1, 2 \);
H5) there exists a function \( G : \mathcal{C} \to R \), such that, \( \forall \phi \in \mathcal{C} \),
\[
L_g L_f^j h(\phi(0)) |p_1(\phi)| = G(H(\phi)),
\]
where \( H : \mathcal{C} \to \mathcal{C} \) is defined as \( H(\phi)(\tau) = h(\phi(\tau)), \tau \in [-\Delta, 0] \);
H6) the functional \( \phi \rightarrow p_1(\phi) \) is not equal to zero \( \forall \phi \in \mathcal{C} \).

Lemma 1. Let \( n \) be a positive integer. Let \( L : \mathcal{C} \times \mathcal{C} \to R^m \) and \( M : \mathcal{C} \times \mathcal{C} \to R^n \) be smooth functions, with \( L(0,0) = M(0,0) = 0 \). Let there exist a positive constant \( l \) such that the inequality holds
\[
|L(x, x + y) - L(x, x)| \leq l \|y\|_\infty, \quad \forall x, y \in \mathcal{C}
\]
(7)

Let there exist a Hurwitz matrix \( H \) such that
\[
L(v, v) = Hv(0), \quad \forall v \in \mathcal{C}
\]
(8)

Consider the system in the unknown variables \( v(t), w(t) \in R^n \),
\[
\begin{align*}
\dot{v}(t) &= L(v, v_1 + w_1) \\
\dot{w}(t) &= M(w_1, v_1)
\end{align*}
\]  
(9)

Let system (9) be such that, for a suitable function \( \beta \) of class \( \mathcal{K} \), the inequality holds
\[
|w(t)| \leq \beta(|w_0|_\infty, t), \quad t \geq 0
\]
(10)

Then, the trivial solution of the system (9) is globally uniformly asymptotically stable.

Proof. Let \( P \) be a symmetric positive definite matrix such that \( HT P + PH = -I \). Let us consider the first equation in system (9), and let us consider \( w_1 \) as a disturbance. Let us choose the Liapunov-Krasovskii functional \( V(\phi) = \phi^T(0) P \phi(0), \phi \in \mathcal{C} \). We have, for all \( \phi, \psi \in \mathcal{C} \),
\[
D^+ V(\phi, \psi) = \frac{\partial V(\phi(0))}{\partial \phi(0)} L(\phi, \phi + \psi) = \\
\frac{\partial V(\phi(0))}{\partial \phi(0)} L(\phi, \phi) + \frac{\partial V(\phi(0))}{\partial \psi(0)} (L(\phi, \phi + \psi) - L(\phi, \phi))
\]
(11)

Therefore, \( \forall \phi, \psi \in \mathcal{C} \), the following inequality holds
\[
\frac{\partial V(\phi(0))}{\partial \phi(0)} L(\phi, \phi + \psi) - \phi^T(0) \phi(0)
\]
\[
+ 2P||\phi(0)||_{} |L(\phi, \phi + \psi) - L(\phi, \phi)| \leq 0
\]
(12)

From condition (7) we get
\[
D^+ V(\phi, \psi) \leq -\phi^T(0) \phi(0) + 2P||\phi(0)||_{} \|\psi\|_\infty
\]
(13)

From results in Pepe & Jiang, 2006, the inequality follows
\[
|v(t)| \leq \beta_1(||v_{0\tau}|_\infty, t - t_0) + \beta_2(||w_{t_{0\tau}}|_\infty, t - t_0) \geq 0,
\]
(14)
where \( \beta_1 \) is a suitable function of class \( KL \) satisfying \( \beta_1(s, 0) \geq s, \forall s \geq 0 \), and \( \beta_2 \) is a suitable function of class \( \mathcal{K} \). From the inequalities (10), (14) it follows that the system (9) is stable. As far as the global attractivity is concerned, it is proved by the inequality

\[
|v(t)| \leq \beta_1(|v_0|, 0) + \beta_2(|w|-\Delta,t_0), t > t_0
\]

\[
+ \beta_2(|v_0|, 0), \quad \forall t \geq t_0 > 0
\]

For, taking \( \epsilon > 0 \), there exists a time \( t_0 \) such that, by the inequality (10), the inequality holds

\[
\beta_2(|v_{t_0}|-\Delta, t_0) < \frac{\epsilon}{2}, \quad \forall t \geq t_0
\]

There exists a time \( \bar{t} \geq t_0 \) such that

\[
\beta_1(|v_{t_0}|, 0) + \beta_2(|w|-\Delta, t_0), \bar{t} - t_0 < \epsilon
\]

Then, \( \forall t \geq \bar{t} \),

\[
|v(t)| \leq \beta_1(|v_{t_0}|, 0) + \beta_2(|w|-\Delta, t_0), \bar{t} - t_0 < \epsilon
\]

Therefore, taking into account the inequality (10), the global attractivity property of the system (9) is proved. The uniformity follows from the fact that the involved functions \( \beta_1 \) and \( \beta_2 \) are independent of the initial conditions.

**Theorem 2.** Consider the system (3) and let the hypotheses \( H1 - H6 \) be satisfied. Then, there exist a row vector \( \Gamma \) and a column vector \( K \) such that the trivial solution of the closed loop system

\[
\dot{x}(t) = f(x(t)) + g(x(t))(p_1(x_1)u(t) + p_2(x_2))
\]

\[
y(t) = h(x(t))
\]

is globally uniformly asymptotically stable.

**Proof.** Let us introduce the variables \( z(t) = \Phi(x(t)), \quad e(t) = \Phi(x(t)) - \Phi(x(t)), \quad t \geq -\Delta \). Then the couple of variables \( z(t), e(t) \) satisfy the equations

\[
\dot{z}(t) = L(z_t, z_t + e_t), \quad \dot{e}(t) = M(e_t, z_t),
\]

where \( L \) is the functional defined as, for \( v, w \in C \),

\[
L(v, w) = Av(t) + B
\]

\[
\left[ L_1^T h(\Phi^{-1}(v(0))) + L_2^T \Phi^{-1}(v(0))p_2(\Psi^{-1}(v)) \right]
\]

\[
- L_1^T h(\Phi^{-1}(w(0))) - L_2^T \Phi^{-1}(w(0)) + \Gamma w(0)
\]

and \( M \) is the functional defined as, for \( v, w \in C \),

\[
M(v, w) = -L(w, v + w) + (A + \nabla)\Phi(0) + w(0)) - KCV(0)
\]

Taking into account the hypotheses \( H1 - H6 \), by the results in Germani, Manes & Pepe, 2001, Germani & Pepe, 2005, it follows that there exist a gain vector \( K \) and positive reals \( a, b \) such that \( |e(t)| \leq a \cdot e^{-bt}|e_0| \). Taking into account that \( L(v, v) = (A + \nabla)\Phi(0) \), and that \( A + \nabla \Phi \) is a controllable pair, we can choose \( \Gamma \) such that \( A + \nabla \Phi \) is Hurwitz. Thus, the hypotheses of Lemma 1 are satisfied by the equations (18). By applying Lemma 1, the uniform global asymptotic stability of the closed-loop system (17) is proved.

**Theorem 3.** Consider the system (3) and let the hypotheses \( H1, H2, H5, H6 \) be satisfied. Let (5) be satisfied \( \forall v_1, v_2 \in R^n \) with \( h(v_1) = h(v_2) \), and let (6) be satisfied \( \forall v_1, v_2 \in C \) with \( h(v_1(\tau)) = h(v_2(\tau)), \tau \in [-\Delta, 0] \). Let \( h(x(0)) = x_i, \) for some \( i \in \{1, 2, \ldots, n\} \). Let the function \( \Phi^{-1} \) be known. Then, there exist a row vector \( \Gamma \) and a column vector \( K \) such that the trivial solution of the closed loop system

\[
\dot{x}(t) = f(x(t)) + g(x(t))(p_1(x_1)u(t) + p_2(x_2))
\]

\[
y(t) = h(x(t))
\]

is globally uniformly asymptotically stable.

**Proof.** The proof is similar to the one given for Theorem 2 and for lack of space is here omitted.

**Remark 4.** \( \Gamma \) and \( K \) must be chosen such that \( A + \nabla \Phi \) and \( A - \nabla \Phi \) are Hurwitz. Hypotheses \( H1 - H4 \) and \( H6 \) concern observability, Lipschitz and full relative degree conditions. Hypothesis \( H5 \) imposes that the term multiplying the input, in the model (3) rewritten in normal form, depends on only the measured output.

**4. LOCAL SEPARATION THEOREM**

Let us introduce in this section the following hypotheses for (3):

1. \( J1 \) there exists a positive real \( H \) such that the triple \( f, g, h \) has full relative degree in the ball \( B_H(0) \); and
2. \( J2 \) the nonlinear delay-free system described by the triple \( (f, g, h) \) is uniformly Lipschitz drift observable (ULDQ, see Dalla Mora, Germani, Manes, 2000), i.e., the function \( \Phi : B_H(0) \to \Omega \) (where \( \Omega \) is a suitable open bounded set in \( R^n \)) defined, for \( x \in B_H(0) \), by

\[
\Phi(x) = [h(x) \quad L_2 h(x) \quad \ldots \quad L_{n-1} h(x)]^T,
\]

is a diffeomorphism, and there exist positive reals \( \gamma_0 \) and \( \gamma_{f-1} \) such that, for all \( v_1, v_2 \in B_H(0), u_1, u_2 \in \Omega, \)

\[
|\Phi(v_1) - \Phi(v_2)| \leq \gamma_0|v_1 - v_2|,
\]

\[
|\Phi^{-1}(w_1) - \Phi^{-1}(w_2)| \leq \gamma_{f-1}|w_1 - w_2|.
\]
there exist a positive real \( \gamma_1 \) such that, for all \( \forall v_1, v_2 \in \Omega \),
\[
|L)^{\varphi}_{\varphi} (\Phi^{-1}(v_1)) - L)^{\varphi}_{\varphi} (\Phi^{-1}(v_2))| \leq \gamma_1|v_1 - v_2|;
\]
there exist a positive real \( \gamma_2 \) such that, for all \( \forall v_1, v_2 \in \mathcal{C} \), with \( v_1(\tau) \in \Omega, \tau \in [-\Delta, 0] \), \( i = 1, 2, \)
\[
|L)^{\varphi}_{\varphi} (\Phi^{-1}(v_1(0)))p_2(\Psi^{-1}(v_1)) - L)^{\varphi}_{\varphi} (\Phi^{-1}(v_2(0)))p_2(\Psi^{-1}(v_2))| \leq \gamma_2||v_1 - v_2||_{\infty},
\] (23)
where \( \Psi^{-1}(v_i)(\tau) = \Phi^{-1}(v_i(\tau)), \tau \in [-\Delta, 0] \), \( i = 1, 2; \)
there exist a positive real \( \gamma_3 \) such that, for all \( \forall v_1, v_2 \in \mathcal{C} \), and \( v_1(\tau) \in \Omega, \tau \in [-\Delta, 0] \), \( i = 1, 2, \)
\[
|L)^{\varphi}_{\varphi} (\Phi^{-1}(v_1(0)))p_1(\Psi^{-1}(v_1)) - L)^{\varphi}_{\varphi} (\Phi^{-1}(v_2(0)))p_1(\Psi^{-1}(v_2))| \leq \gamma_3||v_1 - v_2||_{\infty},
\] (24)
where \( \Psi^{-1}(v_i)(\tau) = \Phi^{-1}(v_i(\tau)), \tau \in [-\Delta, 0] \), \( i = 1, 2; \)
J5) there exist positive integer \( n \) and positive reals \( \Delta_k \), \( k = 1, 2, \ldots, m \), such that, for all \( \forall v \in \mathcal{C} \),
\[
p_1(v) = \overline{p}_1(v(0), v(-\Delta_1), \ldots, v(-\Delta_m)), \quad p_2(v) = \overline{p}_2(v(0), v(-\Delta_1), \ldots, v(-\Delta_m)),
\]
where \( \overline{p}_1, \overline{p}_2 : R^{n \times m+1} \rightarrow R \) are smooth functions (admit continuous partial derivatives of any order);
J6) the functional \( \phi \rightarrow p_1(\phi) \) is not equal to zero for all \( \phi \in \mathcal{C}_R \).

**Lemma 5.** Let
\[
S = \{(z_1, z_2) \in \mathcal{C} \times \mathcal{C} : z_i(\tau) \in \Omega, \tau \in [-\Delta, 0]\}
\]
Let \( L : S \rightarrow R \) be the functional defined, for \( (z_1, z_2) \in S \), as
\[
L(z_1, z_2) = L)^{\varphi}_{\varphi} (\Phi^{-1}(z_2(0))) + L)^{\varphi}_{\varphi} (\Phi^{-1}(z_2(0)))p_1(\Phi^{-1}(z_2)) \cdot
\]
\[
\frac{1}{L)^{\varphi}_{\varphi} (\Phi^{-1}(z_1(0)))p_1(\Phi^{-1}(z_1))}
\]
\[
\left[ -L)^{\varphi}_{\varphi} (\Phi^{-1}(z_1(0))) - L)^{\varphi}_{\varphi} (\Phi^{-1}(z_1(0)))p_2(\Phi^{-1}(z_1)) + W(z_1(0)) + L)^{\varphi}_{\varphi} (\Phi^{-1}(z_2(0)))p_2(\Phi^{-1}(z_2))\right]
\]
with \( W \) any given row vector in \( R^n \). Then, the functional \( L \) admits continuous Fréchet derivative at \((0, 0)\), \( D_F L \), and, moreover, there exist row vectors \( b_k \in R^n, k = 0, 1, \ldots, m \), such that, for all \( y_1, y_2 \in S \), satisfying \( y_1 + y_2 \in S \),
\[
D_F L(y_1, y_1 + y_2) - W(y_1(0)) = b_0 y_2(0) + \sum_{k=1}^{m} b_k y_2(t - \Delta_k)\]
(25)

**Proof.** The Fréchet differentiability property follows from the fact that \( f, g, h, p_1, p_2 \) (see the hypothesis J5) are smooth, \( \Phi \) is a diffeomorphism. As far as the equality (25) is concerned, by the hypothesis J5, there exist row vectors \( a_k, b_k \in R^n, k = 0, 1, \ldots, m \), such that
\[
D_F L(z_1, z_2) = a_0 z_1(0) + b_0 z_2(0) + \sum_{k=1}^{m} a_k z_1(-\Delta_k) + b_k z_2(-\Delta_k)
\] (26)

Since \( L(z_1, z_1) = W(z_1(0)) \), and \( y_1, y_1 + y_2 = (y_1, y_1) + (0, y_2) \), we get
\[
D_F L(y_1, y_1 + y_2) - W(y_1(0)) = D_F L(y_1, y_1 + y_2) - D_F L(y_1, y_1) = D_F L(0, y_2) = b_0 y_2(0) + \sum_{k=0}^{m} b_k y_2(-\Delta_k)
\] (27)

**Theorem 6.** Consider the system (3) and let the hypotheses J1 - J6 be satisfied. Then, there exist a row vector \( \Gamma \) and a column vector \( K \) in \( R^n \) such that the trivial solution of the closed loop system
\[
\dot{x}(t) = f(x(t)) + g(x(t))(p_1(x(t)) u(t) + p_2(x(t)) y(t) = h(x(t))
\]
\[
\dot{x}(t) = f(x(t)) + g(x(t))(p_1(x(t)) u(t) + p_2(x(t)) + \left[ \frac{\partial \Phi(\dot{x}(t))}{\partial \dot{x}(t)} \right]^{-1} K(y(t) - h(x(t)))
\]
(28)
\[
u(t) = -L)^{\varphi}_{\varphi} (\Phi^{-1}(\dot{x}(t))) - L)^{\varphi}_{\varphi} (\Phi^{-1}(\dot{x}(t)))p_2(x(t)) + \Gamma \Phi(\dot{x}(t))
\]

is locally asymptotically stable.

**Proof.** Let us introduce the variables \( \tilde{z}(t) = \Phi(\dot{x}(t)), \epsilon(t) = \Phi(x(t)) - \Phi(\dot{x}(t)), t \geq -\Delta \). Then the couple of variables \( \tilde{z}(t), \epsilon(t) \) satisfies the equations
\[
\dot{\epsilon}(t) = (A + B \Gamma) \epsilon(t) + KC \epsilon(t),
\]
\[
\dot{\epsilon}(t) = (A - KC) \epsilon(t) + B L(\tilde{z}, \tilde{z} + \epsilon) - \Gamma \tilde{z}(t),
\] (29)
where \( L \) is the functional defined in (24), with \( W = \Gamma \). By Lemma 5, we get that the first order approximation of (29) is given by the following equations
\[
\dot{\epsilon}(t) = (A + B \Gamma) \epsilon(t) + KC \epsilon(t),
\]
\[
\dot{\epsilon}(t) = (A - KC) \epsilon(t) + B k_\epsilon(t) + B \sum_{k=1}^{m} b_k \epsilon(t - \Delta_k)
\] (30)

Since we can choose \( \Gamma \) such that \( A + B \Gamma \) has any prescribed eigenvalues in the open left complex semi-plane, the proof of the theorem is completed, provided that the trivial solution of the second equation in (30) is exponentially stable, for some \( K \). We will use the same reasoning used in Baluomo, Pepe, Panmuni & De Gaetano, 2009, for the single-delay glucose-insulin system. We get
\[
\epsilon(t) = \exp((A - KC)t) \epsilon(0)
\]
\[
+ \int_{0}^{t} \exp((A - KC)(t - \tau)) B \cdot
\]
\[
(b_0 \epsilon(\tau) + \sum_{k=1}^{m} b_k \epsilon(\tau - \Delta_k)) d\tau\]
(31)

Since we have an n-pla of negative real distinct eigenvalues, in decreasing order, \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \). As in Ciccarella, Dalla Mora & Germani, 1993, let \( V(\lambda) \) be the diagonalizing (for \( A - KC \)) Vandermonde matrix, let \( \psi(t) = V(\lambda)e(t), t \geq -\Delta \). Then, from (31), we get
\[
|\psi(t)| \leq \exp(\lambda_1 t)|\psi(0)| + \int_{0}^{t} \exp(\lambda_1(t - \tau)) \sqrt{n}\]
\[
\left[ b_0|V^{-1}(\lambda)||\psi(\tau)| + \sum_{k=1}^{m} |b_k||V^{-1}(\lambda)||\psi(\tau - \Delta_k)\right] d\tau
\]
Let \( \xi(t) = \exp(-\lambda_1 t)\psi(t) \). From (32) we obtain, by applying the Bellman–Gronwall inequality,

\[
\xi(t) \leq \exp(\sqrt{m}|b_0||V^{-1}(\lambda)|t)\xi(0) + \int_0^t \exp(\sqrt{m}|b_0||V^{-1}(\lambda)|(t-\tau)) \cdot \\
\exp(-\lambda_1 \tau) \sum_{k=1}^m |b_k||V^{-1}(\lambda)||\psi(\tau - \Delta_k)| d\tau
\]

Turning back to the variable \( \psi(t) \), we get from (33)

\[
|\psi(t)| \leq \exp(\lambda_1 + \sqrt{m}|b_0||V^{-1}(\lambda)|)t|\psi(0)| + \int_0^t \exp((\lambda_1 + \sqrt{m}|b_0||V^{-1}(\lambda)|)(t-\tau)) \cdot \\
\sum_{k=1}^m |b_k||V^{-1}(\lambda)||\psi(\tau - \Delta_k)| d\tau
\]

Let \( M = \sup_{\tau \in [-\Delta,0]} |\psi(\tau)| \). Let \( p(t) = M\exp(\alpha t) \), \( t \geq -\Delta \), \( \alpha \) a suitable negative real which will be found next. Let us see if there exists a negative \( \alpha \) such that \( p(t) \) satisfies (34) with the equality sign, i.e.,

\[
M\exp(\alpha t) = \exp((\lambda_1 + \sqrt{m}|b_0||V^{-1}(\lambda)|)t)M \]

\[
+ \int_0^t \exp((\lambda_1 + \sqrt{m}|b_0||V^{-1}(\lambda)|)(t-\tau)) \cdot \\
\sum_{k=1}^m |b_k||V^{-1}(\lambda)|M\exp(\alpha(\tau - \Delta_k)) d\tau
\]

If (35) holds, then, from the choice of \( M \) and a standard step procedure, it follows that \( |\psi(t)| \leq M\exp(\alpha t) \), \( t \geq 0 \). Therefore, if \( \alpha \) can be chosen negative, the proof of the exponential stability of the trivial solution of the second equation in (30) is complete. Solving the integral in (35), we get that the equality (35) holds if and only if the following condition can be fulfilled:

\[
\sqrt{n} \sum_{k=1}^m |b_k||V^{-1}(\lambda)|\exp(-\alpha \Delta_k) = \\
\alpha - \lambda_1 - \sqrt{m}|b_0||V^{-1}(\lambda)|
\]

The equation (36) in the unknown variable \( \alpha \) admits a unique negative solution if and only if the value at zero of the right-hand side function of \( \alpha \) is greater than the value at zero of the left-hand side function of \( \alpha \), i.e., if and only if

\[
\lambda_1 + |V^{-1}(\lambda)|\sqrt{n} \sum_{k=0}^m |b_k| < 0
\]

It is proved in Ciccarella, Dalla Mora & Germani, 1993, that there exist an n-pla of real negative distinct eigenvalues \( \lambda \) such that the inequality (37) holds. Since, for any choice of \( \lambda \), there exist \( K \) such that the set of eigenvalues of \( A - KC \) is equal to \( \lambda \), the proof of the theorem is complete.

5. A NUMERICAL EXAMPLE

Let us consider the unstable system described by the following equations

\[
\dot{x}_1(t) = x_2(t) + \text{sech}(x_1(t)) - 1 \]
\[
\dot{x}_2(t) = \tanh(x_2(t - \Delta)) + x_1(t) + u(t),
\]
\[
y(t) = x_1(t),
\]

where, as well known, for \( a \in R \), \( \text{sech}(a) = \frac{\exp(a) - \exp(-a)}{\exp(a) + \exp(-a)} \),

\[
\tanh(a) = \frac{\exp(a) - \exp(-a)}{\exp(a) + \exp(-a)}.
\]

We get, for \( x = [x_1 \ x_2] \in R^2 \), \( w \in \mathcal{C} \),

\[
\phi(x) = \begin{bmatrix} x_1 \\ x_2 + \text{sech}(x_1) - 1 \end{bmatrix},
\]
\[
L^2_y h(x) = x_1 - \tanh(x_1)\text{sech}(x_1) (x_2 + \text{sech}(x_1) - 1),
\]
\[
L_q f_h(x) = 1, \quad G(H(w))(\tau) = 1, \quad \tau \in [-\Delta,0],
\]
\[
\frac{\partial \Phi(\tilde{x}(t))}{\partial \tilde{x}(t)} = \begin{bmatrix} 1 & 0 \\ -\text{tanh}(x_1)\text{sech}(x_1) & 1 \end{bmatrix}
\]

All the hypotheses of Theorem 3 are satisfied by the system (38). The observer based control law for the system (38) is given by (see (21))

\[
\begin{bmatrix} \dot{\tilde{x}}_1(t) \\ \dot{\tilde{x}}_2(t) \end{bmatrix} = \begin{bmatrix} \text{tanh}(\tilde{x}_2(t - \Delta)) + \text{sech}(\tilde{x}_1(t)) - 1 \\ \text{tanh}(\tilde{x}_2(t - \Delta)) + \text{sech}(\tilde{x}_1(t)) - 1 \end{bmatrix} + B \begin{bmatrix} y(t) - \tilde{x}_1(t) \\ \text{tanh}(\tilde{x}_1(t)) \end{bmatrix} + u(t)
\]

\[
+ \text{tanh}(\tilde{x}_2(t - \Delta)) + \text{sech}(\tilde{x}_1(t)) - 1 \cdot \text{tanh}(\tilde{x}_1(t)) - \text{tanh}(\text{sech}(\tilde{x}_2(t - \Delta)) - \text{sech}(y(t - \Delta)))
\]

\[
-\text{tanh}(\tilde{x}_2(t - \Delta)) + \text{sech}(\tilde{x}_1(t)) - 1
\]

\[
\begin{bmatrix} \text{tanh}(\tilde{x}_2(t - \Delta)) + \text{sech}(\tilde{x}_1(t)) - 1 \end{bmatrix}
\]

By Theorem 3, there exist a row vector \( \Gamma \) and a column vector \( K \) (chosen such that \( A + BK \) and \( A - KC \) are Hurwitz) such that the trivial solution of the closed loop system (38), (40) is globally uniformly asymptotically stable. The performed simulations validate this result. In fig. 1 the evolutions of the true and estimated state are plotted. The initial state variables are set constant and equal to \( \begin{bmatrix} 10000 \\ 5000 \end{bmatrix} \), the initial estimated state variables are set constant and equal to \( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \). The time-delay \( \Delta \) is set equal to 50. The gain vector \( \Gamma \) and the gain vector \( K \) are chosen such to have eigenvalues \(-0.1, -0.2, -1.1, -1.2, \) for \( A + BK \) and \( A - KC \), respectively. Both the true and estimated state converge to 0.
6. CONCLUSIONS

In this paper the separation principle for a class of retarded nonlinear systems is investigated. Global and local convergence results are provided in the case of global Lipschitz property and local Lipschitz property of the functionals describing the equations, respectively. Full relative degree is assumed. Future work will concern the separation theorem for retarded nonlinear systems with un-matched time-delay and control input. One such a case has been studied in Di Ciccio, Bottini, Pepe & Foscolo, where an observer-based control law is developed for a continuous stirred tank reactor with recycle time-delay, using some results in Germani, Manes & Pepe, 2001. Such application shows the possibility of extending the results of this paper to a larger class of retarded nonlinear systems, and this will be the topic of the forthcoming research. The separation theorem for nonlinear systems with delayed output will be also topic of the forthcoming research.

REFERENCES