Abstract: We consider the problem of controlling a linear system when the state is available with a non-negligible delay (delayed-state-feedback control). In such conditions, the resulting closed-loop system is always a time-delay-system. The solution proposed in this paper consists in partially assigning the spectrum of the closed-loop system while ensuring the exponential zero-state stability with a prescribed decay rate. The proposed approach is appealing for its simplicity and can be applied in both cases of constant and time-varying delays. Sufficient stability conditions, that in some cases are also necessary, are provided. Such conditions allow to easily compute a lower bound, and in some cases the exact value, of the maximum delay that ensures the prescribed closed-loop behavior (Partial Spectrum Assignment with prescribed exponential stability).

1. INTRODUCTION

Since the birth of automatic control the problem of stabilizing systems with delays in the state and/or input variables has been investigated by a multitude of researchers (see e.g. (Chen [1995], Gu and Kharitonov [2003], Mazenc et al. [2008], Richard [2003] and the references therein). For the general case of linear systems with delays in the state equations several control approaches have been developed, including Finite Spectrum Assignment (Jankovic and Kolmanovsky [2009]), Continuous Pole Placement (Michels and Niculescu [2007]), optimization based pole assignment [Michiels et al. [2010], Infinite Spectrum Assignment (Stepan and Insperger [2006]), Lyapunov-Krasovskii functional based methods (He et al. [2007], Pepe et al. [2008]), Matrix Lambert Function (Shinozaki and Mori [2006], Yi et al. [2008, 2010]).

In this paper we are concerned with systems whose dynamics does not include state delays, and only input or output delays are present. In particular, the case of possibly time-varying delays is considered. In the presence of output delay the problem of designing state predictors has been largely investigated and solved, for some classes of nonlinear systems, by means of single or cascade observers (Ahmed-Ali et al. [2012], Cacace et al. [2010], Germani et al. [2002]). Predictor-based approaches for controlling time-delay systems have long been used for systems with delayed input, and have received renewed interest in recent years, even if most approaches consider deal only with the constant delay case (Artstein [1982], Krstic [2010]). Predictor-based state feedback and output feedback controllers are designed in Bekiaris-Liberis and Krstic [2011] for nonlinear systems with time-varying input delays. A predictor for nonlinear systems with delayed input is proposed in Karafyllis [2011]. Traditional predictor-based controllers use infinite-dimensional static feedback laws and this causes difficulties in their practical implementation (see for example Richard [2003], Van Assche et at. [1999]). For this reason it is interesting to develop predictor feedback methods that only involve finite dimensional static state feedback by safely ignoring the input dependent term from the computation of the control gain. This approach, named “truncated prediction feedback” has been used in Zhou et al. [2012] to control linear systems with time-varying input delays that are not exponentially stable. For the application of the proposed technique, the state is assumed entirely known, and also the future values of the delay function are assumed to be completely known.

In this paper we use a similar method for linear systems with scalar input, when the state is available with a possibly time-varying delay. The structure of the feedback gain that we propose here has a simple structure and nice properties that allow to derive sufficient, and in some cases necessary, conditions of stability. Such conditions allow to compute a lower bound (or the exact value, in some cases) of the maximum delay that guarantees the exponential stability of the zero-state with a prescribed exponential decay rate. The time-varying delay does not need to be continuous.

This paper is organized as follows. In Section 2 the control problem is formally stated, and preliminary definitions are given. In Section 3 the proposed approach is illustrated.
2. PROBLEM STATEMENT

For a given linear system of the type
\[ \dot{x}(t) = Ax(t) + Bu(t), \quad t \in \mathbb{R}, \]
where \( x(t) \in \mathbb{R}^n \) is the system state and \( u(t) \in \mathbb{R} \) is the control input, \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^n \), we consider the problem of constructing a stabilizing state-feedback control law when the state is available with possibly time-varying delay \( \delta \). Under the assumption that the pair \( (A, B) \) is controllable, the following class of delayed state feedback laws is considered
\[ u(t) = -K_x(x(t - \delta t)), \quad K \in \mathbb{R}^{1 \times n} \] (2)
We assume that the control law (2) starts operating at \( t = 0 \), and that the delay is upper bounded, i.e. \( \delta_t \in [0, \delta] \). Thus, the closed-loop system is a time-delay system described by the following equations
\[ \dot{x}(t) = Ax(t) - BK_x(x(t - \delta_t)), \quad t \geq 0, \]
\[ x(t) = \phi(t), \quad t \in [-\delta, 0], \]
where \( \phi \in \mathcal{C}_\delta^\alpha \) is the so-called preshape function.

The following scenarios are investigated in this paper:
- constant and known delay: \( \delta_t = \delta \);
- time-varying, known and bounded delay \( \delta_t \in [0, \delta] \).

We are interested in finding control laws that ensure exponential state decay with a prescribed rate. The following definition is useful for our purposes.

**Definition 1.** (\( \alpha \)-exponential stability) Given a real number \( \alpha > 0 \), the feedback system (3) is said to be \( \alpha \)-exponentially stable if the following condition holds
\[ \forall \phi \in \mathcal{C}_\delta^\alpha, \exists \mu_\phi > 0 : \| x(t) \| \leq e^{-\alpha t} \mu_\phi, \forall t \geq 0. \] (4)
For a given \( \alpha > 0 \) the system (3) is said to be strictly \( \alpha \)-exponentially stable if it is \( \alpha \)-exponentially stable and, moreover, condition (4) cannot be satisfied for any \( \alpha > \alpha_0 \).

**Remark 1.** A sufficient condition of \( \alpha \)-exponential stability is the following:
\[ \forall \phi \in \mathcal{C}_\alpha^\alpha, \exists x_\phi > 0 : \| x(t) \| \leq e^{\alpha t} x_\phi, \forall t \geq 0, \]
\[ \text{and } \lim_{t \to \infty} e^{\alpha t} x(t) = 0. \] (5)

**Remark 2.** It is well known that if a LTI time-delay system is asymptotically stable, then it is strictly \( \alpha_0 \)-exponentially stable for some \( \alpha_0 > 0 \) (Thm. 5.2 in Hale [1977]), and it is \( \alpha \)-exponentially stable for all \( \alpha \in (0, \alpha_0) \).

In this paper we are interested in conditions that guarantee that the feedback system (3) is \( \alpha \)-exponentially stable, for a given \( \alpha > 0 \).

2.1 The spectrum of a LTD system

In this paper we will use the following definition of the spectrum of a LTDs of the type (3), in the general case of time-varying delay \( \delta_t \) and feedback gain \( K_t \):

**Definition 2.** The spectrum of the LTD (3) is the set of all the constant complex numbers \( \lambda \) such that the DDE (3) admits a solution of the type \( x(t) = e^{\lambda t} v, \) for \( t \in [-\delta, \infty) \), for some \( v \in \mathbb{C}^n \). The spectrum of the system (3) is denoted by \( \sigma_{\delta_t}(A - BK_t) \).

Note that \( \sigma_{\delta_t}(A - BK_t) \) is required to be independent of time. When \( K_t \) and/or \( \delta_t \) are time-varying the spectrum \( \sigma_{\delta_t}(A - BK_t) \) may be an empty set, but in the case of LTI delay systems the spectrum is always a countable set. The spectrum of a LTI system of the type (1), when \( u(t) = 0 \), coincides with the spectrum of the matrix \( A \), denoted \( \sigma(A) \) (the set of eigenvalues, i.e. the roots of the characteristic polynomial \( p_A(s) = sI_n - A \)). Thus, the spectrum of the system (3) when \( K_t = K \) and \( \delta_t = 0 \) (LTI system) is \( \sigma(A - BK) \). If \( \delta_t = \delta > 0 \), (3) is a LTI system, and the following characteristic function can be defined
\[ \nu_{\delta}(s) = |sI_n - A + BK|e^{-\delta t}. \] (6)
In this case the spectrum \( \sigma_{\delta}(A - BK) \in \mathcal{S} \subset \mathbb{C} \) is made of the countably infinite roots of the characteristic equation \( \nu_{\delta}(s) = 0 \) (i.e., \( \lambda \in \sigma_{\delta}(A - BK) \iff \nu_{\delta}(\lambda) = 0 \)). Let \( \sigma_{\delta} \) denote the maximum of the real parts of the roots of \( \nu_{\delta}(s) \)
\[ \sigma_{\delta} = \max \{ \Re(\lambda), \lambda \in \sigma_{\delta}(A - BK) \}. \] (7)
It is known that there always exists \( \sigma_{\delta} < \infty \) (Lemma 4.1 in Hale [1977]). Thanks to Thm. 6.2 in Hale [1977], if \( \sigma_{\delta} < 0 \) then the system (3) is \( \alpha \)-exponentially stable for any \( \alpha \in [0, -\sigma_{\delta}] \). Moreover, we have the following result.

**Proposition 1.** Consider the system (3), with \( \delta_t = \delta \) and \( K_t = K \). For a given real number \( \alpha > 0 \) the system is \( \alpha \)-exponentially stable if
\[ \nu_{\delta}(s) \neq 0, \quad \forall s \in \mathcal{C}_{2-\alpha}, \] (8)
and only if
\[ \nu_{\delta}(s) \neq 0, \quad \forall s \in \mathcal{C}_{\alpha+\alpha}. \] (9)

**Proof.** If (8) holds true, than \( \Re(\lambda) < -\alpha \), \( \forall \lambda \in \sigma_{\delta}(A - BK) \), and this is sufficient to guarantee the \( \alpha \)-exponential stability (Thm. 6.2 in Hale [1977]). The necessity of (9) is proved by recalling that if \( \nu_{\delta}(\lambda) = 0 \), then there exists \( v \in \mathbb{C}^n \) such that \( x(t) = e^{\lambda t} v \) is a solution of (3), and the norm of such solution is \( \| x(t) \| = e^{\Re(\lambda)t}\|v\| \). It easily follows that if such \( \lambda \) satisfies \( \Re(\lambda) > -\alpha \), i.e. the condition (9) is violated, then the condition (4) of \( \alpha \)-exponential stability cannot be satisfied. Thus, condition (9) is necessary for the \( \alpha \)-exponential stability. \( \blacksquare \)

2.2 The PSA problem with \( \alpha \)-exponential stability

Let \( L = \{ \lambda_1, \ldots, \lambda_n \} \subset \mathcal{S} \subset \mathbb{C} \) denote a set of \( n \) real or complex numbers and let
\[ \bar{\lambda} = \max_{\lambda_t \in L} \{ \Re(\lambda_t) \}. \] (10)
The Partial Spectrum Assignment (PSA) problem with \( \alpha \)-exponential stability consists in finding a feedback law of the type (2) such that the system (3) is \( \alpha \)-exponentially
stable and the set $L$ is included in its spectrum. Of course, a necessary condition to solve the problem is that $\lambda \leq -\alpha$.

In the case of delayless system, i.e. $\delta_t = 0$, the assumption that the pair $(A,B)$ is controllable guarantees that for any choice of $L \in S_C$, there exists $\overline{K} \in \mathbb{R}^n$ such that $\sigma(A - B\overline{K}) = L$.

**Definition 3.** A control gain $\overline{K} \in \mathbb{R}^{1 \times n}$ for the pair $(A,B)$ is said to be trace-left-shifting if it is such that
\[
tr(A - B\overline{K}) < tr(A).
\]

(11)

Recalling that the trace of a matrix is the sum of its eigenvalues, we can define the (real) barycenters of the spectra $\sigma(A)$ and $\sigma(A - B\overline{K})$ as $b_A = tr(A)/n$ and $b_{A - B\overline{K}} = tr(A - B\overline{K})/n$, respectively. Obviously, a trace-left-shifting gain $\overline{K}$ is such that $b_{A - B\overline{K}} < b_A$.

**Proposition 2.** A gain $\overline{K} \in \mathbb{R}^{1 \times n}$ is trace-left-shifting for a pair $(A,B)$ with $B \in \mathbb{R}^{n \times 1}$ if and only if
\[
\overline{K}B > 0.
\]

(12)

**Proof.** From Definition 3, $\overline{K}$ is trace-left-shifting if and only if $tr(A - B\overline{K}) - tr(A) < 0$. Since we have
\[
tr(A - B\overline{K}) - tr(A) = -tr(B\overline{K}) = -\overline{K}B,
\]

(13)

the thesis easily follows. $\square$

In the following, this problem of PSA with exponential stability is first investigated for the case of constant and known delay (Section 3), and then for the case of time-varying known delay (Section 4).

### 3. CONSTANT DELAY, CONSTANT GAIN

The main results of this section are summarized in some theorems and corollaries, listed below. In order to focus the attention on the statements, all the proofs are postponed.

**Theorem 3.** Consider the system (1) and a control law of the type (2), where the delay $\delta_t$ is constant and known, i.e. $\delta_t = \delta$. Let $\overline{K} \in \mathbb{R}^n$ be such to assign the spectrum $L$ to the matrix $A - B\overline{K}$, i.e. $L = \sigma(A - B\overline{K})$. Then, the constant gain matrix

\[
K = \overline{K}e^{\overline{A}t}, \quad \text{where} \quad \overline{A} = A - B\overline{K},
\]

(14)

is such to include the set $L = \sigma(\overline{A})$ in the spectrum of the closed loop system

\[
\dot{x}(t) = Ax(t) - BKx(t - \delta), \quad t \geq 0,
\]

\[
x(t) = \phi(t), \quad t \in [-\delta,0], \quad \phi \in C^n_{\delta},
\]

(15)

i.e., $L \subset \sigma(A - B\overline{K})$.

**Remark 3.** Note that the structure of the gain in (14) is seemingly similar to the one in Lin & Fang [2007]. However, there is a main difference: in (14) the gain $K$ depends on the exponential of the closed-loop stable matrix $\overline{A}$, while in Lin & Fang [2007] the gain depends on the exponential of the open-loop, possibly unstable, matrix $A$.

In the proof of Theorem 3 and in the statement of the subsequent results we need to define the following scalar function
\[
q(t) = \overline{K}e^{\overline{A}t}B, \quad t \geq 0,
\]

(16)

and the Laplace transform of its restriction in $[0,\delta]$:
\[
Q_3(s) = \int_0^\delta q(t)e^{-st}dt = \int_0^\delta \overline{K}e^{\overline{A}t}Be^{-st}dt,
\]

(17)

which is well defined for all $s \in \mathbb{C}$.

**Theorem 4.** Under the same conditions and assumptions of Theorem 3, consider a given $\alpha > 0$ such that $\lambda < -\alpha$. Then, with the feedback gain (14), the closed loop system (15) is $\alpha$-exponentially stable if
\[
Q_3(s) \neq 1, \quad \forall s \in \mathbb{C}_{\geq -\alpha}
\]

(18)

and only if
\[
Q_3(s) \neq 1, \quad \forall s \in \mathbb{C}_{> -\alpha}
\]

(19)

The following corollaries of Theorem 4 provide easy-to-check conditions of $\alpha$-exponential stability.

**Corollary 5.** For a given $\alpha > 0$ such that $\lambda < -\alpha$, under the same conditions and assumptions of Theorem 3, if
\[
\int_0^\delta |\overline{K}e^{\overline{A}t}B|e^{\alpha t}dt < 1,
\]

(20)

then the closed loop system (15) is $\alpha$-exponentially stable.

**Corollary 6.** Under the same conditions and assumptions of Theorem 3, assume that $\overline{K}e^{\overline{A}t}B > 0 \forall t \in [0,\delta]$. Then, for a given $\alpha > 0$, such that $\lambda < -\alpha$, the system (15) is $\alpha$-exponentially stable if
\[
\int_0^\delta \overline{K}e^{\overline{A}t}Be^{\alpha t}dt < 1,
\]

(21)

and only if
\[
\int_0^\delta \overline{K}e^{\overline{A}t}Be^{\alpha t}dt \leq 1
\]

(22)

**Remark 4.** Note that, by definition (16) we have $q(0) = \overline{K}B$, and therefore $q(t)$ is strictly positive in a neighborhood of $t = 0$ if and only if the gain $\overline{K}$ is trace-left-shifting for the given pair $(A,B)$ (see Proposition 2).

**Proof of Theorem 3.** Let $\tilde{p}(s) = |sI_n - \overline{A}|$ denote the characteristic polynomial of $\overline{A} = A - B\overline{K}$. By assumption $\sigma(\overline{A}) = L$. Taking into account the definitions of $\overline{K}$ and of $\overline{A}$ in (14), the characteristic function $\tilde{\nu}_3(s)$ of the system (3) is
\[
\tilde{\nu}_3(s) = |sI_n - A + B\overline{K}e^{\overline{A}s}e^{-s\delta}|.
\]

(23)

Then we have
\[
\tilde{\nu}_3(s) = |sI_n - A + BK(e^{\overline{A}s}e^{-s\delta} - I_n)|
\]
\[
= |sI_n - \overline{A} + BK(e^{(\overline{A} - sI_n)s} - I_n)|
\]
\[
= |sI_n - \overline{A} - sI_n + (sI_n - \overline{A})^{-1}B(K(e^{(\overline{A} - sI_n)s} - I_n))|.
\]

(24)

Finally, using the property $|I_n + aB^T| = 1 + b^T a$, we get
\[
\tilde{\nu}_3(s) = \tilde{p}(s) \left(1 + \overline{K}(e^{(\overline{A} - sI_n)s} - I_n)(sI_n - \overline{A})^{-1}B\right).
\]

(25)

It is not difficult to prove that, for $s \notin L$,
\[
\overline{K}(e^{(\overline{A} - sI_n)s} - I_n)(sI_n - \overline{A})^{-1}B
\]
\[
= -\int_0^\delta \overline{K}e^{(\overline{A} - sI_n)s}Bdt = -Q_3(s),
\]

(26)

where $Q_3(s)$ is defined in (17). Note that the identity (26) holds true even for $s \to \lambda_i \in L$. Thus, we can rewrite the quasipolynomial $\tilde{\nu}_3(s)$ in (25) as follows
\[
\tilde{\nu}_3(s) = \tilde{p}(s)(1 - Q_3(s)).
\]

(27)
Since $Q_{d}(s)$ exists and is finite for any $s \in \mathbb{C}$, equation (27) proves that all the roots of $\bar{p}(s)$ are also roots of $\bar{v}_{0}(s)$, i.e., the inclusion $L \subset \sigma_{d}(A - BK)$ holds true. □

**Proof of Theorem 4.** The identity (27) shows that the spectrum of the closed loop system (15) is the union of the assigned spectrum $L$ with the set of all $s \in \mathbb{C}$ that solve $Q_{d}(s) = 1$. Thus, together with the assumption $\lambda < -\alpha$, the condition (18) ensures that $\bar{v}_{0}(s) \neq 0 \forall s \in \mathbb{C}_{\geq -\alpha}$, which is sufficient for the $\alpha$-exponential stability (see Proposition 1). Similarly, (19) is equivalent to $\bar{v}_{0}(s) \neq 0 \forall s \in \mathbb{C} > -\alpha$, which is necessary for the $\alpha$-exponential stability. □

**Proof of Corollary 5.** If condition (20) is satisfied then $|Q_{d}(s)| < 1$ for all $s \in \mathbb{C}_{\geq -\alpha}$, and therefore condition (18) is satisfied. This ensures the $\alpha$-exponential stability, thanks to Theorem 4. □

**Proof of Corollary 6.** Note first that, by the definition (17) of $Q_{d}(s)$, the conditions (21) and (22) can be written as

\[
Q_{d}(-\alpha) < 1, \quad \text{and} \quad Q_{d}(-\alpha) \leq 1,
\]

respectively. When $q(t) > 0$ in $[0, \delta]$, then the condition (21) is equivalent to the condition (20), and therefore, thanks to Corollary 5, it is sufficient for the $\alpha$-exponential stability of (15). The necessity of condition (22) is readily proven by showing that if it is not satisfied, i.e. if $Q_{d}(-\alpha) > 1$, then there exists a real $\beta > -\alpha$ such that $Q_{d}(\beta) = 1$ (simply because $Q_{d}(\rho)$ is monotonically decreasing in $[-\alpha, \infty)$ and $\lim_{\rho \to \infty} Q_{d}(\rho) = 0$). Thus, the necessary condition of Theorem 4 is not satisfied, and the system is not $\alpha$-exponentially stable. □

**Remark 5.** Thanks to Corollary 5 we can easily compute a lower bound $\delta_{n}$ of the maximum delay that guarantees the $\alpha$-exponential stability of the closed loop system (15) for a given $\alpha > 0$: $\delta_{n}$ is the value such that the integral of $|[\mathbf{K}\bar{e}^{\mathbf{A}t}B]e^{\alpha t}|$ in $[0, \delta_{n}]$ is equal to 1. The Corollary 5 guarantees the $\alpha$-exponential stability of the system (15) for any $\delta \in (0, \delta_{n})$. If, in addition, $q(t) > 0$ in $[0, \delta_{n}]$, the Corollary 6 states that the bound is strict, i.e., if $\delta > \delta_{n}$ the system is not $\alpha$-exponentially stable.

### 4. TIME-VARYING KNOWN DELAY

In the case of time varying delay $\delta_{t}$, a suitable choice of the state feedback gain $K_{t}$ provides stability results very close to those given in Section 3 for the case of constant delay.

**Theorem 7.** Consider the system (1) and a control law of the type (2), with a known and bounded time-varying delay $\delta_{t} \in [0, \delta]$. Let $\mathbf{K} \in \mathbb{R}^{n}$ be such to assign the spectrum $L$ to the matrix $A - BK$. Then, the delay-dependent gain matrix

\[
K_{t}^{T} = \mathbf{K} \bar{e}^{\mathbf{A}t},
\]

is such to include the set $L$ in the spectrum of the closed loop system

\[
\dot{x}(t) = Ax(t) - BK_{t}x(t - \delta_{t}), \quad t \geq 0,
\]

\[
x(t) = \phi(t), \quad t \in [-\delta, 0], \quad \phi \in \mathbb{C}^{n},
\]

i.e. $L \subset \sigma_{d}(A - BK_{t})$.

**Proof.** The proof consists in showing that for any $\lambda_{i} \in \sigma(\mathbf{A})$ there exists $v_{i} \in \mathbb{C}^{n}$ such that the function $x(t) = e^{\lambda_{i}t}v_{i}$, for $t \in [-\delta, \infty)$ satisfies the DDE (30) for $t \in [0, \infty)$, so that we can conclude that $\lambda_{i} \in \sigma_{d}(A - BK_{t}^{T})$ (see Definition 2). To this aim, let $v_{i}$ be the eigenvector of $\mathbf{A}$ associated to $\lambda_{i}$ and replace $\dot{x}(t)$, $x(t)$ and $x(t - \delta_{t})$ in (30) with $\lambda_{i}e^{\lambda_{i}t}v_{i}$, $e^{\lambda_{i}t}v_{i}$ and $e^{\lambda_{i}(t-\delta_{t})}v_{i}$, respectively. The identity (30) is readily verified by taking into account that

\[
\mathbf{A} = A - BK, \quad \bar{A}v_{i} = \lambda_{i}v_{i}, \quad e^{\lambda_{i}t}v_{i} = e^{\lambda_{i}\delta_{t}}v_{i}. \tag{31}
\]

**Remark 6.** Note that the structure of the gain in (29) is seemingly similar to the one in Zhou et al. [2012]. The main difference is that in (29) the constant gain $K$ multiplies a time-varying exponential of the closed-loop stable matrix $\bar{A}$, while in Zhou et al. [2012] the constant gain multiplies the exponential of the open-loop, possibly unstable, matrix $A$.

**Theorem 8.** Under the same conditions and assumptions of Theorem 7, for a given $\alpha > 0$ such that $\lambda < -\alpha$, the system (30) is $\alpha$-exponentially stable for any $\delta_{t} \in [0, \delta]$ if

\[
\int_{0}^{\delta} \mathbf{K} e^{\mathbf{A}t}B e^{\alpha t}dt < 1. \tag{32}
\]

If, in addition, $q(t) > 0 \forall t \in [0, \delta]$, the system (30) is $\alpha$-exponentially stable for any $\delta_{t} \in [0, \delta]$ only if

\[
\int_{0}^{\delta} \mathbf{K} e^{\mathbf{A}t}B e^{\alpha t}dt \leq 1. \tag{33}
\]

(i.e. $Q_{d}(-\alpha) < 1$).

In order to provide a sketch of the proof of Theorem 8 too long to be reported here in full), we need to define the $\alpha$-perturbed state $x^{\alpha}(t)$ as follows

\[
x^{\alpha}(t) = e^{\alpha t}x(t), \quad t \geq -\delta. \tag{34}
\]

Straightforward computations show that if $\delta_{t}(t)$ obeys the system equations (30), with $K_{t}^{T} = \mathbf{K} e^{\mathbf{A}t}$, then $x^{\alpha}(t)$ obeys the following

\[
x^{\alpha}(t) = A_{\alpha}x^{\alpha}(t) - BK_{t}e^{\lambda_{i}\delta_{t}}x^{\alpha}(t - \delta_{t}), \quad t \geq 0,
\]

\[
x^{\alpha}(t) = e^{\alpha t}\phi(t), \quad t \in [-\delta, 0], \quad \phi \in \mathbb{C}^{n},
\]

where

\[
A_{\alpha} = A + \alpha I_{n}, \quad \text{and} \quad \bar{A}_{\alpha} = \bar{A} + \alpha I_{n}. \tag{35}
\]

Adding and subtracting the term $BK_{t}x^{\alpha}(t)$ in (35), and noting that $\bar{A}_{\alpha} = A_{\alpha} - BK$, we get

\[
\dot{x}^{\alpha}(t) = \bar{A}_{\alpha}x^{\alpha}(t) + BK_{t}\bar{e}^{\mathbf{A}(t-\tau)}dt, \tag{36}
\]

\[
\dot{x}^{\alpha}(t) = x^{\alpha}(t) - \bar{e}^{\mathbf{A}\delta_{t}}x^{\alpha}(t - \delta_{t}), \quad t \geq 0.
\]

Let $t_{0} \geq 0$ be such that $t_{0} - \delta_{t_{0}} = 0$. Then we have

\[
\dot{x}^{\alpha}(t) = \int_{t_{0}}^{t} e^{\lambda_{i}(t-\tau)}BK_{t}x^{\alpha}(\tau)dt, \quad t \geq t_{0}. \tag{37}
\]

Defining the scalar variable

\[
c^{\alpha}(t) = \mathbf{K} e^{\lambda_{i}t}x^{\alpha}(t), \quad t \geq 0, \tag{39}
\]

the differential equation (37) can be rewritten as

\[
\dot{c}^{\alpha}(t) = \bar{A}_{\alpha}c^{\alpha}(t) + Bc^{\alpha}(t), \quad t \geq 0, \tag{40}
\]

where the matrix $\bar{A}_{\alpha} = \bar{A} - \alpha I_{n}$ is Hurwitz stable under the assumptions that $L \subset \sigma(\bar{A})$ and $\lambda < -\alpha$. Moreover, by virtue of (38) we have the following

\[
c^{\alpha}(t) = \int_{t_{0}}^{t} \mathbf{K} e^{\lambda_{i}(t-\tau)}Bc^{\alpha}(\tau)dt, \quad t \geq t_{0}. \tag{41}
\]
Defining \( q^a(t) = e^{\alpha t}q(t) \), where \( q(t) \) is defined in (16), it easily follows that
\[
q^a(t) = R e^{\frac{\alpha}{t} B},
\]
so that we can write
\[
e^a(t) = \int_{t-\delta}^{t} q^a(t-\tau) e^{\alpha \tau} d\tau, \quad t \geq t_0.
\] (43)

**Sketch of the Proof of Theorem 8.** The proof of the sufficiency of the condition (32) for the \( \alpha \)-exponential stability is achieved by showing that under such a condition \( e^a(t) \) is bounded and exponentially goes to zero, for any choice of the preshape function \( \phi(t) \) (this implies that \( \|x(t)\| \) is bounded by an exponential function \( e^{-\alpha t} \)). Thanks to equation (40), the boundedness and the exponential decay of \( e^a(t) \) are implied by the boundedness and the exponential decay of \( e^a(t) \). Thus, we only need to prove that if (32) is satisfied, then there exists \( \beta > 0 \) such that for any bounded initial value of \( e^a(t) \) in the interval \([0, t_0]\), there exists \( \kappa > 0 \) such that \( |e^a(t)| \leq e^{-\beta t} \kappa \). This result can be obtained by defining a sequence of intervals \( I_k \), for \( k = 0, 1, 2, \ldots \), defined as \( I_k = [k \delta, (k+1)\delta) \), and a sequence of nonnegative numbers \( \bar{c}_k \), \( k = 0, 1, \ldots \), as
\[
\bar{c}_k = \sup_{t \in I_k} |e^a(t)|.
\] (44)

After some computations we can show that for \( k \geq 0 \)
\[
ev_{k+1} \leq \bar{q}_\alpha \bar{c}_k,
\]
where \( \bar{q}_\alpha = \int_0^\delta R e^{\frac{\alpha}{t} B} e^{\alpha t} dt \). (45)

From this
\[
\bar{c}_k \leq \bar{q}_\alpha \bar{c}_0, \quad k \geq 0.
\] (46)

Noting that under the assumption (32) we have \( \bar{q}_\alpha \in (0, 1) \), further computations provide the following exponential bound
\[
|e^a(t)| \leq e^{-\beta t} e^{\beta \delta} \bar{c}_0, \quad t \geq t_0,
\] (47)
where \( \beta = -\log(\bar{q}_\alpha)/\delta > 0 \). As discussed before, this bound is sufficient for the \( \alpha \)-exponential stability of the system (30).

The necessity of (33) for the \( \alpha \)-exponential stability of the system (30) for any \( \delta \in [0, \delta] \), directly follows from the necessity of the same condition when the delay \( \delta \) is constant and equal to \( \delta \), proved in the Corollary 6. \( \square \)

**4.1 Some comments on the results of Theorem 8**

The sufficient condition (32) for the \( \alpha \)-exponential stability of the time varying system (30) is the same of condition (20) of Corollary 5 for LTI systems. In those cases where \( q(t) > 0 \) in \([0, \delta] \) the sufficient and the necessary conditions of Theorem 8 coincide with those of the Corollary 6.

Note that the sufficient condition (32) can be easily exploited to find a lower bound to the largest delay \( \delta \) for which the control law (2), with \( K_0 \) given in (29), guarantee a prescribed \( \alpha \)-exponential stability.

Conditions of asymptotic stability of the closed loop system (30) are easily derived from the conditions of \( \alpha \)-exponential stability given in Theorem 8 by simply letting \( \alpha = 0 \). Thus, if \( L \subset \mathbb{C}^- \), a sufficient condition of stability is
\[
\int_0^\delta R e^{\frac{\alpha}{t} B} dt < 1.
\] (48)

### 6. CONCLUSIONS

A method for the exponential stabilization of a linear system when the state is known with a possibly time-
varying delay is presented in this paper. The method consists in designing a delayed-state-feedback that assigns a part of the spectrum of the closed-loop delay system while ensuring the exponential stability with a prescribed decay rate (Partial Spectrum Assignment (PSA) with $\alpha$-exponential stability). Sufficient, and in some cases necessary, conditions are provided to ensure that the delay-state-feedback achieves the prescribed system behavior. Such conditions allow to easily compute a lower bound, or in some cases the exact value, of the largest state-delay that preserves the prescribed performance (PSA with $\alpha$-exponential stability).

REFERENCES


