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Input–output linearization with delay cancellation for nonlinear delay systems: the problem of the internal stability

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SUMMARY

This paper investigates the issue of the internal stability of nonlinear delay systems controlled with a feedback law that performs exact input–output, linearization and delay cancellation. In previous works the authors showed that, unlike with the case of systems without state delays, when the relative degree is equal to the number of state variables and the output is forced to be identically zero, delay systems still possess a non-trivial internal state dynamics. Not only, in the same conditions delay systems are also characterized by a non-trivial input dynamics. Obviously, both internal state and input dynamics should give bounded trajectories, otherwise the exact input–output linearization and delay cancellation technique cannot be applied. This paper studies the relationships between the internal state and input dynamics of a controlled nonlinear delay system. An interesting result is that a suitable stability assumption on the internal state dynamics ensures that, when the output is asymptotically driven to zero, both the state and control variables asymptotically decay to zero. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: nonlinear systems; delay systems; output regulation; infinite dimensional systems; delay cancellation

1. INTRODUCTION

Some control problems for nonlinear systems can be solved through a preliminary compensation of nonlinearities, so that virtually all control techniques developed for linear systems can be applied to the *linearized* system (see e.g. [1]). In the same way, when dealing with delay systems, a feasible approach to control problems is the preliminary compensation of all delays, followed by the application of control techniques developed for systems without delays. Nonlinear systems with state and input delays can be the object of various kinds of compensation of nonlinearities and delays: the linearization can be exact or approximated, the delay can be partially compensated through prediction or, in some cases, can be exactly cancelled. The approach of preliminary delay compensation through prediction is pursued in [2–4]. After delay compensation, classical tools of differential geometry can be applied for the

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analysis of nonlinear systems and for control synthesis. A different approach is followed in References [5–10], where preliminary delay compensation is avoided and suitable extensions of differential geometry have been developed for dealing with delay systems. The papers [7, 8] were concerned mainly with the problem of disturbance decoupling, while papers [5,6,9,10] developed the extension to delay systems of the classical technique of input–output (I/O) linearization [1,16]. In these works exact I/O linearization was obtained together with exact delay cancellation by means of a state feedback. After exact I/O linearization and delay cancellation the problem of driving the output exponentially to zero becomes an easy task. However, output stabilization is not sufficient to achieve state stabilization, because the control law that achieves an identically zero output may create an unobservable dynamics, called *zero-dynamics*. As in the case of systems without delays, the stability of the zero-dynamics is a necessary condition for the practical use of the control law that linearizes the I/O map and removes the delay. It is well known that systems without state delay do have non-trivial zero-dynamics if and only if the relative degree is smaller than the number of state variables. This situation is investigated in Reference [10] for the case of delay systems, and it is shown that, under suitable assumptions, the local asymptotic stability of the zero-dynamics implies the local asymptotic stability of the origin of the controlled delay systems. In Reference [6] it has been shown that, different from the case of systems without state delays, nonlinear delay systems with relative degree equal to the number of state variables (full relative degree) may have, in general, a non-trivial zero-dynamics. For such systems a sufficient stability criterion of the state zero-dynamics is discussed in Reference [6]. The existence of the state internal dynamics in the case of full relative degree is due to intrinsic infinite dimensionality of delay systems. A necessary and sufficient condition which characterizes delay systems that do not admit state internal dynamics is reported in Reference [11].

This paper points out that in addition to the internal state dynamics, the output stabilizing control law through I/O linearization and delay cancellation induces also an internal dynamics of the control variable (input dynamics). This means that the true zero-dynamics for nonlinear delay systems is composed of both the internal state and input dynamics, and both the state and input trajectories must be uniformly bounded, otherwise the technique of exact input–output linearization with delay cancellation cannot be applied. The remarkable result obtained in this work is that the stability of the input dynamics is not an additional assumption, but is implied by a suitable stability property of the internal state dynamics. Such property will be proved with reference to the class of nonlinear systems with delay only in the state. Similar results are proved in [12] with reference to the class of linear systems with delays both in the state and in the input; in this case an additional stability condition is also needed.

2. PROBLEM STATEMENT AND PRELIMINARIES

This paper considers delay systems described by the following equations:

$$\dot{x}(t) = f(x(t), x(t-\Delta)) + g(x(t), x(t-\Delta))u(t), \quad t \geqslant 0 \tag{1}$$

$$y(t) = h(x(t)) \tag{2}$$

where $\Delta > 0$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$ and $y(t) \in \mathbb{R}$, the vector functions f and g are C^∞ with respect to both arguments, and h is a C^∞ scalar function. The model description is completed by the knowledge of the function $x(\tau)$, $\tau \in [-\Delta, 0]$, in a suitable function space, which represents the initial state in the classical infinite dimensional description of delay systems. It is assumed that system (1) and (2) is such that

$$f(0,0) = 0, \quad g(0,0) \neq 0, \quad h(0) = 0 \tag{3}$$

These positions imply that the state $x(\tau) = 0$, $\tau \in [-\Delta, 0]$, is an equilibrium point. In the following, some notations are introduced in order to simplify the writing of mathematical expressions. Throughout the paper the symbol $0_{a\times b}$ denotes the zero matrix in $\mathbb{R}^{a\times b}$, while I_a denotes the identity matrix in $\mathbb{R}^{a\times a}$. The symbols $A^B_{p\times q}$, $B^B_{p\times q}$, $C^B_{p\times q}$, will denote the block- Brunowsky triplet defined by

$$A^B_{p,q} = \begin{bmatrix} 0_{q\times q} & I_q & \cdots & 0_{q\times q} \\ 0_{q\times q} & 0_{q\times q} & \cdots & 0_{q\times q} \\ \vdots & \vdots & \cdots & \vdots \\ 0_{q\times q} & 0_{q\times q} & \cdots & I_q \\ 0_{q\times q} & 0_{q\times q} & \cdots & 0_{q\times q} \end{bmatrix} \in \mathbb{R}^{pq\times pq}, \quad B^B_{p,q} = \begin{bmatrix} 0_{q\times q} \\ 0_{q\times q} \\ \vdots \\ I_q \end{bmatrix} \in \mathbb{R}^{pq\times q}$$

$$C^B_{p,q} = [I_q \quad 0_{q\times q} \quad \cdots \quad 0_{q\times q}] \in \mathbb{R}^{q\times pq} \tag{4}$$

The symbol $\mathscr{X}_{i,j}$, with i, j integer numbers, will denote a vector in $\mathbb{R}^{(|i-j|+1)n}$, composed of subvectors $\chi_k \in \mathbb{R}^n$ as follows:

$$\text{if } i \leqslant j: \quad \mathscr{X}_{i,j} = \begin{bmatrix} \chi_i \\ \chi_{i+1} \\ \vdots \\ \chi_j \end{bmatrix}; \quad \text{if } j < i: \quad \mathscr{X}_{i,j} = \begin{bmatrix} \chi_i \\ \chi_{i-1} \\ \vdots \\ \chi_j \end{bmatrix} \tag{5}$$

In the same way, the symbol $\mathscr{V}_{i,j}$, will denote a vector in $\mathbb{R}^{(|i-j|+1)}$, composed of scalar v_k as follows:

$$\text{if } i \leqslant j: \quad \mathscr{V}_{ij} = \begin{bmatrix} v_i \\ v_{i-1} \\ \vdots \\ v_j \end{bmatrix}; \quad \text{if } j < i: \quad \mathscr{V}_{i,j} = \begin{bmatrix} v_i \\ v_{i-1} \\ \vdots \\ v_j \end{bmatrix} \tag{6}$$

As in the case of nonlinear systems without delay, an important concept for the development of an I/O linearizing feedback control law is the concept of *relative degree*, the integer that indicates how many time derivatives of the output should be made in order to have a direct relationship with the input. The concept of relative degree for nonlinear delay systems was introduced independently in [5,7,10,13], with little differences. Indeed, many extensions of this concept can be made for nonlinear delay systems. Based on References [5,7,10,13], three possible definitions of relative degree are reported here.

Definition 2.1 (Type-I relative degree)
The nonlinear delay system (1), (2) is said to have type-I relative degree r in an open set $\Omega_r \in \mathbb{R}^{n(r+1)}$ if, defining

$$F(\mathscr{X}_{0,r}) = \begin{bmatrix} f(\chi_0,\chi_1) \\ f(\chi_1,\chi_2) \\ \vdots \\ f(\chi_{r-1},\chi_r) \\ 0_{n\times 1} \end{bmatrix}, \quad G(\mathscr{X}_{0,r}) = \begin{bmatrix} \mathrm{diag}_{i=0}^{r-1}\ \{g(\chi_i,\chi_{i+1})\} \\ 0_{n\times r} \end{bmatrix}$$

$$H(\mathscr{X}_{0,r}) = h(\chi_0) \tag{7}$$

the following conditions are satisfied $\forall \mathscr{X}_{0,r} \in \Omega_r$

$$\begin{aligned} &L_G L_F^k H(\mathscr{X}_{0,r}) = 0, \quad k = 0,1,\ldots,r-2 \\ &L_G L_F^{r-1} H(\mathscr{X}_{0,r}) \neq 0 \end{aligned} \tag{8}$$

where

$$\begin{aligned} L_F^0 H(\mathscr{X}_{0,r}) &= H(\mathscr{X}_{0,r}) \\ L_F^k H(\mathscr{X}_{0,r}) &= \left(\frac{\partial}{\partial \mathscr{X}_{0,r}} L_F^{k-1} H\right) F(\mathscr{X}_{0,r}), \quad k \leqslant r \\ L_G L_F^k H(\mathscr{X}_0, r) &= \left(\frac{\partial}{\partial \mathscr{X}_{0,r}} L_F^{k} H\right) G(\mathscr{X}_{0,r}), \quad k \leqslant r-1 \end{aligned} \tag{9}$$

If $\Omega_r = \mathbb{R}^{n(r+1)}$, the system is said to have uniform type-I relative degree r.

Remark 2.2
Note that the Lie derivative $L_F^k H(\mathscr{X}_{0,r})$ is well defined only for $k \leqslant r$ and is a function of $\mathscr{X}_{0,k}$, a sub-vector of $\mathscr{X}_{0,r}$. Similarly, the term $L_G L_F^k H(\mathscr{X}_{0,r})$ is well defined only for $k \leqslant r-1$ and is a function of $\mathscr{X}_{0,k+1}$.

Definition 2.3 (Type-II relative degree)
The nonlinear delay system (1), (2) is said to have type-II relative degree r in an open set $\Omega_r \in \mathbb{R}^{n(r+1)}$ if, defining the vector functions $F(\mathscr{X}_{0,r})$, $G(\mathscr{X}_{0,r})$, $H(\mathscr{X}_{0,r})$ as in (7), the following conditions are verified $\forall \mathscr{X}_{0,r} \in \Omega_r$

$$\begin{aligned} &L_G L_F^k H(\mathscr{X}_{0,r}) = 0, \quad k = 0,1,\ldots,r-2 \\ &\gamma_0(\mathscr{X}_{0,r}) \neq 0 \end{aligned} \tag{10}$$

where

$$\gamma_0(\mathscr{X}_{0,r}) = \left(\frac{\partial}{\partial \chi_0} L_F^{r-1} H(\mathscr{X}_{0,r})\right) g(\chi_0,\chi_1) = L_G L_F^{r-1} H(\mathscr{X}_{0,r}) \begin{bmatrix} 1 \\ 0_{(r-1)\times 1} \end{bmatrix} \tag{11}$$

If $\Omega_r = \mathbb{R}^{n(r+1)}$, the system is said to have uniform type-II relative degree r.

Definition 2.4 (type-III relative degree)
The nonlinear delay system (1), (2) is said to have type-III relative degree r in an open set $\Omega_r \in \mathbb{R}^{n(r+1)}$ if, defining the vector functions $F(\mathcal{X}_{0,r})$, $G(\mathcal{X}_{0,r})$, $H(\mathcal{X}_{0,r})$ as in (7), the following conditions are satisfied $\forall \mathcal{X}_{0,r} \in \Omega_r$:

$$L_G L_F^k H(\mathcal{X}_{0,r}) = 0, \quad k = 0, 1, \dots, r-2 \quad \gamma_0(\mathcal{X}_{0,r}) \neq 0 \tag{12}$$

$$\gamma_i(\mathcal{X}_{0,r}) = 0, \quad i = 1, \dots, r-1$$

where

$$\gamma_i(\mathcal{X}_{0,r}) = \left(\frac{\partial}{\partial \chi_i} L_F^{r-1} H(\mathcal{X}_{0,r})\right) g(\chi_i, \chi_{i+1}) \tag{13}$$

If $\Omega_r = \mathbb{R}^{n(r+1)}$, the system is said to have uniform type-III relative degree r.

Remark 2.5
The computation of the (types I–III) relative degree of a nonlinear delay system is made by constructing the vector functions $F(\mathcal{X}_{0,r})$, $G(\mathcal{X}_{0,r})$, $H(\mathcal{X}_{0,r})$ defined in (7), for increasing values of the integer r, starting from $r = 1$, and checking for each r if condition (8) is satisfied. If an r is found such that (8) holds, then the system has type-I relative degree r. If, moreover, condition (10) holds, then the system has type-II relative degree r. If, in addition, condition (12) is satisfied, then the system has type-III relative degree r.

In order to study the role of the three types of relative degrees in the input–output relationship, it is useful to define a *stack operator* as follows. For a given function $q(t) \in \mathbb{R}^m$, the symbol $q_{i\Delta}(t)$, with i a non-negative integer, will denote its translation by $-i\Delta$, i.e. $q_{i\Delta}(t) = q \times (t - i\Delta)$. $x(t)$ being defined for $t \geqslant -\Delta$, the delayed function $x_{i\Delta}(t)$ is defined for $t \geqslant (i-1)\Delta$, while $u_{i\Delta}(t)$ is defined for $t \geqslant i\Delta$, $u(t)$ being defined for $t \geqslant 0$.

Definition 2.6
Consider a function $q(t) \in \mathbb{R}^m$, defined for $t \in [t_1, t_2] \subseteq \mathbb{R}$. The symbol $\mathrm{Stack}_{i,j}(q)$, with i, j such that $0 \leqslant |j - i| \leqslant (t_2 - t_1)/\Delta$, denotes a vector function, defined for $t \in [t_1 + j\Delta, t_2 + i\Delta]$, *if* $i \leqslant j$, and for $t \in [t_1 + i\Delta, t_2 + j\Delta]$ if $j > i$, defined as follows:

$$\text{if } i \leqslant j\text{:} \quad \mathrm{Stack}_{i,j}(q)(t) = \begin{bmatrix} q_{i\Delta}(t) \\ q_{(i+1)\Delta}(t) \\ \vdots \\ q_{j\Delta}(t) \end{bmatrix}; \quad \text{if } j < i: \mathrm{Stack}_{i,j}(q)(t) = \begin{bmatrix} q_{i\Delta}(t) \\ q_{(i-1)\Delta}(t) \\ \vdots \\ q_{j\Delta}(t) \end{bmatrix}. \tag{14}$$

Using the stack operator, the following vector functions can be defined:

$$X_{i,j}(t) = \mathrm{Stack}_{i,j}(x)(t), \quad U_{i,j}(t) = \mathrm{Stack}_{i,j}(u)(t) \tag{15}$$

that collect the values of the system variable x and of the input u at different time instants.

Lemma 2.7
Assume that system (1), (2) has relative degree equal to r (of type-I, type-II or type-III) in an open set $\Omega_r \subseteq \mathbb{R}^{n \cdot (r+1)}$. Then for $t \geqslant (r-1)\Delta$ the time derivatives of the output until order r can

be written in Ω_r as

$$y^{(k)}(t) = L_F^k H(X_{0,k}(t)), \quad k = 0, 1, \ldots, r-1 \tag{16}$$

$$\begin{aligned} y^{(r)}(t) &= L_F^r H(X_{0,r}(t)) + L_G L_F^{r-1} H(X_{0,r}(t)) U_{0,r-1}(t) \\ &= L_F^r H(X_{0,r}(t)) + \gamma_0(X_{0,r}(t)) u(t) + \Gamma(X_{0,r}(t)) U_{1,r-1}(t) \\ &= L_F^r(X_{0,r}(t)) + \sum_{i=0}^{r-1} \gamma_i(X_{0,r}(t)) u_{i\Delta}(t) \end{aligned} \tag{17}$$

where $\gamma_i(\mathscr{X}_{0,r})$ has been defined in (13), and

$$\Gamma(\mathscr{X}_{0,r}) = L_G L_F^{r-1} H(\mathscr{X}_{0,r}) \begin{bmatrix} 0_{1\times(r-1)} \\ I_{(r-1)} \end{bmatrix} = [\gamma_1(\mathscr{X}_{0,r}) \quad \cdots \quad \gamma_{r-1}(\mathscr{X}_{0,r})] \tag{18}$$

Proof
The proof is readily obtained by direct calculations, taking into account the definitions of relative degree.

As said before, the concept of relative degree for nonlinear delay systems was introduced independently in References [5,7,10,13]. In particular, the definition given in Reference [7] corresponds to type-I relative degree, while the one given in Reference [5] corresponds to type-II relative degree and the one in References [10,13] is of a type-III relative degree. A system with type-I relative degree r is such that the output derivative of order r at time t is an affine function of the inputs at time $t - i\Delta$, for *some* of integers $i \in [0, r-1]$. A system with type-II relative degree r is such that the rth output derivative at time t is an affine function of the input at time t and possibly of the inputs at times $t - i\Delta$, for *some* of the integers $i \in [1, r-1]$. A system with type-III relative degree r has the rth output derivative at time t that is an affine function of the input at time t and *is not* function of the input at times $t - i\Delta$, *for all* integers $i \in [1, r-1]$. In Reference [14] an *observation delay relative degree* was defined and that is a type-I relative degree. For a nonlinear delay system the assumption to have a type-III relative degree is rather strong. In this paper, following the approach of [5], we will consider systems with type-II relative degree.

It is well known that for nonlinear systems without delay when the relative degree is equal to the dimension of the state space n, the existence of a state feedback that achieves exact linearization of the input–output map implies the existence of the solution of the problem of exact linearization of the system through a static state feedback and a nonlinear change of co-ordinates (see Reference [1]). The new co-ordinates are the output derivatives up to order $n-1$. The stabilization of the system is obtained assigning the eigenvalues to the system in the linear form. If the relative degree r is strictly less than n, only a subsystem of dimension r can be linearized and stabilized through linearization and stabilization of the input–output map. r eigenvalues can be assigned in this case. The linearizing feedback induces an unobservable dynamics, the so-called zero-dynamics that is unaffected by the assigned eigenvalues. The control via exact linearization can be pursued only if the zero-dynamics is stable. On the other hand, systems with full relative degrees do not have zero-dynamics, and therefore the exact linearization approach can be always pursued. Unfortunately, *this is not the case for nonlinear* *delay systems*. Also, when the relative degree is equal to the dimension n of the system variable x, exact linearization of the input–output map does not imply exact linearization of the system.

The control law that linearizes the input–output map with delay-cancellation is described in the following proposition, whose simple proof can be found in References [5, 6, 10].

Proposition 2.8

Assume that the nonlinear delay system (1), (2) has type-II relative degree n in an open set Ω_n. Moreover, assume that the initial state $x_0 \in C([-\Delta, 0]; \mathbb{R}^n)$ and the initial choice of the input u in the time interval $[0, (n-1)\Delta)$ are such to guarantee the existence and uniqueness of a continuous solution $x(t)$ on $[0, (n-1)\Delta)$ and that $X_{0,n}((n-1)\Delta) \in \Omega_n$. Then, defining a new input function $v(t)$, the feedback control law

$$u(t) = \frac{v(t) - L_F^n H(X_{0,n}(t)) - \Gamma(X_{0,n}(t))U_{1,n-1}(t)}{\gamma_0(X_{0,n}(t))}, \quad t \geqslant (n-1)\Delta \tag{19}$$

is such that the input–output map becomes

$$y^{(n)}(t) = v(t), \quad t \geqslant (n-1)\Delta \tag{20}$$

provided that, with the chosen $v(t)$, $X_{0,n}(t)$ is unique continuous and remains in Ω_n.

The output derivatives up to order $n-1$ can be written by defining the following map $\Phi : \mathbb{R}^{n^2} \to \mathbb{R}^n$:

$$z = \begin{bmatrix} L_F^0 H(\chi_0) \\ L_F^1 H(\mathcal{X}_{0,1}) \\ \vdots \\ L_F^{n-1} H(\mathcal{X}_{0,n-1}) \end{bmatrix} = \Phi(\mathcal{X}_{0,n-1}) \tag{21}$$

After substitution of $\mathcal{X}_{0,k}$ with $X_{0,k}(t)$ in (21), that is giving to χ_i the value $x(t - i\Delta)$, the map Φ gives a vector $z(t)$ that collects the output derivatives up to order $n-1$:

$$z(t) = \begin{bmatrix} y(t) \\ \vdots \\ y^{(n-1)}(t) \end{bmatrix} = \Phi(X_{0,n-1}(t)), \quad t \geqslant (n-1)\Delta \tag{22}$$

After feedback (19) the input–output dynamics can be put in the form

$$\begin{aligned} \dot{z}(t) &= A_{n,1}^B z(t) + B_{n,1}^B v(t) \\ y(t) &= C_{n,1}^B z(t), \quad t \geqslant (n-1)\Delta \end{aligned} \tag{23}$$

where $(A_{n,1}^B, B_{n,1}^B, C_{n,1}^B)$ is a Brunowsky triplet, as defined in (4). Applying a linear feedback law of the form

$$\begin{aligned} v(t) &= -k^{\mathrm{T}} z(t) \\ &= -k^{\mathrm{T}} \Phi(X_{0,n-1}(t)) \end{aligned} \tag{24}$$

the output dynamics is governed by the autonomous linear system

$$\dot{z}(t) = (A^B_{n,1} - B^B_{n,1}k^{\mathrm{T}})z(t)$$

$$y(t) = C^B_{n,1}z(t), \quad t \geqslant (n-1)\Delta \tag{25}$$

If k assigns all the eigenvalues of matrix $A^B_{n,1} - B^B_{n,1}k^{\mathrm{T}}$ in the open left-half complex plane (i.e. k is Hurwitz) the output is exponentially stabilized, i.e. there exist positive γ, β such that

$$\|z(t)\| \leqslant \gamma e^{-\beta(t-(n-1)\Delta)} \|z((n-1)\Delta)\|, \quad t \geqslant (n-1)\Delta \tag{26}$$

The feedback law that achieves exponential output stabilization, after linearization and delay cancellation of the input–output map (as long as $X_{0,n}(t) \in \Omega_n$), is obtained replacing the variable v in (19) with expression (24), obtaining

$$u(t) = \frac{-k^{\mathrm{T}}z(t) - L^n_F H(X_{0,n}(t)) - \Gamma(X_{0,n}(t))U_{1,n-1}(t)}{\gamma_0(X_{0,n}(t))}, \quad t \geqslant (n-1)\Delta \tag{27}$$

This equation describes the dynamics of the control variable $u(t)$ for $t \geqslant (n-1)\Delta$ in closed loop. If type-III relative degree is assumed, as in [9], it is $\gamma(\mathscr{X}_{0,n}) \neq 0$ and $\Gamma(\mathscr{X}_{0,n}) \equiv 0$ and it is evident that if $z(t)$ and $x(t)$ asymptotically go to zero, then $u(t)$ also goes to zero asymptotically. This is the reason why in Reference [10] the issue of the boundedness of the control variable is not addressed. If the less-restrictive assumption of type-II relative degree is made, Equation (27) is a continuous-time algebraic delay equation, where the value of the control variable u at time t depends on $n-1$ previous values of the same variable and on old and present values of the state. Equation (27) can be put in the form

$$u(t) = -\frac{1}{\gamma_0(X_{0,n}(t))} k^{\mathrm{T}}\_z(t) - p_0(X_{0,n}(t)) - \sum_{j=1}^{n-1} p_j(X_{0,n}(t))u(t-j\Delta), \quad t \geqslant (n-1)\Delta \tag{28}$$

where $p_j : \Omega_n \to \mathbb{R}$, $j = 0, 1, \ldots, n-1$ are defined by

$$p_0(\mathscr{X}_{0,n}) = \frac{L^n_F H(\mathscr{X}_{0,n})}{\gamma_0(\mathscr{X}_{0,n})}$$

$$p_j(\mathscr{X}_{0,n}) = \frac{\gamma_j(\mathscr{X}_{0,n})}{\gamma_0(\mathscr{X}_{0,n})}, \quad j = 1, \ldots, n-1 \tag{29}$$

Note that the vector of output derivatives behaves as an input for the input dynamics. For this reason, throughout the paper, the dynamics described by Equation (28) will be also denoted as the *output-driven* input dynamics.

In the following, it is shown that the internal dynamics of the system variable $x(t)$ is governed by the map $z = \Phi(\mathscr{X}_{0,n-1})$ defined in (21). In Reference [14] this map is called *observability map* of system (1) and (2), because suitable assumptions on this map allow the construction of an observer for nonlinear time delay systems. The observability map can be seen as a square map from χ_0 to z, in which the sub-vector $\mathscr{X}_{1,n-1} \in \mathbb{R}^{n(n-1)}$ is considered as a vector of parameters. To stress this point of view, in the following the map Φ will be rewritten as follows:

$$z = \Phi(\chi_0, \mathscr{X}_{1,n-1}) \tag{30}$$

For systems with type-II and type-III relative degree in a set Ω_n, it is

$$\det\left(\frac{\partial\Phi(\chi_0, \mathscr{X}_{1,n-1})}{\partial\chi_0}\right) \neq 0 \quad \forall \mathscr{X}_{0,n-1} \in \Omega_n \tag{31}$$

The proof can be found in Reference [10] (Lemma 1). This implies that the map Φ is locally invertible with respect to the first component χ_0 (local partial invertibility).

From here on we suppose that the following assumption holds for the observability map Φ (*Global partial invertibility*):

(H1) System (1), (2) has uniform type-II relative degree equal to n (the dimension of the system variable x) and there exists the inverse Φ^{-1} of function (30) w.r.t. the first component χ_0 for all $z \in \mathbb{R}^n$ and $\mathscr{X}_{1,n-1} \in \mathbb{R}^{n(n-1)}$, i.e.

$$\chi_0 = \Phi^{-1}(z, \mathscr{X}_{1,n-1}) \quad \forall z \in \mathbb{R}^n, \ \mathscr{X}_{1,n-1} \in \mathbb{R}^{n(n-1)} \tag{32}$$

In paper [14] assumption H_1 is a necessary condition for the construction of an asymptotic observer for a delay system.

The state dynamics of the system variable $x(t)$ for $t \geqslant (n-1)\Delta$ is governed by the following continuous-time algebraic delay equation forced by the vector $z(t)$ of output derivatives:

$$x(t) = \Phi^{-1}(z(t), X_{1,n-1}(t)) \tag{33}$$

$z(t)$ acts as an input in Equation (33), and therefore such dynamics will be denoted throughout this paper as the *output-driven state dynamics*. The stability of this dynamics has been studied in [6, 15], where sufficient conditions on the map Φ that ensure convergence of $x(t)$ to zero when the output together with its $n-1$ derivatives, is driven to zero by the control law (27), are provided. The output dynamics (25), the state dynamics (33) and the input dynamics (28) of the closed loop system, form a triangular system of differential-algebraic equations, well defined for $t \geqslant (n-1)\Delta$:

$$\dot{z}(t) = (A^B_{n,1} - B^B_{n,1}k^{\mathrm{T}})z(t) \tag{34a}$$

$$x(t) = \Phi^{-1}(z(t), X_{1,n-1}(t)) \tag{34b}$$

$$u(t) = -\frac{k^{\mathrm{T}}z(t)}{\gamma_0(X_{0,n}(t))} - p_0(X_{0,n}(t)) - p^{\mathrm{T}}(X_{0,n}(t))U_{n-1,1}(t) \tag{34c}$$

where

$$p^{\mathrm{T}}(\mathscr{X}_{0,n}) = [p_{n-1}(\mathscr{X}_{0,n}) \ \cdots \ p_1(\mathscr{X}_{0,n})] \tag{35}$$

The output dynamics (34a) is autonomous and can be made stable by a suitable choice of the gain vector k. Obviously, the stability of the state dynamics and of the input dynamics is a necessary condition for the overall stability of the controlled delay system.

Now consider an (open-loop) input function $u(t)$, $t \in [0, (n-1)\Delta]$, to bring $z(t)$ to zero at time $t = (n-1)\Delta$, and then apply the feedback law (27), that keeps $z(t) = 0$ for $t \geqslant (n-1)\Delta$. Equations (33) and (28) become, for $t \geqslant (n-1)\Delta$,

$$x(t) = \Phi^{-1}(0, X_{1,n-1}(t)) \tag{36a}$$

$$u(t) = -p_0(X_{0,n}(t)) - p^{\mathrm{T}}(X_{0,n}(t))U_{n-1,1}(t) \tag{36b}$$

Equation (36) is a continuous-time algebraic delay equation that describes the system *zero-dynamics*, i.e. the internal state and input dynamics when the output is identically zero. Note that the dynamics of $x(t)$, given by (36a), is only a part of the zero-dynamics and is completely characterized by the partial inverse map Φ^{-1}, and in the following will be denoted the *state zero-dynamics*. We will refer to Equation (36b) as the *input zero-dynamics*, and we will say that the system zero-dynamics is composed of the state and input zero-dynamics. Here is a simple example of delay system with full uniform Type-II relative degree with unstable zero-dynamics.

Example

Consider the following nonlinear delay systems:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) + \sigma x_2(t-\Delta) \\ \dot{x}_2(t) &= (1 + x_1^2(t))u(t) \\ y(t) &= x_1(t) \end{aligned} \tag{37}$$

where $\sigma \in \mathbb{R}$ is a parameter. Instead to test the conditions given in Definitions 1–4 for $r = 1, 2, \ldots$, the relative degree can be computed by repeatedly differentiated the output with respect to time. The output and its first derivative are

$$\begin{aligned} y(t) &= x_1(t) \\ \dot{y}(t) &= x_2(t) + \sigma x_2(t-\Delta) \end{aligned} \tag{38}$$

Since they do not depend on the input $u(t)$, the relative degree must be greater than 1.

The second-order derivative is

$$\ddot{y}(t) = (1 + x_1^2(t))u(t) + \sigma(1 + x_1^2(t-\Delta))u(t-\Delta) \tag{39}$$

and depends on both $u(t)$ and $u(t-\Delta)$. Since the coefficient of $u(t)$ is never zero, the system has uniform type-II relative degree $r = 2$. Equation (38) is the map $\Phi(X_{0,1}(t))$ defined in (22) that gives the output derivatives. The coefficient $\gamma_0(\mathscr{X}_{0,2})$ defined in (9) is $(1 + \chi_{0,1}^2)$. The output stabilizing control law (27) is

$$u(t) = \left(-\sigma(1 + x_1^2(t-\Delta))u(t-\Delta) - k^{\mathrm{T}} \begin{bmatrix} x_1(t) \\ x_2(t) - 2x_2(t-\Delta) \end{bmatrix} \right) \frac{1}{(1 + x_1^2(t))} \tag{40}$$

This control law, with k such that $A_{2,1}^B - B_{2,1}^B k^{\mathrm{T}}$ is stable, is such to drive $y(t)$ and $\dot{y}(t)$ exponentially to zero. Equations (36) of the zero-dynamics are the following:

$$\begin{aligned} x_1(t) &= 0 \\ x_2(t) &= -\sigma x_2(t-\Delta) \\ u(t) &= -\sigma \frac{1 + x_1^2(t-\Delta)}{1 + x_1^2(t)} u(t-\Delta) \quad t \geqslant \Delta \end{aligned} \tag{41}$$

Considering that $x_1(t) = 0$, the third equation becomes $u(t) = -\sigma u(t-\Delta)$. It follows that the zero-dynamics for system (37) is stable for $|\sigma| \leqslant 1$ and unstable for $|\sigma| > 1$. In the latter case, the closed-loop system is unstable.

This example shows that it is not sufficient to have relative degree equal to the dimension of the system vector x to stabilize a nonlinear delay system by means of exponential output stabilization, after exact input–output linearization with delay cancellation. In general there exists a zero-dynamics that may be unstable. In the case of systems with type-III relative degree the input $u(t)$ is a continuous function of only $X_{0,n}(t)$, and *is not* a function of the past values $u(t-i\Delta)$, so that the stability of the state zero-dynamics trivially implies the stability of the zero-dynamics.

3. THE CASE OF LINEAR DELAY SYSTEMS

This section shows the application of the exact input–output linearization with delay cancellation to the case of linear delay systems. Obviously, in the linear case the interest of the approach is in the delay cancellation. Moreover, this section presents some results on the stability of the zero-dynamics that will be needed later in the paper to prove more general results for nonlinear systems.

Consider a linear delay system of the form

$$\dot{x}(t) = A_0x(t) + A_1x(t-\Delta) + Bu(t)$$

$$y(t) = Cx(t) \tag{42}$$

with matrices $A_0, A_1 \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$. The computation of the Lie derivatives defined in (9) gives

$$L_F^i H(\mathscr{X}_{0,i}) = \sum_{j=0}^{i} C(A_0, A_1)^{[i,j]}\chi_j \tag{43}$$

where

$$(A_0, A_1)^{[i,j]} = 0_{n\times n}, \quad \text{if } i<0 \text{ or } j<0$$

$$(A_0, A_1)^{[0,0]} = I_n$$

$$(A_0, A_1)^{[i,j]} = (A_0, A_1)^{[i-1,j]}A_0 + (A_0, A_1)^{[i-1,j-1]}A_1 \tag{44}$$

From definition (3) it follows that

$$i<j \Rightarrow (A_0, A_1)^{[i,j]} = 0$$

$$(A_0, A_1)^{[i,0]} = A_0^i, \quad (A_0, A_1)^{[i,i]} = A_1^i \tag{45}$$

All definitions (Type-I, II or III) of relative degree r require that

$$C(A_0, A_1)^{[i,j]}B = 0, \quad i, j = 0, 1, \ldots, r-2 \tag{46}$$

In the case $r = n$, the output derivatives up to order $n-1$ can be written as

$$y^{(i)}(t) = \sum_{j=0}^{i} C(A_0, A_1)^{[i,j]}x_{j\Delta}(t), \quad i = 0, 1, \ldots, n-1 \tag{47}$$

while the nth order derivative is

$$y^{(n)}(t) = \sum_{j=0}^{n} C(A_0, A_1)^{[n,j]} x_{j\Delta}(t) + \sum_{j=1}^{n-1} C(A_0, A_1)^{[n-1,j]} u_{j\Delta}(t) + CA_0^{n-1}Bu(t) \tag{48}$$

In the case of linear systems having type-II relative degree n it is $CA_0^{n-1}B \neq 0$, and the control law (19) becomes

$$u = \frac{v - \sum_{j=0}^{n} C(A_0, A_1)^{[n,j]} x_{j\Delta} - \sum_{j=1}^{n-1} C(A_0, A_1)^{[n-1,j]} B u_{j\Delta}}{CA_0^{n-1}B} \tag{49}$$

Defining

$$s_j = \frac{C(A_0, A_1)^{[n-1,j]}B}{CA_0^{n-1}B}, \quad j = 1, \ldots, n-1 \tag{50}$$

$$q_j = \frac{C(A_0, A_1)^{[n,j]}}{CA_0^{n-1}B}, \quad j = 0, \ldots, n \tag{51}$$

the control law (49) is written as

$$u(t) = -\frac{v(t)}{CA_0^{n-1}B} - \sum_{j=0}^{n} q_j x_{j\Delta}(t) - \sum_{j=1}^{n-1} s_j u_{j\Delta}(t) \tag{52}$$

In the case of linear systems the relative degree (of any type) is always uniform, and the transformation of the I/O map in a chain of n integrators, i.e. $y^{(n)}(t) = v(t)$, is global.

The map Φ defined in (21) is as follows:

$$z = \sum_{j=0}^{n-1} Q_j \chi_j \tag{53}$$

where the $n \times n$ matrix Q_0 is the observability matrix of the pair (A_0, C), i.e.

$$Q_0 = \begin{bmatrix} C \\ CA_0 \\ \vdots \\ CA_0^{n-1} \end{bmatrix} \tag{54}$$

and the $n \times n$ matrices Q_j, $j = 1, 2, \ldots, n-1$, can be computed using (43), and are given by

$$\begin{aligned} \{Q_j\}_{(i,:)} &= 0 && \text{for } i = 1, \ldots, j \\ \{Q_j\}_{(i,:)} &= C(A_0, A_1)^{[i-1,j]} && \text{for } i = j+1, \ldots, n \end{aligned} \tag{55}$$

where $\{Q_j\}_{(i,:)}$ denotes the ith row of matrix Q_j.

Exponential convergence of the output $y(t)$ to zero is obtained with the feedback law (52) with v given by (24), so that the control law (28), defined for $t \geqslant (n-1)\Delta$, becomes

$$u(t) = -\frac{k^{\mathrm{T}} z(t)}{CA_0^{n-1}B} - q^{\mathrm{T}} X_{n,0}(t) - s^{\mathrm{T}} U_{n-1,1}(t) \tag{56}$$

where

$$q^{\mathrm{T}} = [q_n \quad q_{n-1} \quad \cdots \quad q_0] \tag{57}$$

$$s^{\mathrm{T}} = [s_{n-1} \quad s_{n-2} \quad \cdots \quad s_1] \tag{58}$$

with q_j and s_j defined in (51) and (50), respectively. With this control law, the vector of output derivatives $z(t)$ exponentially goes to zero, i.e. satisfies (26) for suitable positive γ and β. The partial invertibility property of the map Φ with respect to the χ_0 (assumption H_1) is equivalent to the observability of the pair (A_0, C). The inverse map Φ^{-1} defined in (32) is

$$\chi_0 = Q_0^{-1} z - \sum_{j=1}^{n-1} Q_0^{-1} Q_j \chi_j \tag{59}$$

and the state dynamics (33) is

$$x(t) = Q_0^{-1} z(t) - \sum_{j=1}^{n-1} Q_0^{-1} Q_j x(t - j\Delta) \tag{60}$$

For our purposes it is convenient to define a co-ordinate transformation $w(t) = Q_0 x(t)$, so that the state dynamics can be rewritten as

$$w(t) = z(t) - \sum_{j=1}^{n-1} Q_j Q_0^{-1} w(t - j\Delta) \tag{61}$$

The continuous-time algebraic delay equations (56) and (61), that describe the input and state dynamics can be written as discrete-time equations on suitable Banach spaces. Let $\mathscr{B} = L_\infty \times ([-\Delta, 0], \mathbb{R})$ and define

$$\tilde{u}(k) \in \mathscr{B}, \quad \tilde{u}(k)(\tau) = u(k\Delta + \tau),\ \tau \in [-\Delta, 0]$$

$$\tilde{x}(k) \in \mathscr{B}^n, \quad \tilde{x}(k)(\tau) = x(k\Delta + \tau),\ \tau \in [-\Delta, 0]$$

$$\tilde{w}(k) \in \mathscr{B}^n, \quad \tilde{w}(k)(\tau) = w(k\Delta + \tau),\ \tau \in [-\Delta, 0]$$

$$\tilde{z}(k) \in \mathscr{B}^n, \quad \tilde{z}(k)(\tau) = z(k\Delta + \tau),\ \tau \in [-\Delta, 0] \tag{62}$$

Then, Equation (56) can be put in the form

$$\tilde{u}(k+n) = -\frac{k^{\mathrm{T}} \tilde{z}(k+n)}{CA_0^{n-1} B} - \sum_{j=0}^{n} q_j Q_0^{-1} \tilde{w}(k+n-j) - \sum_{j=1}^{n-1} s_j \tilde{u}(k+n-j), \quad k \geqslant 0 \tag{63}$$

and defining $\tilde{z}(k) \in \mathscr{B}^n$ by $\tilde{z}(k)(\tau) = z(k\Delta + \tau)$, $\tau \in [-\Delta, 0]$, (61) can be written as

$$\tilde{w}(k+n) = -\sum_{j=1}^{n-1} Q_j Q_0^{-1} \tilde{w}(k+n-j) + \tilde{z}(k+n), \quad k \geqslant 0 \tag{64}$$

When $\tilde{z}(k+n)=0, \forall k \geqslant 0$, equations (63) and (64) describe the system zero-dynamics, while (64) describes the state zero-dynamics. Defining the extended vectors

$$\tilde{U}(k)=\begin{bmatrix}\tilde{u}(k+1)\\ \vdots \\ \tilde{u}(k+n-1)\end{bmatrix} \in \mathscr{B}^{n-1}, \quad \tilde{W}(k)=\begin{bmatrix}\tilde{w}(k)\\ \vdots \\ \tilde{w}(k+n)\end{bmatrix} \in \mathscr{B}^{(n+1)\times n} \tag{65}$$

(note that $\tilde{U}(k)(\tau)=Stack_{n-1,1}(u(k\Delta+\tau))$ and $\tilde{W}(k)(\tau)=Stack_{n,0}(w(k\Delta+\tau)))$ from (63) the following equations can be written:

$$\begin{aligned}\tilde{U}(k+1) &= A^B_{n-1,1}\tilde{U}(k)-B^B_{n-1,1}(\tilde{q}^{\mathrm{T}}\tilde{W}(k)+s^{\mathrm{T}}\tilde{U}(k))+D_0\tilde{v}_0(k)\\ \tilde{u}(k) &= C^B_{n-1,1}\tilde{U}(k), \quad k\geqslant 0\end{aligned} \tag{66}$$

where $\tilde{v}_0(k)=\tilde{z}(k+n)$, $(A^B_{n-1,1}, B^B_{n-1,1}, C^B_{n-1,1})$ is a Brunowsky triplet, as defined in (4), s^{T} has been defined in (58), and

$$\begin{aligned}\tilde{q}^{\mathrm{T}} &= [q_nQ_0^{-1} \quad q_{n-1}Q_0^{-1} \quad \cdots \quad q_0Q_0^{-1}]\\ D_0 &= -\frac{1}{CA_0^{n-1}B}B^B_{n-1,1}k^{\mathrm{T}}\end{aligned} \tag{67}$$

In a similar way, from (64) the following equations can be written:

$$\begin{aligned}\tilde{W}(k+1) &= A^B_{n+1,n}\tilde{W}(k)-B^B_{n+1,n}\Sigma\tilde{W}(k)+B^B_{n+1,n}\tilde{v}_1(k)\\ \tilde{w}(k) &= C^B_{n+1,n}\tilde{W}(k), \quad k\geqslant -1\end{aligned} \tag{68}$$

where $\tilde{v}_1(k)=\tilde{z}(k+n+1)$, $(A^B_{n+1,n}, B^B_{n+1,n}, C^B_{n+1,n})$ is a Brunowsky triplet and

$$\Sigma=[0_{n\times n} \quad 0_{n\times n} \quad Q_{n-1}Q_0^{-1} \quad Q_{n-2}Q_0^{-1} \quad \cdots \quad Q_1Q_0^{-1}] \tag{69}$$

(note that the first two blocks of matrix Σ are zero because $\tilde{w}(k+n+1)$ is not a function of $\tilde{w}(k)$ and $\tilde{w}(k+1)$).

From (68) and (66) the state and input dynamics for the linear delay systems (42) in closed loop, admits the following state-space representation (for $k\geqslant 0$):

$$\begin{aligned}\xi(k+1) &= \begin{bmatrix}A^B_{n+1,n}-B^B_{n+1,n}\Sigma & 0_{(n+1)n\times(n-1)}\\ -B^B_{n-1,1}q^{\mathrm{T}} & A^B_{n-1,1}-B^B_{n-1,1}s^{\mathrm{T}}\end{bmatrix}\xi(k)+\begin{bmatrix}0_{(n+1)n\times n} & B^B_{n+1,n}\\ D_0 & 0_{(n-1)\times n}\end{bmatrix}\tilde{v}(k)\\ \begin{bmatrix}\tilde{w}(k)\\ \tilde{u}(k)\end{bmatrix} &= \begin{bmatrix}C^B_{n+1,n} & 0_{(n+1)\times(n-1)}\\ 0_{1\times(n+1)n} & C^B_{n-1,1}\end{bmatrix}\xi(k)\end{aligned} \tag{70}$$

where

$$\xi(k)=\begin{bmatrix}\tilde{W}(k)\\ \tilde{U}(k)\end{bmatrix} \in \mathscr{B}^{(n+1)n+n-1}, \qquad \tilde{v}(k)=\begin{bmatrix}\tilde{v}_0(k)\\ \tilde{v}_1(k)\end{bmatrix} \in \mathscr{B}^{2n} \tag{71}$$

System (70) forced by $\tilde{v}(k) = 0$, $k \geqslant 0$, describes the zero-dynamics of the delay system (42) on the Banach space $\mathscr{B}^{(n+1)n+n-1}$. A representation of the *state zero-dynamics* is

$$\tilde{W}(k+1) = (A^B_{n+1,n} - B^B_{n+1,n}\Sigma)\tilde{W}(k), \quad k \geqslant 0 \tag{72}$$

The first result is as follows:

Lemma 3.9

Consider the triangular transition matrix of system (70). The eigenvalues of the matxix $A^B_{n-1,1} - B^B_{n-1,1}s^{\mathrm{T}}$ are a subset of the eigenvalues of matrix $A^B_{n+1,n} - B^B_{n+1,n}\Sigma$.

Proof

The $(n+1)n \times (n+1)n$ matrix $A^B_{n+1,n} - B^B_{n+1,n}\Sigma$ can be put in a block-triangular form as follows:

$$A^B_{n+1,n} - B^B_{n+1,n}\Sigma = \begin{bmatrix} A^B_{2,n} & \Pi_n \\ 0_{(n-1)n\times 2n} & A^B_{n-1,n} - B^B_{n-1,n}\bar{\Sigma} \end{bmatrix} \tag{73}$$

where

$$\Pi_n = \begin{bmatrix} 0_{n\times n} & 0_{n\times n(n-2)} \\ I_n & 0_{n\times n(n-2)} \end{bmatrix} \tag{74}$$

($0_{n\times n(n-2)}$ vanishes for $n = 2$) and

$$\bar{\Sigma} = [Q_{n-1}Q_0^{-1} \quad Q_{n-2}Q_0^{-1} \quad \cdots \quad Q_1Q_0^{-1}] \tag{75}$$

so that $2n$ eigenvalues are the eigenvalues of $A^B_{2,n}$ (all zero), and the remaining $n(n-1)$ eigenvalues are those of $A^B_{n-1,n} - B^B_{n-1,n}\bar{\Sigma}$. The proof that the $n-1$ eigenvalues of the matrix $A^B_{n-1,1} - B^B_{n-1,1}s^{\mathrm{T}}$ are a subset of the $n(n-1)$ eigenvalues of the stable matrix $A^B_{n-1,n} - B^B_{n-1,n}\bar{\Sigma}$ is obtained by showing that there exists a matrix $M \in \mathbb{R}^{(n-1)n\times(n-1)}$ such that

$$(A^B_{n-1,n} - B^B_{n-1,n}\bar{\Sigma})M = M(A^B_{n-1,1} - B^B_{n-1,1}s^{\mathrm{T}}) \tag{76}$$

Note first that the last column of matrix Q_0^{-1} is as follows:

$$\{Q_0^{-1}\}_{(:,n)} = B\frac{1}{CA_0^{n-1}B} \tag{77}$$

This happens because the assumption of relative degree equal to n implies that the triplet (C, A_0, B) has relative degree n, that is

$$Q_0B = (CA_0^{n-1}B)d_n, \quad \text{where } d_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^n \tag{78}$$

and $\{Q_0^{-1}\}_{(:,n)}$, by definition, is the unique vector such that $Q_0\{Q_0^{-1}\}_{(:,n)} = d_n$. Recalling definition (55) of matrices Q_j appearing in matrix $\bar{\Sigma}$ (see (75)) it is trivially verified that $\{Q_j\}_{(i,:)}\{Q_0^{-1}\}_{(:,n)} = 0$, $i = 1,\ldots,j$. Moreover, being the. delay relative degree equal to n, for $i = j+1, \ldots, n-1$, we have

$$\{Q_j\}_{(i,:)}\{Q_0^{-1}\}_{(:,n)} = \{Q_j\}_{(i,:)}B\frac{1}{CA_0^{n-1}B} = \frac{C(A_0,A_1)^{[i-1,j]}B}{CA_0^{n-1}B} = 0 \tag{79}$$

At last, for $i = n$, we have

$$\{Q_j\}_{(n,:)}\{Q_0^{-1}\}_{(:,n)} = \{Q_j\}_{(n,:)}B\frac{1}{CA_0^{n-1}B} = \frac{C(A_0,A_1)^{[n-1,j]}B}{CA_0^{n-1}B} = s_j \tag{80}$$

where the reals s_j, defined in (50), are the components of vector s^{T}. Thus, the last columns of the products $Q_jQ_0^{-1}$, for $j = 1, \ldots, n-1$ are

$$\{Q_jQ_0^{-1}\}_{(:,n)} = Q_j\{Q_0^{-1}\}_{(:,n)} = Q_jB\frac{1}{CA_0^{n-1}B} = s_jd_n \tag{81}$$

so that the matrix $A^B_{n-1,n} - B^B_{n-1,n}\bar{\Sigma}$ has the structure

$$A^B_{n-1,n} - B^B_{n-1,n}\bar{\Sigma} = \begin{bmatrix} 0_{n\times n} & I_n & \cdots & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & \cdots & 0_{n\times n} \\ \vdots & \vdots & \cdots & \vdots \\ 0_{n\times n} & 0_{n\times n} & \cdots & I_n \\ -[\star\ s_{n-1}d_n] & -[\star\ s_{n-2}d_n] & \cdots & -[\star\ s_1d_n] \end{bmatrix} \tag{82}$$

where the asterisks denote inessential $n\times(n-1)$ matrices. On the other hand, the structure of matrix $A^B_{n-1,1} - B^B_{n-1,1}s^{\mathrm{T}}$ is

$$A^B_{n-1,1} - B^B_{n-1,1}s^{\mathrm{T}} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \\ -s_{n-1} & -s_{n-2} & \cdots & -s_1 \end{bmatrix} \tag{83}$$

From these it is easily verified that matrix

$$M = \begin{bmatrix} d_n & 0_{n\times 1} & \cdots & 0_{n\times 1} \\ 0_{n\times 1} & d_n & \cdots & 0_{n\times 1} \\ \vdots & \vdots & \vdots & \vdots \\ 0_{n\times 1} & 0_{n\times 1} & \cdots & d_n \end{bmatrix} \in \mathbb{R}^{(n-1)n\times(n-1)} \tag{84}$$

satisfies identity (76), and the lemma is proved. □

Theorem 3.10
Consider the linear delay system (42) and its state zero-dynamics, defined by (60) with $z(t) \equiv 0$, and the zero-dynamics, defined by both (60) and (56) with $z(t) \equiv 0$. The following statements are true:

(i) the state zero-dynamics is exponentially stable if and only if all eigenvalues of matrix $A^B_{n-1,n} - B^B_{n-1,n}\bar{\Sigma}$ defined in (73) are inside the open unit circle;
(ii) if the state zero-dynamics is exponentially stable then also the zero-dynamics is exponentially stable.

Proof
The first assertion is proved by considering that, as previously discussed, the state zero-dynamics of system (42) can be represented by the discrete time equation (72) on the Banach space $\mathscr{B}^{(n+1)n}$. Then, the eigenvalues of $A^B_{n+1,n} - B^B_{n+1,n}\Sigma$ inside the open unit circle of the complex plane provide a necessary and sufficient condition for exponential stability. The second assertion is proved by considering the zero-dynamics represented by (70) on $\mathscr{B}^{(n+1)n+n-1}$. By the assumption of exponential stability of the state zero-dynamics it follows that all eigenvalues of $A^B_{n+1,n} - B^B_{n+1,n}\Sigma$ are inside the open unit circle, and by Lemma 3.9 it follows that also all eigenvalues of matrix $A^B_{n-1,1} - B^B_{n-1,1}s^{\mathrm{T}}$ are inside the open unit circle. As a consequence also the transition matrix of (70), thanks to its triangular structure, has all eigenvalues in the open unit circle. This implies exponential stability of the zero-dynamics of system (42). □

Theorem 3.11
If the linear delay system (42) has on exponentially stable state zero-dynamics, then the output stabilizing feedback law (56) is such that both $x(t)$ and $u(t)$ exponentially go to zero.

Proof
By Theorem 3.10, the exponential stability assumption of the state zero-dynamics implies the exponential stability of the zero-dynamics. This means that all eigenvalues of the transition matrix in the representation (70) are inside the open unit circle. Standard results on linear discrete time systems on Banach spaces allow to state that if the input $\tilde{v}(k)$ is such that

$$\|\tilde{v}(k)\| \leqslant \rho\lambda^k, \quad k \geqslant 0 \tag{85}$$

for some $\rho > 0$ and $\lambda \in (0, 1)$ then there exist $\mu > 0$ and $\tilde{\lambda} \in [\lambda, 1)$ such that

$$\|\xi(k)\| \leqslant \mu(\rho + \|\tilde{W}(0)\|)\tilde{\lambda}^k, \quad k \geqslant 0 \tag{86}$$

Recall that $\tilde{v}(k)^{\mathrm{T}} = [\tilde{z}^{\mathrm{T}}(k+n) \ \tilde{z}^{\mathrm{T}}(k+n+1)]$ and that the control law (56) achieves exponential decay of $z(t)$, so that inequality (85) holds for some $\rho > 0$ and $\lambda \in (0, 1)$.

Moreover, from definitions (62), (65) and (71)

$$\begin{aligned} \|\tilde{x}(k)\| &\leqslant \|Q_0^{-1}\| \cdot \|\tilde{W}(k)\| \leqslant \|Q_0^{-1}\| \cdot \|\xi(k)\| \\ \|\tilde{u}(k)\| &\leqslant \|\tilde{U}(k)\| \leqslant \|\xi(k)\|, \end{aligned} \tag{87}$$

so that from (86) exponential convergence to zero of both $x(t)$ and $u(t)$ easily follows. □

4. MAIN RESULTS FOR NONLINEAR DELAY SYSTEMS

As in the case of linear delay systems, the state and input dynamics of nonlinear delay systems in closed-loop with the feedback law (27) admit a representation with a discrete time system on a Banach space. Recall that the assumptions (3) imply that the state $x(\tau) = 0$, $\tau \in [-\Delta, 0]$, is an equilibrium point for system (1) and (2). Let

$$A_0 = \left.\frac{\partial f(\chi_0, \chi_1)}{\partial \chi_0}\right|_{(\chi_0,\chi_1)=(0,0)}, \quad A_1 = \left.\frac{\partial f(\chi_0, \chi_1)}{\partial \chi_1}\right|_{(\chi_0,\chi_1)=(0,0)}, \quad B = g(0,0)$$

$$C = \left.\frac{\partial h(\chi_0)}{\partial \chi_0}\right|_{\chi_0=0} \tag{88}$$

so that the linear system (42) can be seen as the linear approximation of system (1) and (2) around the origin.

Recalling definitions (9) and considering that $f(0,0) = 0$ and $h(0) = 0$,

$$L_F^i H(0) = 0, \quad i = 0, 1, \ldots$$

It follows that the map $z = \Phi(\mathscr{X}_{0,n-1})$ defined in (21) is such that $\Phi(0) = 0$, and its linear approximation around $\mathscr{X}_{0,n-1} = 0$ provides the linear map defined in (53), with matrices Q_j built up using Jacobian (88). In particular

$$\left.\frac{\partial L_F^i H(\chi_0, \ldots, \chi_i)}{\partial \chi_0}\right|_{\mathscr{X}_{0,i}=0} = CA_0^i, \quad i = 0, 1, \ldots, n-1 \tag{89}$$

and from this

$$\left.\frac{\partial \Phi(\chi_0, \ldots, \chi_i)}{\partial \chi_0}\right|_{\mathscr{X}_{0,i}=0} = Q_0 \tag{90}$$

where Q_0 is the observability matrix of the pair (A_0, C), defined in (54), non-singular if a type-II relative degree n is assumed around the origin. Moreover, note that

$$\gamma_0(0) = CA_0^{n-1}B, \quad \text{and} \quad p_0(0) = \left.\frac{L_F^n H(\mathscr{X}_{0,n})}{\gamma_0(\mathscr{X}_{0,n})}\right|_{\mathscr{X}_{0,n}=0} = 0$$

The functions $p_j(\mathscr{X}_{0,n})$ for $j = 1, \ldots, n-1$, in the control law (28), when computed at $\mathscr{X}_{0,n} = 0$ give back $p_j(0) = s_j$, the coefficients defined in (50). The input dynamics can be written as

$$u(t) = \mu(z(t), X_{0,n}(t)) - (s^{\mathrm{T}} + \tilde{p}^{\mathrm{T}}(X_{0,n}(t)))U_{n-1,1}(t) \tag{91}$$

where

$$\mu(z, \mathscr{X}_{0,n}) = -\frac{k^{\mathrm{T}} z}{\gamma_0(\mathscr{X}_{0,n})} - p_0(\mathscr{X}_{0,n})$$

$$\tilde{p}^{\mathrm{T}}(\mathscr{X}_{0,n}) = \tilde{p}^{\mathrm{T}}(\mathscr{X}_{0,n}) - s^{\mathrm{T}} \tag{92}$$

so that $\tilde{p}^{\mathrm{T}}(0) = 0$. The linear approximation of Equation (91) around the solution $z(t) \equiv 0$, $x(t) \equiv 0$, $u(t) \equiv 0$, gives back Equation (56).

The state dynamics (33) can be written in the co-ordinates $w(t) = Q_0 x(t)$ as

$$w(t) = \Psi(z(t), w_\Delta(t), \ldots, w_{(n-1)\Delta}(t)) \tag{93}$$

where

$$\Psi(z,\omega_1,\ldots,\omega_{n-1}) = \Phi^{-1}(z, Q_0^{-1}\omega_1,\ldots,Q_0^{-1}\omega_{n-1}) \tag{94}$$

The linear approximation of (93) around the solution $z(t)\equiv 0,\ x(t)\equiv 0$ gives Equation (61). It follows that all the results presented in the previous section devoted to linear delay systems can be applied to the stability analysis of the linear approximation of the state and input equations of nonlinear delay systems with the output stabilizing control law (27).

In the following it will be shown that some assumptions on the global stability of the state zero-dynamics imply a kind of global internal stability of the controlled nonlinear delay system. The proof of this result is obtained using the linear stability analysis presented in the previous section applied to the linear approximation of the state zero-dynamics. It should not be a surprise that a local analysis helps in the proof of a global stability property: a stronger global stability assumption has been made on the state zero-dynamics.

A first useful lemma is the following.

Lemma 4.12

Consider a nonlinear delay system (1)–(2) with uniform type-II relative degree and globally partially invertible observability map (assumption H_1). Assume that the equilibrium $x(t)\equiv 0$ of the state zero-dynamics (36a) is exponentially stable. Then, the linear approximation of the zero-dynamics (36) around $x(t)=0,\ u(t)\equiv 0$ is exponentially stable.

Proof

From the previous discussion, Equation (70), with $\tilde{v}(k)\equiv 0$ is a representation of the linear approximation of the zero-dynamics (36) around $x(t)\equiv 0,\ u(t)\equiv 0$. Moreover, the assumption of exponential stability of the nonlinear state zero-dynamics implies exponential stability of the linear approximation of the state zero-dynamics. From the assertion (ii) of Theorem 3.10 the linear approximation of the zero-dynamics is exponentially stable. □

Differently from the case of linear systems, it is not obvious if the exponential stability of the nonlinear state zero-dynamics (36a) implies that if $z(t)$ asymptotically goes to zero, then also $x(t)$ asymptotically goes to zero. As a consequence, this kind of *input-state* stability for the output-driven state dynamics (33) is a property that must be explicitly assumed, together with the exponential stability of the state zero-dynamics.

Definition 4.13

The output-driven state dynamics (33) is said to be globally input-state asymptotically (exponentially) stable if for all $z(t)$ that asymptotically (exponentially) go to zero and for all initial states, $x(t)$ asymptotically (exponentially) goes to zero.

Remark 4.14

If the output-driven state dynamics (33) is globally input-state exponentially stable, then the state zero-dynamics is exponentially stable and the transition matrix of the linear approximation (68) has all eigenvalues inside the unit circle. On the other hand, if the linear approximation is exponentially stable, then the state zero-dynamics is *locally* exponentially stable, and the output-driven state dynamics (33) is *locally* input-state exponentially stable.

As done for the output-driven input dynamics of linear delay systems with control law (56) also the (output-driven) input dynamics of nonlinear delay systems can be written on the Banach space $\mathscr{B}^{(n-1)}$ exploiting the same definitions given in (62) and (65), but in this case the transition operator cannot be represented simply by a matrix in $\mathbb{R}^{(n-1)\times(n-1)}$. Regarding $X_{0,n}(t)$ as a time-varying parameter, Equation (41) can be written as a time-varying system on the Banach Space $\mathscr{B}^{(n-1)}$ as follows:

$$\tilde{U}(k+1) = (\mathscr{A} + \mathscr{S}(k))\tilde{U}(k) - B^B_{n-1,1}\tilde{\mu}(k) \tag{95}$$

where $\mathscr{A}$ and $\mathscr{S}(k)$ are operators from $\mathscr{B}^{n-1}$ to $\mathscr{B}$, defined by

$$\begin{aligned} &[\mathscr{A}\tilde{U}](\tau) = (A^B_{n-1,1} - B^B_{n-1,1}s^{\mathrm{T}})[\tilde{U}(k)](\tau) \\ &[\mathscr{S}(k)\tilde{U}(k)](\tau) = \tilde{p}^{\mathrm{T}}(X_{0,n}((k+n)\Delta+\tau))[\tilde{U}(k)](\tau), \quad \tau \in [-\Delta, 0] \end{aligned} \tag{96}$$

and the sequence $\tilde{\mu}(k) \in \mathscr{B}$ is defined by

$$[\tilde{\mu}(k)](\tau) = \mu(z((k+n)\Delta+\tau), X_{1,n}((k+n)\Delta+\tau)), \quad \tau \in [-\Delta, 0], \tag{97}$$

where $\tilde{p}^{\mathrm{T}}(\mathscr{X}_{0,n})$ and $\mu(z, \mathscr{X}_{1,n})$ have been defined in (92).

The following theorem is the main result on the internal stability of nonlinear delay systems controlled with the output-stabilizing law (27).

Theorem 4.15

Consider the control law (27), with k Hurwitz, applied to a non-linear delay system (1)–(2) with uniform type-II relative degree and globally partially invertible observability map (assumption H_1). Assume that the state zero-dynamics is globally exponentially stable and that the output-driven state dynamics is globally input-state asymptotically stable (Definition 4.13). Then, both the system variable $x(t)$ and the input variable $u(t)$ asymptotically go to zero.

Proof

Thanks to the control law (27), with k Hurwitz, the vector $z(t)$ exponentially goes to zero. By the assumption of global input-state asymptotic stability of the output-driven state dynamics, also $x(t)$ asymptotically goes to zero. The assumption of global exponential stability of the state zero-dynamics implies that the linear approximation of the state zero-dynamics on the Banach space $\mathscr{B}^{(n+1)n}$ is exponentially stable, and therefore that all eigenvalues of matrix $A^B_{n+1,n} - B^B_{n+1,n}\Sigma$, are inside the open unit circle of the complex plane. Thanks to Lemma 3.9, also the eigenvalues of matrix $A^B_{n-1,1} - B^B_{n-1,1}s^{\mathrm{T}}$ that govern the linear approximation of the input dynamics (95) on $\mathscr{B}^{n-1}$, are inside the open unit circle. Now, note that representation (95) is of the same type of the one described by Equation (A1)) in the appendix. Since $z(t), x(t)$, and therefore $X_{0,n}(t)$, asymptotically go to zero, recalling that, from (92), $\tilde{p}^{\mathrm{T}}(0) = 0$ and $\mu(0,0) = 0$, the sequence $\|\tilde{\mu}(k)\|$ and $\|\mathscr{S}(k)\|$ are bounded by sequences convergent to zero. Since the matrix $A^B_{n-1,1} - B^B_{n-1,1}s^{\mathrm{T}}$ has all eigenvalues inside the open unit circle, thanks to Lemma A.1, proved in the appendix, it follows that the sequence $\|\tilde{U}(k)\|$ asymptotically tends to zero. This trivially implies that $u(t)$ asymptotically goes to zero, and the proof is complete. □

Remark 4.16

The *local* exponential stability of the state zero-dynamics, and hence the *local* input-state exponential stability of the output-driven state dynamics, can be tested by evaluating the eigenvalues of matrix $A^B_{n-1,n} - B^B_{n-1,n}\bar{\Sigma}$. Global stability properties are much harder to investigate.

5. AN EXAMPLE

Consider the following delay system of type (1) and (2)

$$\begin{aligned}\dot{x}_1(t) &= \frac{x_2(t)}{1 + x_1^2(t-\Delta)} + \sigma x_2(t-\Delta)\\ \dot{x}_2(t) &= x_1(t)x_2(t-\Delta) + u(t)\\ y(t) &= x_1(t)\end{aligned} \tag{98}$$

where σ is a constant parameter. According to definition 2.3 it is not difficult to show that this system has uniform Type-II relative degree $r = n = 2$. Here follows the expression of $\gamma_0(\mathscr{X}_{0,2})$, as defined in (11),

$$\gamma_0(\mathscr{X}_{0,2}) = \frac{1}{1 + \chi_{1,1}^2} \neq 0, \quad \forall \mathscr{X}_{0,2} \in \mathbb{R}^6 \tag{99}$$

The substitution $\mathscr{X}_{0,2} = X_{0,2}(t)$ in the previous equation yields

$$\gamma_0(X_{0,2}(t)) = \frac{1}{1 + x_1^2(t-\Delta)} \tag{100}$$

Following definition (21) the map $z = \Phi(\mathscr{X}_{0,2})$ can be easily constructed and can be shown to be globally partially invertible, i.e. $\chi_0 = \Phi^{-1}(z, \mathscr{X}_{1,2})$ exists $\forall z \in \mathbb{R}^2$ and $\forall \mathscr{X}_{1,2} \in \mathbb{R}^4$. After the substitutions of $\mathscr{X}_{0,2}$ with $X_{0,2}(t)$ and of $\mathscr{X}_{1,2}$ with $X_{1,2}(t)$, the maps Φ and Φ^{-1}, for the given example, are as follows:

$$z(t) = \Phi(X_{0,2}(t)) = \begin{bmatrix} x_1(t) \\ \dfrac{x_2(t)}{1 + x_1^2(t-\Delta)} + \sigma x_2(t-\Delta) \end{bmatrix} \tag{101}$$

$$x(t) = \Phi^{-1}(z(t), X_{1,2}(t)) = \begin{bmatrix} z_1(t) \\ (z_2(t) - \sigma x_2(t-\Delta))(1 + x_1^2(t-\Delta)) \end{bmatrix} \tag{102}$$

The control law (27) has the following expression:

$$u(t) = \alpha(z(t), x(t), x(t-\Delta), x(t-2\Delta)) + \beta(x(t), x(t-\Delta))u(t-\Delta) \tag{103}$$

with

$$\begin{aligned}\alpha = (1 + x_1^2(t-\Delta))\Bigg[&-k^{\mathrm{T}}z(t) - \sigma x_1(t-\Delta)x_2(t-2\Delta)\\ &+ \frac{x_2(t)x_1(t-\Delta)}{(1 + x_1^2(t-\Delta))^2}\left(\frac{x_2(t-\Delta)}{1 + x_1^2(t-2\Delta)} + \sigma x_2(t-2\Delta)\right)\Bigg]\end{aligned} \tag{104}$$

$$\beta = -\sigma(1 + x_1^2(t-\Delta)) \tag{105}$$

The gain vector $k^{\mathrm{T}} = [k_1 \;\; k_2]$ is such to assign stable eigenvalues to the matrix

$$A^B_{2,1} - B^B_{2,1} k^{\mathrm{T}} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} \tag{106}$$

Equation (102), when $z(t) \equiv 0$, for $t \geqslant \Delta$, defines the state-zero-dynamics of the system, and together with (103) define the system zero-dynamics (see Equation (36)). According to the results of Theorem 4.15, the convergence of the system variables (state and input) to zero is guaranteed if the state-zero-dynamics is globally exponentially stable and if the dynamics given by (102) is globally input-state asymptotically stable. Thanks to the continuity of the map Φ^{-1}, and suitably exploiting the first scalar equation ($x_1(t) = z_1(t)$, for $t \geqslant \Delta$), it is not difficult to prove that the null element of $L_\infty([-\Delta, \Delta], \mathbb{R}^2)$ is a globally exponentially stable equilibrium point of the state-zero-dynamics if and only if $|\sigma| < 1$. Under the same condition on σ it can be shown that if $z(t)$ asymptotically converges to zero then also $x(t)$ asymptotically decays to zero. Thanks to Theorem 4.15 it follows that if $|\sigma| < 1$ then the feedback law (103), with stabilizing gain k, is such to asymptotically drive to zero both the state and the input of system (98). All computer simulations have shown that all system variables asymptotically go to zero when $|\sigma| < 1$.

The results of two numerical simulations are presented below, in which two different values of the parameter σ are used. The delay Δ used in the simulations is $\Delta = 0.1$. In both simulations the initial state for the system has been chosen as

$$x(\tau) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \tau \in [-0.1, 0] \tag{107}$$

The gain matrix k^{T} in (103) used is $k^{\mathrm{T}} = [20 \;\; 9]$ (assigns eigenvalues $\lambda_1 = -4$ and $\lambda_2 = -5$ to matrix (106)). The control law (103) is applied starting at time $t = 0.1$.

Figures 1 and 2 report simulation results when $\sigma = 0.5$. Figure 1 shows the state evolution of the controlled system. Note that the first state component, the system output, asymptotically goes to zero with a typical two modes exponential decay. Also, the second system variable asymptotically goes to zero, together with the input, depicted in Figure 2.

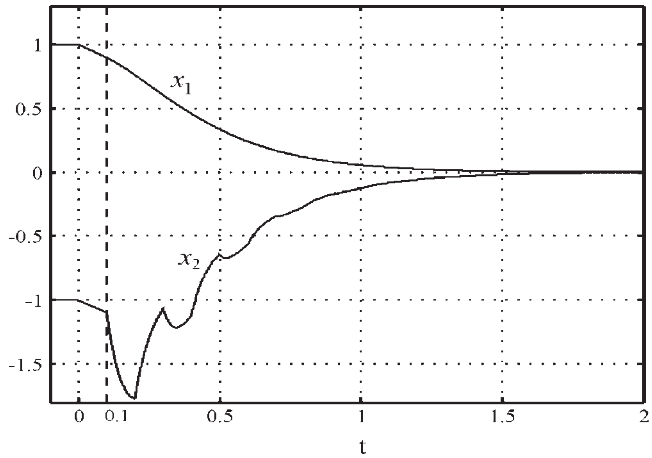

Figure 1. State evolution of the system for $\sigma = 0.5$.

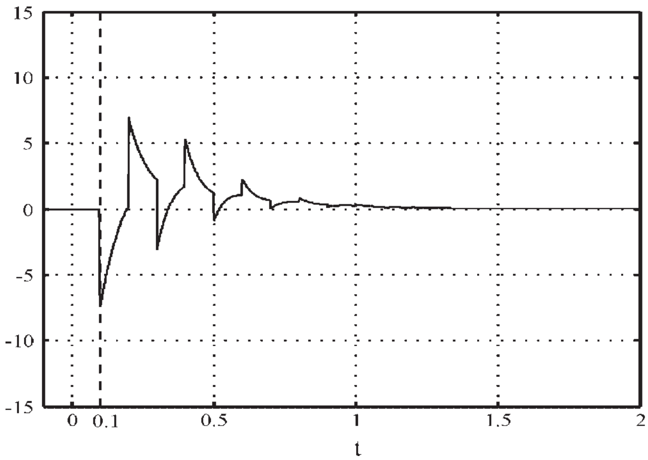

Figure 2. Input evolution of the system for $\sigma = 0.5$.

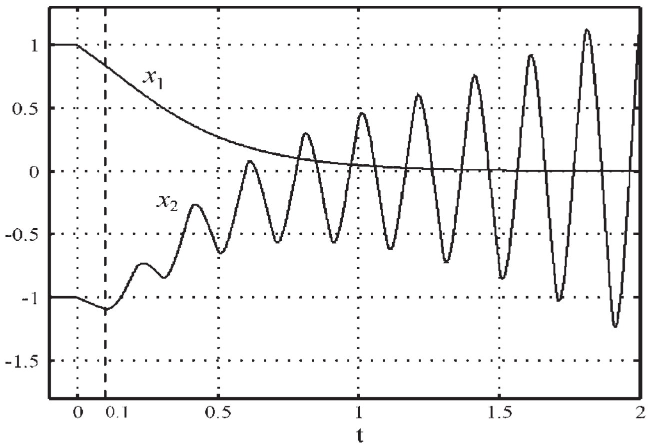

Figure 3. State evolution of the system for $\sigma = 1.1$.

Figures 3 and 4 report simulation results when $\sigma = 1.1$. In this case some system variables diverge. Figure 3 shows that the first state component exponentially goes to zero, exactly as in the previous simulation, while variable x_2 diverges. This happens because the state-zero-dynamics is unstable. Also the input variable diverges in this case, as shown in Figure 4.

6. CONCLUDING REMARKS

The technique of exact I/O linearization, originally developed for nonlinear systems without state delay, has been recently applied to nonlinear delay systems by means of a suitable extension of the tools of the standard differential geometry [5–10]. Simultaneous I/O

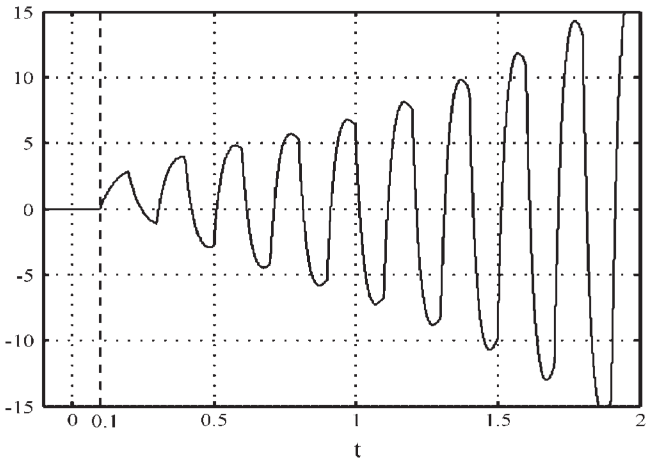

Figure 4. Input evolution of the system for $\sigma = 1.1$.

linearization and delay cancellation can be obtained for systems that have a relative degree and have stable zero-dynamics (internal state and input dynamics when the output is forced to be zero). In this paper we pointed out that, unlike the case of systems without delay, for delay systems the full relative degree property does not imply the absence of the zero-dynamics. The stability of the state zero-dynamics in the case of full relative degree delay systems has been studied for the first time in Reference [6]. In Reference [10] the authors discussed the issue of stability of what in this paper is called *state zero-dynamics* (Equation (36a)) in the case of non-full relative degree, while the issue of stability of the *total zero-dynamics* (Equations (36a) and (36b)) was not investigated because the authors considered a class of systems (those with type-Ill relative degree) that do not have input zero-dynamics. This paper explicitly considers delay systems with full relative degree and with non-trivial state and input zero-dynamics. The main result presented here is that a suitable stability assumption on the state zero-dynamics, a necessary assumption for the applicability of the technique of exact I/O linearization with delay cancellation, in the case of full relative degree implies the stability of the input zero-dynamics. This implies the closed-loop stability of the nonlinear delay system, with the output zeroing controller. Future work will involve the study of the internal state and input dynamics in the case of not full relative degree.

ACKNOWLEDGEMENTS

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APPENDIX A

This appendix reports a convergence result on linear time-varying systems on Banach spaces that is needed in the proof of Theorem 4.15). Note that the symbols used in this appendix do not refer to quantities defined in the main body of the paper.

Consider the Banach space $\mathscr{S} = L_\infty([-\delta, 0], \mathbb{R})$, and a linear time-varying system described by the equation

$$x_{k+1} = (\mathscr{A} + \mathscr{E}_k)x_k + \mathscr{B}u_k, \quad k \geqslant 0 \tag{A1}$$

where $x_k \in \mathscr{S}^n$ is the state, $u_k \in \mathscr{S}^p$ is the input sequence, and the operators $\mathscr{A}, \mathscr{E}_k$ and $\mathscr{B}$ are defined by

$$\begin{aligned} &[\mathscr{A}x_k](\tau) = Ax_k(\tau), \quad A \in \mathbb{R}^{n\times n} \\ &[\mathscr{E}_k x_k](\tau) = \mathscr{E}_k(\tau)x_k(\tau), \quad \mathscr{E}_k(\tau) \in \mathbb{R}^{n\times n}, \ \tau \in [-\delta, 0] \\ &[\mathscr{B}u_k](\tau) = Bx_k(\tau), \quad B \in \mathbb{R}^{n\times p} \end{aligned} \tag{A2}$$

where $x_k(\tau) \in \mathbb{R}^n$ and $u_k(\tau) \in \mathbb{R}^p, \tau \in [-\delta, 0]$. The time-varying operator $\mathscr{E}_k$ belongs to $\mathscr{S}^{n\times n}$.

The norms $\|x_k\|$ and $\|u_k\|$ are defined by

$$\|x_k\| = \sup_{\tau\in[-\delta,0]} \|x_k(\tau)\|, \quad \|u_k\| = \sup_{\tau\in[-\delta,0]} \|u_k(\tau)\| \tag{A3}$$

where $\|x_k(\tau)\|$ and $\|x_k(\tau)\|$ are the Euclidian norms in $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively.

Lemma A.1

Consider the linear time-varying discrete-time system described by Equation (A1), together with a positive real $\bar{x}_0$ and two bounded sequences of positive reals $\{\eta(k)\}, \{v(k)\}, k \geqslant 0$, such that

$$\begin{aligned} &\eta(k) \leqslant \tilde{\eta} \quad \text{and} \quad \lim_{k\to\infty} \eta(k) = 0 \\ &v(k) \leqslant \bar{v} \quad \text{and} \quad \lim_{k\to\infty} v(k) = 0 \end{aligned} \tag{A4}$$

for some positive $\bar{\eta}$ and $\bar{v}$. If the matrix A that defines the operator $\mathscr{A}$ in (A2) has all eigenvalues inside the open unit circle, then there exists a sequence of positive reals $\{c(k)\}$ satisfying

$$\lim_{k\to\infty} c(k) = 0 \tag{A5}$$

such that $\forall x(0), \mathscr{E}_k, u_k$ bounded by $\bar{x}_0, \eta(k), v(k)$, respectively, i.e.

$$\|x_0\| \leqslant \bar{x}_0, \quad \|\mathscr{E}_k\| \leqslant \eta(k), \quad \|u_k\| \leqslant v(k) \quad \forall k \geqslant 0 \tag{A6}$$

the following inequality is satisfied:

$$\|x(k)\| \leqslant c(k) \quad \forall k \geqslant 0 \tag{A7}$$

Proof

When the input sequence $\{u_k\}$ is identically zero in the linear system (A1), the following inequality holds:

$$\|x_k\| \leqslant (\|A\| + \tilde{\eta})^{k-j} \|x_j\| \quad \forall k \geqslant j \geqslant 0 \tag{A8}$$

and the free state evolution is characterized by a bounded transition operator:

$$x_k = \Phi(k, j)x_j, \quad k \geqslant j \geqslant 0 \tag{A9}$$

It follows that

$$\|\Phi(k, j)\| \leqslant (\|A\| + \bar{\eta})^{k-j} \quad \forall k \geqslant j \geqslant 0 \tag{A10}$$

The first step to prove the lemma is to show that the assumption on the eigenvalues of matrix A implies existence of a positive μ and of a $\lambda \in (0,1)$ such that

$$\|\Phi(k,j)\| \leqslant \mu\lambda^{k-j}, \quad k \geqslant j \geqslant 0 \tag{A11}$$

From the stability of matrix A it follows that there exists a pair of symmetric positive definite matrices P and Q such that

$$A^{\mathrm{T}}QA - Q = -P \quad \text{and} \quad \frac{\lambda_{\min}(P)}{\lambda_{\mathrm{Max}}(Q)} < 1 \tag{A12}$$

The inequality in (A12) is true because $Q \geqslant P$ (note that $Q - P = A^{\mathrm{T}}QA \geqslant 0$), and therefore $\lambda_{\mathrm{Max}}(Q) \geqslant \lambda_{\min}(P)$.

Consider now the positive definite function $V(x_k)$ defined

$$V(x_k) = \sup_{\tau \in [-\delta, 0]} x_k^{\mathrm{T}}(\tau) Q x_k(\tau) \tag{A13}$$

Note that

$$\lambda_{\min}(Q)\|x_k\|^2 \leqslant V(x_k) \leqslant \lambda_{\mathrm{Max}}(Q)\|x_k\|^2 \tag{A14}$$

From the identity

$$x_{k+1}^{\mathrm{T}}(\tau)Qx_{k+1}(\tau) = x_k^{\mathrm{T}}(\tau)A^{\mathrm{T}}QAx_k(\tau) + x_k^{\mathrm{T}}(\tau)(2E_k^{\mathrm{T}}(\tau)QA + E_k^{\mathrm{T}}(\tau)QE_k(\tau))x_k^{\mathrm{T}}(\tau) \tag{A15}$$

after substitution of AQA^{T} with $Q - P$, and considering that, for $x \in \mathbb{R}^n$,

$$-x^{\mathrm{T}}Px \leqslant -\frac{\lambda_{\min}(P)}{\lambda_{\mathrm{Max}}(Q)} x^{\mathrm{T}}Qx, \quad \|x\|^2 \leqslant \frac{1}{\lambda_{\min}(Q)} x^{\mathrm{T}}Qx \tag{A16}$$

it follows that

$$x_{k+1}^{\mathrm{T}}(\tau)Qx_{k+1}(\tau) \leqslant \left(1 - \frac{\lambda_{\min}(P)}{\lambda_{\mathrm{Max}}(Q)} + \frac{2\|E_k(\tau)\| \cdot \|Q\|(\|A\| + \|E_k(\tau)\|)}{\lambda_{\min}(Q)}\right) x_k^{\mathrm{T}}(\tau)Qx_k(\tau) \tag{A17}$$

Taking the supremum over $[-\delta, 0]$

$$V(x_{k+1}) \leqslant \left(1 - \frac{\lambda_{\min}(P)}{\lambda_{\mathrm{Max}}(Q)} + \frac{2\|\mathscr{E}_k\| \cdot \|Q\|(\|A\| + \|\mathscr{E}_k\|)}{\lambda_{\min}(Q)}\right) V(x_k) \tag{A18}$$

By assumptions (A4) and (A6) $\|\mathscr{E}_k\| \to 0$, so that there exists an integer $\bar{k}$ and $\rho \in (0,1)$ such that

$$\frac{2\|\mathscr{E}_k\| \cdot \|Q\|(\|A\| + \|\mathscr{E}_k\|)}{\lambda_{\min}(Q)} \leqslant \rho < \frac{\lambda_{\min}(P)}{\lambda_{\mathrm{Max}}(Q)} \quad \forall k \geqslant \bar{k} \tag{A19}$$

It follows that

$$V(x_{k+1}) \leqslant \sigma V(x_k) \quad \forall k \geqslant \bar{k} \tag{A20}$$

where $\sigma \in (0,1)$ is defined by

$$\sigma = 1 - \frac{\lambda_{\min}(P)}{\lambda_{\mathrm{Max}}(Q)} + \rho \tag{A21}$$

and for $k \geqslant j \geqslant \bar{k}$, we have

$$V(x_k) \leqslant \sigma^{k-j} V(x_j) \tag{A22}$$

and from (A14)

$$\|x_k\| \leqslant \sqrt{\frac{\lambda_{\mathrm{Max}}(Q)}{\lambda_{\min}(Q)}}\,\lambda^{k-j}\|x_j\| \tag{A23}$$

with $\lambda = \sqrt{\sigma} < 1$. Recalling inequality (A9) and defining

$$\mu = \max\left\{\left(\frac{(\|A\| + \bar{\eta})}{\lambda}\right)^{\bar{k}}, \frac{1}{\lambda^{\bar{k}}} - \sqrt{\frac{\lambda_{\mathrm{Max}}(Q)}{\lambda_{\min}(Q)}}\right\} \tag{A24}$$

it is easy to check that

$$\|x_k\| \leqslant \mu\lambda^{k-j}\|x(j)\| \quad \forall k \geqslant j \geqslant 0 \tag{A25}$$

and from this inequality (A11) is proved for all $k \geqslant j \geqslant 0$.

When the input is not identically zero, the state evolution is written

$$x_k = \Phi(k,0)x_0 + \sum_{j=0}^{k-1} \Phi(k, j+1)\mathscr{B}u_j \tag{A26}$$

and therefore

$$\|x_k\| \leqslant \mu\lambda^k\|x_0\| + \sum_{j=0}^{k-1} \mu\lambda^{k-j-1}\|B\| \cdot \|u_j\| \tag{A27}$$

Let

$$c(k) = \mu\lambda^k\bar{x}_0 + \sum_{j=0}^{k-1} \mu\lambda^{k-j-1}\|B\|v(j) \tag{A28}$$

By construction $c(k)$ is such that (A7) holds. It remains to prove that $c(k)$ asymptotically goes to zero, i.e., for any $\varepsilon > 0$ there exists a k_ε such that $\forall k \geqslant k_\varepsilon$ it is $c(k) < \varepsilon$. To this aim, rewrite $c(k)$ setting $k - j - 1 = i$

$$c(k) = \mu\lambda^k\bar{x}_0 + \sum_{i=0}^{k-1} \mu\lambda^i\|B\|v(k-i-1) \tag{A29}$$

and split the summations as follows:

$$c(k) = \mu\lambda^k\bar{x}_0 + \sum_{i=0}^{v_\varepsilon} \mu\lambda^i\|B\|v(k-i-1) + \sum_{i=v_\varepsilon+1}^{k-1} \mu\lambda^i\|B\|v(k-i-1) \tag{A30}$$

The first term in expression (A30) tends to zero (recall that $\lambda = \sqrt{\sigma} \in (0,1)$), therefore there exists $k_{1,\varepsilon}$ such that $\forall k \geqslant k_{1,\varepsilon}$ we have $\mu\lambda^k\bar{x}_0 \leqslant \varepsilon/3$. As for the third term in (A30), we have

$$\sum_{i=v_\varepsilon+1}^{k-1} \mu\lambda^i\|B\|v(k-i-1) \leqslant \sum_{i=v_\varepsilon+1}^{\infty} \mu\lambda^i\|B\|\bar{v} = \frac{\mu\|B\|\bar{v}\lambda}{1-\lambda}\lambda^{v_\varepsilon} \tag{A31}$$

Since $\lambda \in (0,1)$, it is easy to choose v_ε such that

$$\frac{\mu\|B\|\bar{v}\lambda}{1-\lambda}\lambda^{v_\varepsilon} \leqslant \frac{\varepsilon}{3} \tag{A32}$$

As for the second term in (A30), defining

$$\bar{w}(k) = \sup_{i\in[0,v_\varepsilon]} v(k-i-1) \tag{A33}$$

we have

$$\sum_{i=0}^{v_\varepsilon} \mu\lambda^i\|B\|\bar{u}(k-i-1) \leqslant (v_\varepsilon+1)\mu\|B\|\bar{w}(k) \tag{A34}$$

Now, by assumption (A4), $\bar{w}(k)$ goes to zero and therefore there exists $k_{2,\varepsilon}$ such that

$$\sum_{i=0}^{v_\varepsilon} \mu\lambda^i\|B\|v(k-i-1) \leqslant \frac{\varepsilon}{3} \quad \forall k \geqslant k_{2,\varepsilon} \tag{A35}$$

As a result, denoting $k_\varepsilon = \max\{k_{1,\varepsilon}, k_{2,\varepsilon}\}$ we have that

$$c(k) \leqslant \varepsilon \quad \forall k \geqslant k_\varepsilon \tag{A36}$$

and this concludes the proof. □

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