A method for Internally Positive Realization of continuous time systems

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Abstract—The concept of Internally Positive Realization (IPR) of a system generalizes the idea of realization of a generic transfer function through combination of positive filters: an IPR is a positive state space representation, endowed with input, state and output transformations, that realizes the dynamics of a given generic (i.e. with no positivity constraint) system. Techniques for the construction of IPRs of discrete-time linear systems are available in the literature. This paper presents a method for the construction of IPRs for continuous-time systems, and provides a theoretical characterization of the stability of the resulting realization. The results presented are relevant in the field of positive continuous-time systems, like compartmental systems and for the implementation of analog filters using VLSI distributed RC interconnects.

I. INTRODUCTION

A positive system is a system whose state and output evolutions are always nonnegative provided that both the initial state and input functions are nonnegative. Positive systems are quite common in applications where input, output, and state variables represent intrinsically positive quantities such as populations, consumption of goods, densities of chemical species and so on. In the continuous-time framework, a relevant application area for positive systems is that of compartmental systems, that are effectively used in many different application fields of mathematical modeling (see e.g. [1], [10]). In [3] it is shown that the class of linear compartmental systems is equivalent to the class of linear positive stable systems.

It is known that state-space representations of discrete-time positive linear systems are characterized by nonnegativity of the system’s matrices \{A, B, C, D\}, while continuous-time positive systems are characterized by nonnegativity of \{B, C, D\}, and of the off-diagonal entries of A (see [13], [16] for details). An interesting problem for positive linear systems is the so-called positive realization problem [2], [7], [12]: given a prescribed transfer matrix, or rational function, \(H(s)\) determine (i) whether there exists a positive system of finite dimension that realizes the transfer function and (ii) the minimal dimension of a positive realization. The first problem has been solved in [2], [7], [12] for the SISO case. The solution proposed can be extended to MIMO systems. Minimality has been investigated, mainly in the discrete-time framework (see e.g. [4], [6], [15]), although a general result is still not available.

The concept of Internally Positive Realization (IPR) of discrete-time systems has been introduced in [8], [9] as an extension of the idea of realization of generic (i.e. non positive) transfer functions through the combination of positive filters [5], [15]. An IPR is a positive state space representation, endowed with input, state and output transformations, that realizes the dynamics of a given non positive system.

This paper extends to the case of continuous-time systems the construction method presented in [8], [9] for the discrete-time case. Some features of the proposed technique are the following:

- the method is very simple and straightforward, in that it does not require the numerical solution of optimization problems to compute the matrices of the positive realization;
- the method applies naturally to MIMO systems, without restriction on the multiplicity of poles or eigenvalues;
- the dimension of the IPR is always the double of the dimension of the system to be positively realized, regardless to the multiplicity and position of its poles.

When applying the proposed methodology to a generic stable system (i.e. with no positivity constraint), it may happen that the resulting IPR is unstable. Such a limitation is investigated in the paper, and conditions are given that ensure the stability of the positive realization. Due to space constraints, most proofs of Lemmas and Theorems are omitted, and will be published elsewhere.

The paper is organized as follows. In Section II the concept of Internally Positive Realization (IPR) of systems is introduced, and the IPR realization algorithm is presented. The stability analysis is worked out in Section III. In Section IV an example is presented to illustrate the method. Conclusions follow.

Notation. \(\mathbb{R}\) is the set of real numbers, \(\mathbb{R}_+\) is the set of nonnegative real numbers. \(\mathbb{C}\) is the set of complex numbers (complex plane), and \(\mathbb{C}^-\) is the open left complex plane. \(\mathbb{R}_+^n\) denotes the nonnegative orthant of \(\mathbb{R}^n\), i.e. the set of all nonnegative \(n\)-vectors, while \(\mathbb{R}_+^{n \times m}\) denotes the set of nonnegative \(n \times m\)-matrices. \(I_n \in \mathbb{R}_+^{n \times n}\) is the identity matrix. For \(M \in \mathbb{R}_+^{n \times m}\), the notation \(M \succeq 0\) means that all components of \(M\) are nonnegative, i.e. \(M \in \mathbb{R}_+^{n \times m}\), while \(M \succ 0\) means that all components are strictly positive. Similarly, \(M \preceq 0\) means that all components of \(M\) are nonpositive (i.e. \(-M \in \mathbb{R}_+^{n \times m}\), while \(M \prec 0\) means that all components are strictly negative. The symbol |\(M|\) denotes...
the matrix of absolute values of the elements of $M$, while $M^+$ and $M^-$ denote the positive and negative parts of $M$, respectively, defined as

$$M^+ = \frac{1}{2}(M + |M|), \quad M^- = \frac{1}{2}(|M| - M). \quad (1)$$

$M^+$ and $M^-$ are both nonnegative matrices and are such that $M = M^+ - M^-$. The same formalism is used for vectors. For a given square matrix $A \in \mathbb{R}^{n \times n}$, let $d(A)$ denote a diagonal matrix, whose diagonal is the diagonal of $A$. The symbols $\text{tr}(A)$ and $\sigma(A)$ denote the trace and the spectrum (i.e., the set of eigenvalues) of a square matrix $A$, respectively. $A$ is said to be stable if $\sigma(A) \subset \mathbb{C}^-$. $A$ is said to be a Metzler matrix if all its off-diagonal elements are nonnegative, i.e. if and only if $(A - d(A)) \in \mathbb{R}^{n \times n}_+$. $\mathcal{L}^p_+$ denotes the set of $p$-vector locally integrable functions defined on $\mathbb{R}$, i.e. $\mathcal{L}^p_+ = L_{1,\mathrm{loc}}(\mathbb{R}, \mathcal{R}^p)$. Similarly, $\mathcal{L}^p_{+,+} \subset \mathcal{L}^p_+$ denotes the set of nonnegative $p$-vector locally integrable functions defined on $\mathbb{R}$, i.e. $\mathcal{L}^p_{+,+} = L_{1,\mathrm{loc}}(\mathbb{R}, \mathcal{R}^p_+)$.

II. INTERNALLY POSITIVE REPRESENTATION OF SYSTEMS

A state-space representation of a causal, stationary, finite-dimensional, continuous-time linear system $S$ is made of four matrices $\{A, B, C, D\}$, and three linear spaces $(X, \mathcal{U}, Y)$, denoted the state space, the space of input functions and the space of output functions, respectively, such that for any given $t_0 \in \mathbb{R}$, for any $x(t_0) \in X$ and $u \in \mathcal{U}$, the state and output trajectories obey the equations

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t), \quad \forall t \geq t_0, \quad (2)$$

with $x(t) \in X$ and $y(t) \in Y$. If $X = \mathbb{R}^n$, $\mathcal{U} = \mathcal{L}^p_+$ and $Y = \mathbb{R}^q$, and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{q \times n}$ and $D \in \mathbb{R}^{q \times p}$, the system is said to be real. In the following, a system will be identified with the set $S = \{A, B, C, D; X, \mathcal{U}, Y\}$.

Throughout the paper, the state and output trajectories of system $S$, originated at a given initial time $t_0 \in \mathbb{R}$ by an initial state $x(t_0) \in X$ and input function $u \in \mathcal{U}$, will be symbolically written as follows

$$(x(t), y(t)) = \Phi_S(t, t_0, x(t_0), u) \quad (3)$$

According to the definition, and following the nomenclature in [11], an internally positive linear system is identified by a set $S$ where the system matrices $(A, B, C, D)$ are such that for any given $t_0 \in \mathbb{R}$, the following implication holds

$$\begin{cases} \{x(t_0) \in \mathbb{R}^n_+, \quad u \in \mathcal{L}^p_{+,+}\} \\ \{x(t) \in \mathbb{R}^n_+, \quad y(t) \in \mathbb{R}^q_+\}, \quad \forall t \geq t_0 \end{cases} \quad (4)$$

that means that for any nonnegative initial state and nonnegative input function, the whole state and output trajectories are nonnegative, so that $X = \mathbb{R}^n_+$, $\mathcal{U} = \mathcal{L}^p_{+,+}$, and $Y = \mathbb{R}^q_+$. A well known theorem [16] is the following:

**Theorem 1.** A continuous time linear system $S$ is internally positive if and only if matrices $B$, $C$ and $D$ are nonnegative and $A$ is Metzler.

In [9] the concept of Internal Positive Representation of non positive systems has been introduced. The definition has been initially given for discrete-time systems in state-space form, and generalizes the concept of realization of (non positive) filters as a combination of positive filters [5].

**Definition 1.** An Internally Positive Realization (IPR) of a real linear system $S = \{A, B, C, D; \mathbb{R}^n, \mathcal{L}^p_+, \mathcal{R}^q\}$ is an internally positive system $\tilde{S} = \{\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}; \tilde{R}^n_+, \tilde{L}^p_{+,+}, \tilde{R}^q_+\}$ together with four transformations $\{T^y_x, T^b_x, T^u_x, T^u_y\}$,

$$T^f_x : \mathbb{R}^n \mapsto \tilde{R}^n_+, \quad T^b_x : \mathbb{R}^n_+ \mapsto \mathbb{R}^n_+, \quad T^u_x : \mathcal{R}^q \mapsto \mathcal{R}^q_+, \quad T^y_y : \mathcal{R}^q \mapsto \mathcal{R}^q_+, \quad (5)$$

such that $\forall t_0 \in \mathbb{R}$, $\forall (x(t_0), u) \in \mathbb{R}^n \times \mathcal{L}^p_+$, the following implication holds:

$$\begin{cases} \{\tilde{x}(t_0) = T^f_x(x(t_0))\} \\ \tilde{u}(t) = T^u_x(u(t)), \quad \forall t \geq t_0 \end{cases} \quad (6)$$

where

$$\begin{cases} \{x(t) = T^b_x(\tilde{x}(t)), \quad y(t) = T^y_y(\tilde{y}(t))\}, \quad \forall t \geq t_0 \end{cases} \quad (7)$$

Fig. 1 illustrates what is an IPR. The concept of IPR can also be extended to I/O representations of linear systems, like transfer matrices. It is known that a proper rational transfer matrix $H(s) \in \mathbb{C}^{q \times p}$ characterizes the I/O behavior of a system in that the Laplace transforms of input and output functions starting at $t_0 = 0$, when $x(0) = 0$, are related by the equation

$$Y(s) = H(s)U(s). \quad (8)$$

**Definition 2.** Given a linear system represented by a proper rational transfer matrix $H(s) \in \mathbb{C}^{q \times p}$, an Internally Positive Realization (IPR) of $H(s)$ is a positive system $\tilde{S} = \{\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}; \tilde{R}^n_+, \tilde{L}^p_{+,+}, \tilde{R}^q_+\}$, together with two maps $\{T^u_x, T^y_y\}$

$$T^u_x : \mathcal{R}^p \mapsto \tilde{R}^p_+, \quad T^y_y : \mathcal{R}^q \mapsto \tilde{R}^q_+, \quad (9)$$
such that, for any input function \( u \in L_1^p \) which admits Laplace transform, the following implication holds
\[
\begin{align*}
\dot{x}(0) &= 0, \\
\dot{u}(t) &= T_u(u(t)), \quad \forall t \geq 0,
\end{align*}
\]
where the pair \((u(t), y(t)) \in \mathbb{R}^p \times \mathbb{R}^q\) denotes the input and output of the system represented by \( H(s) \), and \((\hat{u}(t), \hat{y}(t)) \in \mathbb{R}^p \times \mathbb{R}^q_+\) is the input/output pair of the IPR \( \bar{S} \).

Before to provide the construction method of an IPR of a linear system, it useful to give the following lemma (the proof is omitted due to space reasons):

**Lemma 2.** For a given linear system \( S = \{ A, B, C, D; \mathbb{R}^n, L_1^p, \mathbb{L}^q \} \), let the pairs of matrices \((A_1, A_2), (B_1, B_2), (C_1, C_2)\) and \((D_1, D_2)\) be such that
\[
A = A_1 - A_2, \quad B = B_1 - B_2, \quad C = C_1 - C_2, \quad D = D_1 - D_2,
\]
and consider the system \( \bar{S} = \{ \bar{A}, \bar{B}, \bar{C}, \bar{D}; \mathbb{R}^{2n}, L_1^{2p}, \mathbb{L}^{2q} \} \), where
\[
\bar{A} = \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B_1 & B_2 \\ B_2 & B_1 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C_1 & C_2 \\ C_2 & C_1 \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} D_1 & D_2 \\ D_2 & D_1 \end{bmatrix}.
\]

For any given \( t_0 \in \mathbb{R} \) and for any two pairs \((x_i(t_0), u_i) \in \mathbb{R}^n \times L_1^p, i = 1, 2\), consider the initial states and input functions of \( S \) and \( \bar{S} \) as follows
\[
x(t_0) = x_1(t_0) - x_2(t_0), \quad u = u_1 - u_2, \quad \text{for } S, \\
\bar{x}(t_0) = \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix}, \quad \bar{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \text{for } \bar{S}.
\]

Then, for any choice of the pairs \((x_i(t_0), u_i)\), \( i = 1, 2 \), the state and output trajectories of the systems \( S \) and \( \bar{S} \) are such that
\[
\begin{align*}
(x(t), y(t)) &= \Phi_S(t, t_0, x(t_0), u), \\
(\bar{x}(t), \bar{y}(t)) &= \Phi_{\bar{S}}(t, t_0, \bar{x}(t_0), \bar{u}).
\end{align*}
\]
are such that
\[
\begin{align*}
&x(t) = [I_n \quad -I_n] \bar{x}(t), \quad \forall t \geq t_0, \\
y(t) = [I_q \quad -I_q] \bar{y}(t), \quad \forall t \geq t_0.
\end{align*}
\]

Note that the state of system \( S \) in Lemma 2 can be computed as a linear transformation of the state of the larger system \( \bar{S} \). This means that system \( S \) is embedded into the dynamics of system \( \bar{S} \). The following Lemma provides a relationship between the eigenvalues of the two system matrices \( A \) and \( \bar{A} \):

**Lemma 3.** Given \( n \times n \) matrices \( A_1 \) and \( A_2 \), the matrix
\[
\bar{A} = \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix},
\]
is such that
\[
\sigma(\bar{A}) = \sigma(A_1 - A_2) \cup \sigma(A_1 + A_2).
\]

The following theorem provides a method for the construction of a positive IPR for any given linear system when its state space description \( S = \{ A, B, C, D; \mathbb{R}^n, L_1^p, \mathbb{L}^q \} \) is available.

**Theorem 4.** (IPR construction) Consider a linear system \( S = \{ A, B, C, D; \mathbb{R}^n, L_1^p, \mathbb{L}^q \} \). Let \( V \in \mathbb{R}^{n \times n} \) be any given diagonal matrix and let \( A_V = A - V \).
Then, the system \( S_V = \{ A_V, B, C, D; \mathbb{R}^{n_+}, L_1^{p_{+}}, \mathbb{L}^{q_{+}} \} \) with matrices:
\[
A_V = \begin{bmatrix} V + A^+_1 & A^-_V \\ A^-_V & V + A^+_1 \end{bmatrix}, \quad B = \begin{bmatrix} B^+ & B^- \\ B^- & B^+ \end{bmatrix}, \quad C = \begin{bmatrix} C^+ & C^- \\ C^- & C^+ \end{bmatrix}, \quad D = \begin{bmatrix} D^+ & D^- \\ D^- & D^+ \end{bmatrix},
\]

together with the four transformations:
\[
\begin{align*}
T_u : \mathbb{R}^n &\mapsto \mathbb{R}^{2n}, \quad T_u(x) = \begin{bmatrix} x^+ \\ -x^- \end{bmatrix}, \\
T_u : \mathbb{R}^{2n} &\mapsto \mathbb{R}^n, \quad T_u(x) = [I_n - I_n]x, \\
T_u : \mathbb{R}^p &\mapsto \mathbb{R}^{2p}, \quad T_u(u) = \begin{bmatrix} u^+ \\ -u^- \end{bmatrix}, \\
T_y : \mathbb{R}^{2q} &\mapsto \mathbb{R}^q, \quad T_y(y) = [I_q - I_q]y.
\end{align*}
\]
defines an Internally Positive Realization of \( S \).

Note that matrix \( A_V \) in (18) is Metzler for any choice of the diagonal matrix \( V \), and \( B, C, \) and \( D \) are all nonnegative. From Theorem 1 it follows that \( S \) is a positive system.

The proof of Theorem 4 is obtained by proving the implication (6) of Definition 1, for any given \( t_0 \in \mathbb{R} \), \( x(t_0) \in \mathbb{R}^n \) and \( u \in L_1^p \). Using the definitions (19)–(22), the implication (6) is written as
\[
\begin{align*}
&\left\{ X(t_0) = \begin{bmatrix} x^+(t_0) \\ x^-(t_0) \end{bmatrix}, \quad U(t) = \begin{bmatrix} u^+(t) \\ u^-(t) \end{bmatrix}, \quad \forall t \geq t_0 \right\} \\
&\implies \left\{ x(t) = [I_n - I_n]X(t), \\
y(t) = [I_q - I_q]Y(t), \quad \forall t \geq t_0 \right\}
\end{align*}
\]
are such that
\[
\begin{align*}
(x(t), y(t)) &= \Phi_S(t, t_0, x(t_0), u), \\
(X(t), Y(t)) &= \Phi_{S}(t, t_0, X(t_0), U),
\end{align*}
\]
are such that
\[
\begin{align*}
&x(t) = \begin{bmatrix} I_n & -I_n \end{bmatrix} \bar{x}(t), \quad \forall t \geq t_0, \\
y(t) = \begin{bmatrix} I_q & -I_q \end{bmatrix} \bar{y}(t), \quad \forall t \geq t_0.
\end{align*}
\]

The implication can be proved using the result of Lemma 2, by setting
\[
A_1 = V + A^+_1, \quad A_2 = A^-_V,
\]
and
\[
B_1 = B^+, \quad C_1 = C^+, \quad D_1 = D^+, \\
B_2 = B^-, \quad C_2 = C^-, \quad D_2 = D^-,
\]
(recalling that \( A_V = A - V \), it easily follows that \( A = A_1 - A_2 \), and moreover \( B = B_1 - B_2, C = C_1 - C_2, \) and \( D = D_1 - D_2 \)).
A block diagram describing the IPR scheme of Theorem 4 is reported in Fig. 2. Note that The IPR construction method of Theorem 4 provides a positive system whose size is twice the size of the original system.
Theorem 4 also provides a mean for the construction of IPRs for transfer matrices $H(s)$.

**Corollary 1.** Consider a linear system represented by a transfer matrix $H(s) \in \mathbb{C}^{q \times p}$. For any given (zero-state) state space realization $S = \{A, B, C, D; \mathbb{R}^n, \mathbb{L}_1^n, \mathbb{R}^q\}$ of $H(s)$, and for any given diagonal matrix $V$ of the same size of $A$, the four matrices (18) and the two transformations $T_u$ in (21) and $T_y$ in (22) are an IPR for $H(s)$.

**Remark 1.** Note that in principle any diagonal matrix $V$ can be used to define the matrix $A_V$ in (18). Two straightforward choices are: $V = 0$ and $V = d(A)$. The choice $V = 0$ is the simplest one, and is such that $A_V^+ = A^+$ and $A_V^- = A^-$. This choice, which is a good one for the case of discrete-time systems [8], [9], is not good for continuous time systems, because the resulting IPR is always unstable (note that in this case $A$ in (18) is a positive matrix, and therefore, thanks to the Perron-Frobenius Theorem, has a dominant eigenvalue which is positive and real). The choice $V = d(A)$ is such that $A_{d(A)} = A - d(A)$ has all zeros on the main diagonal and $\text{tr}(A_{d(A)}) = 2\text{tr}(A)$. Although this choice does not guarantee that if $A$ is stable then $A_{d(A)}$ is stable, there are good chances because if $A$ is stable, then $\text{tr}(A) < 0$ so that also $\text{tr}(A_{d(A)}) < 0$.

**Definition 3.** Given a system $S = \{A, B, C, D; \mathbb{R}^n, \mathbb{L}_1^n, \mathbb{R}^q\}$, the normal form IPR is the positive system $S_{d(A)} = \{A_{d(A)}, B, C, D; \mathbb{R}^n_+, \mathbb{L}_1^{n_+}, \mathbb{R}^q_+\}$, where the system matrices are those in (18) where $V = d(A)$, and the four transformations $T_u^+, T_y^+, T_u, T_y$ are the ones in (19)–(22).

Note that the $A_{d(A)}$ is as follows

$$A_{d(A)} = \begin{bmatrix} d(A) + (A - d(A))^+ & (A - d(A))^-
(A - d(A))^+ & d(A) + (A - d(A))^+ \end{bmatrix}$$

**III. Stability Analysis**

This section investigates under what conditions the construction of Theorem 4 provides a stable IPR. Since matrix $A_V$ is Metzler by construction, it is useful to recall some stability conditions for Metzler matrices that will be used in this section (see [16] for further details).

A necessary stability condition is given below:

**Proposition 5.** If $A$ is Metzler and stable, then necessarily $d(A) \prec 0$.

From this proposition, a necessary condition for a stable IPR of the type given in Theorem 4 is that $d(A_V) \prec 0$, which is equivalent to $V \prec 0$ (note that $d(A_V) = \text{diag}(V, V)$). Similarly, a necessary condition for a normal form IPR to be stable is that $d(A) \prec 0$.

It follows that the normal form IPR can not be used if $a_{ii} \geq 0$ for some $i \in [1, n]$. In such cases two alternatives can be explored:

- find a nonsingular matrix $Q$ such that $d(QAQ^{-1}) \prec 0$
- construct an IPR using a diagonal matrix $V$ such that $V + d(A) \prec 0$

Note that the above alternatives only ensure the necessary stability condition.

For what follows we need to state the following Lemma, which is a straightforward consequence of the Gershgorin’s Theorem (see e.g. [14]), and provides a sufficient stability condition very easy to check.

**Lemma 6.** If a matrix $A \in \mathbb{R}^{n \times n}$ is diagonal dominant and such that $d(A) \prec 0$, i.e.

$$a_{ii} + \sum_{j \neq i} |a_{ij}| < 0, \quad i = 1, \ldots, n, \quad (28)$$

then $\sigma(A) \subset \mathbb{C}^-$.

A first result is the following:

**Theorem 7.** Given a system $S = \{A, B, C, D; \mathbb{R}^n, \mathbb{L}_1^n, \mathbb{R}^q\}$, and given a diagonal matrix $V \in \mathbb{R}^{n \times n}$, the construction of Theorem 4 provides a stable IPR if and only if

$$\sigma(A) \cup \sigma(V + |A - V|) \subset \mathbb{C}^- \quad (29)$$

**Proof.** The IPR of Theorem 4 is stable if and only if $\sigma(A_V) \subset \mathbb{C}^-$, where $A_V$ is the matrix defined in (18). Note that $A_V$ has the same block structure of matrix (16) of Lemma 3, and therefore $A_{d(A_V)} = \sigma(A) \cup \sigma(V + A_V^+ + A_V^-)$. Being $V + A_V^+ + A_V^- = V + (A - V)^+ + (A - V)^- = V + |A - V|$, the inclusion (29) follows.

**Remark 2.** It must be stressed that, given a square matrix $A$, a diagonal matrix $V$ and a nonsingular matrix $Q$, in general

$$\sigma(V + |A - V|) \neq \sigma(V + |QAQ^{-1} - V|)$

It follows that changes of coordinates play an important role in the stability of IPRs of systems.

Note that for any given diagonal $V \in \mathbb{R}^{n \times n}$, the matrix $V + |A - V|$ is Metzler. It follows that the stability conditions of Metzler matrices can be used to study the stability of $V + |A - V|$, and therefore of $A_V$. As a consequence of Theorem 7, a normal form IPR of a system $S$ is stable if and only if both $A$ and $d(A) + |A - d(A)|$ are stable matrices. Let

$$M_A = d(A) + |A - d(A)| \quad (31)$$
Denoting with $a_{ij}$ the components of matrix $A$, the structure of the Metzler matrix $M_A$ is as follows

$$M_A = \begin{bmatrix}
  a_{11} & |\ldots| & a_{1n} \\
  |a_{21}| & a_{22} & |\ldots| & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  |a_{n1}| & |\ldots| & a_{nn}
\end{bmatrix} \quad (32)$$

**Theorem 8.** Consider a system $S$ whose system matrix $A$ is diagonal dominant and moreover $d(A) \prec 0$. Then, $S$ admits a stable normal form IPR.

**Proof.** Note that if $A$ is dominant diagonal then also $M_A$ is dominant diagonal, as it is evident by comparing condition (28) with formula (32). Moreover $d(M_A) = d(A) \prec 0$. From Lemma 6 it follows that both $A$ and $M_A$ are stable, i.e. $\sigma(A) \cup \sigma(M_A) \subset \mathbb{C}^-$. The stability of the normal form IPR follows by recalling that $\sigma(A(d_A)) = \sigma(A) \cup \sigma(M_A)$. ■

In those cases where $A$ is stable and $M_A$ is not, it may be useful to look for a nonsingular matrix $Q$ such that the similar state matrix $\tilde{A} = QAQ^{-1}$ is such that $M_{\tilde{A}} = d(\tilde{A}) + |\tilde{A} - d(\tilde{A})|$ is stable.

To this purpose, it is useful to consider systems represented in a form in which the matrix $A$ has the Real Jordan block-diagonal structure. This representation allows to give a necessary and sufficient condition for the stability of the normal form IPR.

Consider the following notation. Let $\lambda_i$ be an eigenvalue associated to a matrix $M$, and $\tilde{\eta}_i = \{\eta_{i1}, \ldots, \eta_{im_i}\}$ be the multi-index associated to $\lambda_i$, with $m_i$ the number of chains of generalized eigenvectors associated to $\lambda_i$, and $\eta_{ih}$ is the length of the $h$-th chain, $1 \leq h \leq m_i$, related to $\lambda_i$. The modulus $|\tilde{\eta}_i|$ of the multi-index $\tilde{\eta}_i$ defined as $|\tilde{\eta}_i| = \sum_{h=1}^{m_i} \eta_{ih}$, gives back the algebraic multiplicity of $\lambda_i$. Moreover, we denote with $J_{\eta_{i1}}(\lambda_i)$ the Jordan block of order $\eta_{i1}$ associated to $\lambda_i$, and with:

$$J_{\tilde{\eta}_i}(\lambda_i) = \text{diag}\{J_{\eta_{i1}}(\lambda_i), \ldots, J_{\eta_{im_i}}(\lambda_i)\}, \quad (33)$$

the complex Jordan block-diagonal matrix of order $|\tilde{\eta}_i|$ associated to $\lambda_i$. In the following it will be useful to write $J_{\eta_{i1}}(\lambda_i)$ and $J_{\tilde{\eta}_i}(\lambda_i)$ as

$$J_{\eta_{i1}}(\lambda_i) = \lambda_i I_{\eta_{i1}} + J_{\eta_{i1}}(0),$$

$$J_{\tilde{\eta}_i}(\lambda_i) = \lambda_i I_{\tilde{\eta}_i} + J_{\tilde{\eta}_i}(0). \quad (34)$$

In order to introduce the notations needed for the Real Jordan Form, consider a square matrix $M \in \mathbb{R}^{n \times n}$, with even $n$, with only one pair of complex-conjugate eigenvalues, $\lambda_1 = \alpha + j\beta$ and $\lambda_2 = \lambda_1^* = \alpha - j\beta$, both of them associated to the multi-index $\tilde{\eta} = \{\eta_1, \ldots, \eta_n\}$, with $n = 2|\tilde{\eta}|$. Then, $M$ is similar to the block-diagonal (complex) Jordan form:

$$J = \text{diag}\{J_1(\lambda_1), J_2(\lambda_2)\},$$

with $J_k(\lambda_k)$ the block-diagonal Jordan matrix associated to $\lambda_k$, $k = 1, 2$. Then, there exists a coordinate transformation matrix $T \in \mathbb{C}^{n \times n}$ such that the Jordan block-diagonal matrix $M$ is transformed into the following block matrix (Real Jordan Form)

$$J_\eta(\alpha, \beta) = TJT^{-1} = \begin{bmatrix}
  J_\eta(\alpha) & \beta I_{|\eta|} \\
  -\beta I_{|\eta|} & J_\eta(\alpha)
\end{bmatrix}. \quad (36)$$

A useful expression for $J_\eta(\alpha, \beta)$ is the following

$$J_\eta(\alpha, \beta) = \begin{bmatrix}
  \alpha I_{|\eta|} + J_\eta(0) & \beta I_{|\eta|} \\
  -\beta I_{|\eta|} & \alpha I_{|\eta|} + J_\eta(0)
\end{bmatrix}. \quad (37)$$

Now consider a linear system $S = \{A, B, C, D; \mathbb{R}^n, \mathbb{R}^d, \mathbb{R}^q\}$, let $\nu$ be the real number eigenvalues of $A$ and $\mu$ the number of pairs of complex eigenvalues of $A$, where $A$ is in the following block Real Jordan Form:

$$A = \text{diag}\{J_{\eta_1}, \ldots, J_{\eta_K}, \mathcal{J}_{\sigma_1}, \ldots, \mathcal{J}_{\sigma_\mu}\}, \quad (38)$$

where $J_{\eta_i}$, $i = 1, \ldots, \nu$, are the Real Jordan block-diagonal matrices associated to the real eigenvalues $\lambda_i$ with the multi-indexes $\tilde{\eta}_i = \{\eta_{i1}, \ldots, \eta_{im_i}\}$, and $\mathcal{J}_{\sigma_k}$ are the block-matrices of order $2|\sigma_k|$ of the type of (36), associated to complex pairs of eigenvalues $\alpha_k \pm j\beta_k$ with the multi-index $\sigma_k = \{\sigma_{k1}, \ldots, \sigma_{km_k}\}$, $k = 1, \ldots, \mu$.

**Definition 4.** A $J$-class IPR is the normal form IPR provided by the construction of Theorem 4, when the original representation of the system is in the Real Jordan form (38).

**Theorem 9.** Consider a linear system $S = \{A, B, C, D; \mathbb{R}^n, \mathbb{R}^d, \mathbb{R}^q\}$ where $A$ is the Real Jordan Form (38). Then, the normal form IPR provided by eq.’s (18) is stable if and only if all eigenvalues $\lambda_i$ of $A$ are such that $\Re(\lambda_i) + |\Im(\lambda_i)| < 0$.

Instead of giving a complete proof of Theorem 9 (too long to be reported here), let us consider what happens to a simple stable system of dimension 2, with a pair of conjugate eigenvalues $\alpha \pm j\beta$, with $\alpha < 0$ and $\beta > 0$. The system matrix in the Real Jordan form is

$$A = \begin{bmatrix}
  \alpha & \beta \\
  -\beta & \alpha
\end{bmatrix}. \quad (39)$$

The system matrix $A(d_A)$ of the normal form IPR, eq. (27), which is Metzler because $\beta > 0$, is

$$A(d_A) = \begin{bmatrix}
  \alpha & 0 & 0 \\
  0 & \beta & 0 \\
  0 & 0 & \alpha
\end{bmatrix}. \quad (40)$$

According to Theorem 7, $A(d_A)$ is stable if and only if $\sigma(A) \cup \sigma(d(A)) + |A - d(A)| \subset \mathbb{C}^-$. In our example $\sigma(A) = \{\alpha \pm j\beta\}$ and

$$\sigma(d(A)) + |A - d(A)| = \sigma \left( \begin{bmatrix}
  \alpha & \beta \\
  \beta & \alpha
\end{bmatrix} \right) = \{\alpha \pm \beta\}. \quad (41)$$

It follows that the $J$-class IPR (40) is stable if and only if $\alpha \pm \beta < 0$ (i.e. $\Re(\lambda) + |\Im(\lambda)| < 0$, as stated in Theorem 9).

In Figure 3 the stability region of a $J$-class IPR is represented.
IV. AN EXAMPLE

The IPR of a second-order Bessel filter is reported as an example. It is known that the Bessel filter can be characterized by the order and the cut-off frequency $\omega_c$ by the frequency $\omega_0$ up to which the filter's group delay is approximately constant (it is $\omega_0/\omega_c = 1.274$). The second order filter has the following structure

$$H(s) = \frac{1.622\omega_0^2}{s^2 + 2.260\omega_c s + 1.622\omega_0^2} = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2},$$

(42)

with $\zeta = 0.866$. The poles are $p_{1,2} = (-0.866 \pm j0.5)\omega_0$. $H(s)$ admits the following non-positive real-Jordan state space representation

$$A = \begin{bmatrix} -\zeta & \sqrt{1-\zeta} \\ \sqrt{1-\zeta} & -\zeta \end{bmatrix} \omega_0 = \begin{bmatrix} -0.866 & 0.5 \\ -0.5 & -0.866 \end{bmatrix} \omega_0,$$

(43)

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \sqrt{1-\zeta} \end{bmatrix} \omega_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \omega_0.$$

(44)

The normal form IPR is

$$A_{d(A)} = \begin{bmatrix} -0.866 & 0.5 & 0 & 0 \\ 0 & -0.866 & 0.5 & 0 \\ 0 & 0 & -0.866 & 0.5 \\ 0.5 & 0 & 0 & -0.866 \end{bmatrix} \omega_0,$$

(45)

and

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \omega_0^2.$$

(46)

The eigenvalues of $A_{d(A)}$ are $\lambda_{1,2} = (-0.866 \pm j0.5)\omega_0$, $\lambda_3 = -0.366\omega_0$, $\lambda_3 = -1.366\omega_0$, and therefore the IPR is stable.

Figure 4 reports the output $y(t)$ of the filter when $\omega_0 = 1$ and the input is $u(t) = \cos(2t)$. Also the two nonnegative outputs $y_1(t)$ and $y_2(t)$ of the IPR when the input is $\tilde{u}(t) = [u^+(t) \ u^-(t)]^T$. The two outputs are such that $y(t) = y_1(t) - y_2(t)$.

V. CONCLUSIONS

A methodology for the construction of Internal Positive Realizations of generic (i.e., non positive) state-space or transfer function system representations has been presented. The method is very simple and straightforward, and takes inspiration from the one presented in [8] and [9] for discrete-time systems. The dimension of the IPR is the double of the one of the original system. Conditions are derived for the stability of the resulting IPR.

REFERENCES