Representation of a Class of MIMO Systems \textit{via} Internally Positive Realization

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In some technological frameworks, such as Charge Routing Networks (CRNs), or fiber-optic filters, only positive state-space realizations of digital signal processing algorithms, such as filters or control laws, can be implemented. On the other hand, the imposition of an a priori positivity constraint on the processing algorithm is a too strong design limitation. For this reason, some authors studied the problem of state-space realization of generic stationary filters through combination of positive systems, in the discrete-time framework. The single-input/single-output (SISO) case has been widely investigated and important results are available in the literature. On the contrary, no specific theoretical results exist for multi-input/multi-output (MIMO) systems. In this paper, the problem of the Internal Positive Realization (IPR) of MIMO systems and filters is formulated and a straightforward method for the construction of IPRs is proposed. The stability properties of the resulting positive realization are also investigated. The method is illustrated on two engineering applications.

Keywords: Positive systems, State-space realization, Linear systems

1. Introduction

A positive system is a system whose state and output evolutions are always nonnegative provided that both the initial state and the input sequence are nonnegative [17, 21]. The design of positive realizations plays a crucial role when coping with the necessity of providing a computing scheme that is implementable in a particular technology. Indeed, there are some cases in which a positive realization design is the only implementable choice. Important examples of digital filter technologies that operate under positivity constraints are the charge coupled devices, such as Charge Routing Networks (CRNs) [10, 11, 14, 16, 22], and the fiber-optic filters [4, 9, 18]. In both these technological frameworks, non-negativity is a consequence of the underlying physical mechanisms of data storage and processing. In these cases, the constraint of positive system may well impair the properties of the designed signal processing algorithm (filter or control law). For instance, the design of positive filters heavily restricts the performances w.r.t. other filters such as Butterworth or Chebyshev, which have no sign limitation on their impulse response, as discussed in [3]. In the literature, it has been shown that approximating a given system by means of a positive realization, in general does not lead to satisfactory results [2]. These drawbacks are even enhanced when the
system implements an algorithm that solves an unconstrained theoretical problem (e.g. optimal control, optimal filtering, etc.).

A way to overcome such a drawback is to design the signal processing scheme without any positive constraints, and to realize it by means of combination of positive systems. The case of single input/single output (SISO) transfer functions with simple poles has been successfully investigated in [6] and [4] in a CRN and in a fiber-optic framework, respectively. The extension to SISO transfer functions with possibly multiple poles is also available in the literature [19, 20]. All the mentioned positive realization schemes are based on the assumption that the input signal is nonnegative, in order to ensure that the entries of the state of the positive subsystems remain nonnegative. Although this condition is motivated by the fact that the input to CRN or to fiber optical filters are intrinsically nonnegative, nevertheless it can be a drawback when the processing of signals whose sign is not definite is demanded. Moreover, even if in principle any of the previous SISO-positive realization schemes can be applied to get a positive realization of \( p \times q \) multi input/multi output (MIMO) filters, by simply realizing \( pq \) independent filters, one for each entry of the transfer matrix (highly redundant realization), it must be noted that no specific realization method has been developed until now for MIMO systems.

This paper proposes a novel methodology to solve the problem of finding the representation of discrete time systems or signal processing schemes, that are indefinite in sign, by means of positive state–space realizations (preliminary results on this subject have been presented in [13]). The first novelty is the introduction of the new concept of **Internally Positive Realization (IPR)** of systems that are, in general, indefinite in sign. Such a concept applies both to state–space representation and to I/O transfer matrix representation of systems. The IPR of a system generalizes the concept of transfer function realization through combination of positive systems given in [6].

The proposed methodology for the construction of IPRs of systems has the following advantages over the existing techniques:

- the proposed construction method is very simple and straightforward: it does not require the numerical solution of optimization problems to compute the matrices of the positive system, as in all other approaches;
- the proposed construction method applies naturally to MIMO systems, without restriction on the multiplicity of poles or eigenvalues, whereas all existing approaches are specifically devised for SISO systems, and many of them are restricted to simple poles or eigenvalues;
- differently from all other approaches, that deal with nonnegative input sequences, the proposed construction method applies to systems forced by inputs that are indefinite in sign.

As in [4, 6, 20], an upper bound on the order of the system is given: the dimension of the IPR is always the double of the dimension of the system to be positively realized, regardless to the multiplicity and position of its poles.

When applying the proposed methodology to a stable system, it may happen that the resulting positive realization is unstable. Such a limitation is investigated in the paper, and a necessary and sufficient condition is given which ensures the stability of the positive realization.

The paper is organized as follows. In Section 2, the concept of IPR of systems is introduced, and the IPR realization algorithm is presented. In Section 3 it is shown that in some circumstances a reduced order IPR can be constructed. The stability analysis of the IPR made up with the proposed method is worked out in Section 4, while in Section 5 a procedure for the construction of stable IPR of state observers is given. In Section 6, two examples of engineering applications are worked out to illustrate the method. Conclusions follow.

### 2. Internally Positive Representation of Systems

Throughout the paper, \( I_n \) denotes the identity matrix of order \( n \). A matrix/vector is called **nonnegative** if it is componentwise nonnegative, i.e. when all its components are nonnegative. The symbol \( \mathbb{R}^+ \) denotes the interval \([0, +\infty)\), and, more in general, \( \mathbb{R}^+_n \) denotes the positive orthant of \( \mathbb{R}^n \), \( Z \) is the set of integer numbers.

Given a matrix \( M \), the symbol \( M^+ \) denotes its **positive part**, the nonnegative matrix whose components are defined as:

\[
[M^+]_{ij} = \begin{cases} 
| M_{ij} | & \text{if } M_{ij} \geq 0, \\
0 & \text{if } M_{ij} < 0.
\end{cases}
\]

The **negative part** of \( M \) is the nonnegative matrix \( M^- \) defined as \( M^- = (-M)^+ \), equivalently defined as \( M^- = M^+ - M \). Then, by construction, \( M = M^+ - M^- \).

The **absolute value** of \( M \) is the nonnegative matrix \( |M| = M^+ + M^- \).
In this paper, following the abstract definition of systems [23, 24], a real, causal, stationary, finite-dimensional, discrete-time linear system $S$ is identified by a set $S = \{A,B,C,D;X,U,Y\}$, where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{q \times n}$ and $D \in \mathbb{R}^{q \times p}$ are the system matrices, $X \subseteq \mathbb{R}^n$ is the state-space, and $U \subseteq \mathbb{R}^p$ and $Y \subseteq \mathbb{R}^q$ are the input and output spaces, respectively, such that for any input sequence starting at a given $t_0 \in \mathbb{Z}$, $\{u(t) \in U : t \geq t_0\}$, and for any initial state $x(t_0) \in X$, the state and output sequences for $t \geq t_0$ are such that

$$x(t + 1) = Ax(t) + Bu(t), \quad x(t) \in X$$
$$y(t) = Cx(t) + Du(t), \quad y(t) \in Y. \quad (2)$$

The set $S$ is a representation of the abstract linear system $\Sigma$, but for simplicity throughout the paper we will often refer to $S$ as the system itself (from now on, when referring to linear systems, the attributes real, causal, time-invariant, finite-dimensional and discrete-time will be omitted, for brevity).

According to the previous definition, and following the nomenclature defined in [15], an internally positive linear system is identified by a set $\Sigma$ where the system matrices $(A,B,C,D)$ are all nonnegative, and the three spaces $X, U, Y$ are all subsets of the positive orthants of $\mathbb{R}^n, \mathbb{R}^p, \mathbb{R}^q$, respectively, i.e. $X \subseteq \mathbb{R}^n_+, U \subseteq \mathbb{R}^p_+$ and $Y \subseteq \mathbb{R}^q_+$. Let $U^{(t_0,\infty)}$ denote the set of all input sequences starting at $t_0 \in \mathbb{Z}$, with $u(t) \in U, \forall t \geq t_0$, and let $u^{(t_0,\infty)}$ denote a sequence in $U^{(t_0,\infty)}$.

**Definition 1:** An Internally Positive Realization (IPR) of a linear system $S = \{A,B,C,D;X,U,Y\}$ is a positive linear system $\tilde{S} = \{A,B,C,D;\tilde{X},\tilde{U},\tilde{Y}\}$ together with four transformations $\{\tilde{T}_\alpha, \tilde{T}_u, \tilde{T}_y\}$

$$\tilde{T}_\alpha : \tilde{X} \rightarrow \tilde{X}, \quad \tilde{T}_u : \tilde{U} \rightarrow \tilde{U}, \quad \tilde{T}_y : \tilde{Y} \rightarrow \tilde{Y}$$

such that $\forall \tilde{t}_0 \in \mathbb{Z}, \forall (\tilde{x}(\tilde{t}_0), u^{(\tilde{t}_0,\infty)}) \in X \times U^{(\tilde{t}_0,\infty)}$, the following implication holds:

$$\{ \tilde{x}(\tilde{t}_0) = \tilde{T}_\alpha^{\tilde{t}_0}(x(0)) \}$$
$$\{ \tilde{u}(\tilde{t}_0) = \tilde{T}_u(u(t)), \forall \tilde{t} \geq \tilde{t}_0 \}$$
$$\Rightarrow \{ \tilde{x}(\tilde{t}) = \tilde{T}_\alpha^{\tilde{t}}(\tilde{x}(0)) \}, \tilde{y}(\tilde{t}) = \tilde{T}_y(\tilde{y}(\tilde{t})), \forall \tilde{t} \geq 0.$$  \quad (4)

where $(x(t),u(t),y(t)) \in X \times U \times Y$ denote the state, input, and output of system $S$, and $(\tilde{x}(\tilde{t}),\tilde{u}(\tilde{t}),\tilde{y}(\tilde{t})) \in \tilde{X} \times \tilde{U} \times \tilde{Y}$ the state, input, and output of $\tilde{S}$.

**Remark 1:** The superscripts $f$ and $b$ in the transformations $\tilde{T}_\alpha$ and $\tilde{T}_u$ stand for forward and backward. Note that, for consistency, the backward map $\tilde{T}_u$ must be the left-inverse of the forward map $\tilde{T}_\alpha$ in that, by applying the implication (4) at the initial time $\tilde{t}_0$:

$$x(\tilde{t}_0) = \tilde{T}_\alpha^{\tilde{t}_0}(x(0)) = \tilde{T}_u^{\tilde{t}_0}(\tilde{x}(0)).$$
$$\forall x(\tilde{t}_0) \in X.$$  \quad (5)

The concept of IPR can be also extended to I/O representations of linear systems, like transfer matrices. It is known that a proper rational transfer matrix $H(z) \in \mathbb{C}^{q \times p}$ characterizes the I/O behavior of a system because the $Z$-transforms of input and output sequences starting at $t_0 = 0$, when $x(0) = 0$, are related by the equation

$$y(z) = H(z)u(z).$$  \quad (6)

**Definition 2:** Given a linear system with input space $U \subseteq \mathbb{R}^p$ and output space $Y \subseteq \mathbb{R}^q$, represented by a proper rational transfer matrix $H(z) \in \mathbb{C}^{q \times p}$, and Internally Positive Realization (IPR) of $H(z)$ is a positive system $\tilde{S} = \{A,B,C,D;\tilde{X},\tilde{U},\tilde{Y}\}$, together with two maps $\{\tilde{T}_u, \tilde{T}_y\}$

$$\tilde{T}_u : \tilde{U} \rightarrow \tilde{U}, \quad \tilde{T}_y : \tilde{Y} \rightarrow \tilde{Y}.$$  \quad (7)

**Fig. 1.** Block diagram of an Internally Positive Realization.
such that, for any input sequence \( u(0, \infty) \in U^{[0, \infty)} \) it is:
\[
\begin{cases}
\hat{x}(0) = 0 \\
\hat{u}(t) = T_u(u(t)), \quad \forall t \geq 0,
\end{cases}
\]
\[
\Rightarrow \begin{cases}
y(t) = T_y(\hat{y}(t)), \\
\forall t \geq 0,
\end{cases}
\]
(8)
where \((u(t), y(t)) \in U \times Y\) denotes the input and output of the system represented by \(H(z)\), and \((\hat{u}(t), \hat{y}(t)) \in \hat{U} \times \hat{Y}\) the input and output of \(\hat{S}\).

**Remark 2:** The approaches presented in \([4, 6, 20]\) that provide state-space realizations of a given transfer function \(H(z)\) by means of the differences of two positive systems, can be recast into the IPR framework. First, it must be noted that in these approaches only the SISO case is considered, and that the input sequence is assumed to be nonnegative, in order to have nonnegative state variables. Then, the input space is \(U = \mathbb{R}^+\) and the output space is \(Y = \mathbb{R}\). In the cited approaches, the transfer function \(H(z)\) is decomposed as \(H(z) = H_1(z) - H_2(z)\), in such a way that both \(H_1(z)\) and \(H_2(z)\) admit positive state–space representations, with \(H_2(z)\) of order one (one real pole). Let \(n\) be the number of poles of \(H(z)\), and \(N \geq n\) the number of poles of \(H_1(z)\). In this case, once a pair of positive state–space realizations is found for \(H_1(z)\) and \(H_2(z)\), named \(S_1 = \{A_1, B_1, C_1, D_1; \mathbb{R}^{n_1}_+; \mathbb{R}^+_+, \mathbb{R}^+_+\}\) and \(S_2 = \{A_2, B_2, C_2, D_2; \mathbb{R}^{n_2}_+; \mathbb{R}^+_+, \mathbb{R}^+_+\}\), respectively, then an IPR of \(H(z)\) is given by \(\hat{S} = \{\hat{A}, \hat{B}, \hat{C}, \hat{D}; \mathbb{R}^{n_1+1}_+; \mathbb{R}^+_+, \mathbb{R}^+_+\}\) with:
\[
\hat{A} = \begin{bmatrix} A_1 & O_{n \times 1} \\ O_{1 \times N} & A_2 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},
\]
\[
\hat{C} = \begin{bmatrix} C_1 \\ O_{1 \times N} \\ C_2 \end{bmatrix}, \quad \hat{D} = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix},
\]
(9)
together with the following two transformations:
\[
T_u : \mathbb{R}^+ \mapsto \mathbb{R}^+, \quad T_u(u) = u, \text{(identity)}
\]
\[
T_y : \mathbb{R}^+ \mapsto \mathbb{R}, \quad T_y(y) = [1 - 1]y.
\]
(10)

All cited papers provide a bound on the order \(N\) of the positive realization of \(H_1(z)\). Such a bound depends on the position of the poles of \(H(z)\) in the complex plane, and the computation of the positive realization of \(S_1\) is made through the numerical solution of suitable optimization problems. It must be noted that no paper addresses the issue of minimality of the representation in general. Conditions for minimality in the special case of positive real poles have been investigated in \([7, 19]\).

The following theorem states how to construct a positive IPR for any given linear system.

**Theorem 1:** Consider a linear system \(S = \{A, B, C, D; \mathbb{R}^n, \mathbb{R}, \mathbb{R}^d\}\). The positive system \(\hat{S} = \{\hat{A}, \hat{B}, \hat{C}, \hat{D}; \mathbb{R}^{n_1+1}_+; \mathbb{R}^+_+, \mathbb{R}^+_+\}\) with matrices:
\[
A = \begin{bmatrix} A^+ & A^- \\ A^- & A^+ \end{bmatrix}, \quad B = \begin{bmatrix} B^+ & B^- \\ B^- & B^+ \end{bmatrix},
\]
\[
C = \begin{bmatrix} C^+ & C^- \\ C^- & C^+ \end{bmatrix}, \quad D = \begin{bmatrix} D^+ & D^- \\ D^- & D^+ \end{bmatrix},
\]
(11)
together with the four transformations:
\[
T_x^f : \mathbb{R}^n \mapsto \mathbb{R}^{n_1+1}_+, \quad T_x(y) = \begin{bmatrix} x^+ \\ x^- \end{bmatrix},
\]
(12)
\[
T_y^f : \mathbb{R}^n \mapsto \mathbb{R}^d, \quad T_y^f(x) = [I_n - I_d]x,
\]
(13)
\[
T_u : \mathbb{R}^n \mapsto \mathbb{R}^{n_1}_+, \quad T_u(u) = \begin{bmatrix} u^+ \\ u^- \end{bmatrix},
\]
(14)
\[
T_y : \mathbb{R}^n \mapsto \mathbb{R}^d, \quad T_y(y) = [I_n - I_d]y.
\]
(15)
defines an Internally Positive Realization of \(S\).

**Proof:** The proof is achieved by proving the implication (4) of Definition 1, using the matrices defined in (11) and the transformations defined in (12)–(15). Then, by using the definitions (12)–(15), the following implication must be proved
\[
\begin{cases}
\dot{X}(t_0) = \begin{bmatrix} x^+(t_0) \\ x^-(t_0) \end{bmatrix}, \quad \dot{U}(t) = \begin{bmatrix} u^+(t) \\ u^-(t) \end{bmatrix}, \quad \forall t \geq t_0 \\
y(t) = [I_n - I_d]X(t), \\
\forall t \geq t_0
\end{cases}
\]
(16)
where \((x(t), u(t), y(t)) \in \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^d\) denotes the aggregate of the state, input, and output of system \(S\), and, similarly, \((\dot{X}(t), \dot{U}(t), \dot{Y}(t)) \in \mathbb{R}^{n_1}_+ \times \mathbb{R}^d \times \mathbb{R}^d\) denotes the aggregate of state, input, and output of \(\hat{S}\). The first identity implied in (16), the one involving the states \(x(t)\) and \(\dot{X}(t)\), is proven by induction. The initial step of the induction, \(x(t_0) = T_x^f(\dot{X}(t_0))\), directly follows from the assumption \(x(t_0) = T_x^f(\dot{X}(t_0))\) and from the consistency property of \(T_x^f\) and \(T_y^f\). Consider the following subvectors of \(\dot{X}(t)\) and \(\dot{Y}(t)\):
\[
\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix},
\]
\[
\begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix},
\]
(17)
The one-step state transition of the positive system can be written in terms of $(\mathcal{X}_1(t), \mathcal{X}_2(t))$ and $(u^+(t), u^-(t))$ as

\[
\begin{align*}
\mathcal{X}_1(t+1) &= A^+ \mathcal{X}_1(t) + A^- \mathcal{X}_2(t) + B^+ u^+(t) + B^- u^-(t), \\
\mathcal{X}_2(t+1) &= A^- \mathcal{X}_1(t) + A^+ \mathcal{X}_2(t) + B^+ u^+(t) + B^- u^-(t).
\end{align*}
\]

Then, if $x(t) = T_x^b(\mathcal{X}(t)) = \mathcal{X}_1(t) - \mathcal{X}_2(t)$ is true at a given time instant $t$, then

\[
T_x^b(\mathcal{X}(t+1)) = \mathcal{X}_1(t+1) - \mathcal{X}_2(t+1)
= (A^+ - A^-)(\mathcal{X}_1(t) - \mathcal{X}_2(t)) + (B^+ - B^-)(u^+(t) - u^-(t))
= Ax(t) + Bu(t) = x(t+1)
\]

Thus, the first identity implied in (16) is proved. The second identity is proved below:

\[
T_y^b(y(t)) = y_1(t) - y_2(t)
= C^+ \mathcal{X}_1(t) + C^- \mathcal{X}_2(t) + D^+ u^+(t) + D^- u^-(t)
= (C^+ - C^-)(\mathcal{X}_1(t) - \mathcal{X}_2(t)) + (D^+ - D^-)(u^+(t) - u^-(t))
= Cx(t) + Du(t) = y(t).
\]

A block diagram describing the proposed IPR scheme is reported in Fig. 2.

**Remark 3:** The IPR construction method of Theorem 1 provides a positive system whose size is twice the size of the original system. We stress that this dimension is not proved to be minimal, and reduction procedures could be applied. However, note that standard reduction procedures, aimed to the elimination of unobservable and uncontrollable dynamics, in general do not guarantee the positivity of the reduced system.

Theorem 1 also provides a mean for the construction of IPRs for transfer matrices, as it formalized in the following Corollary:

\[
\begin{align*}
\mathcal{X}(t_0) &\xrightarrow{T_x^f} \mathcal{X}(t) \\
\mathcal{X}(t) &\xrightarrow{T_x^b} \mathcal{X}(t) \\
\mathcal{X}(t) &\xrightarrow{T_y^b} y(t)
\end{align*}
\]

**Corollary 2:** Consider a transfer matrix $H(z) \in \mathbb{C}^{n \times p}$, with input space $U = \mathbb{R}^p$ and output space $Y = \mathbb{R}^q$. The IPR construction presented in Theorem 1 of any zero-state representation $S = \{A, B, C, D; \mathbb{R}^p, \mathbb{R}^p, \mathbb{R}^q\}$ of $H(z)$, without considering the state transformations $T_x^f$ and $T_x^b$ given in (12) and (13), is an IPR for $H(z)$.

**Remark 4:** It must be noted that, differently from the existing approaches [4, 6, 20], the positive representation of $H(z)$ given in Corollary 2 is readily computed, without solving any optimization problem, both in the SISO and in the MIMO case. The state–space dimension of the IPR is twice the order of the system.
Remark 5: In applications, when an IPR of a transfer matrix \( H(\alpha) \) is needed, in order to have an IPR of the lowest dimension, it is important to consider a minimal zero-state representation of \( H(\alpha) \) in the IPR construction of Theorem 1.

Remark 6: Note that the IPR construction of Theorem 1 applies also to time-varying and switching systems.

3. IPR for Systems with Nonnegative Inputs

In order to provide a realization method of [6, 4, 20], let us consider the case of systems described by transfer matrices and forced by nonnegative input sequences, i.e., \( u(t) \in U \subseteq \mathbb{R}_+^n \). Under these conditions, a reduction of the dimension of the IPR may be possible. Before getting into the details, consider the following notation. Let \( \lambda_i \) be an eigenvalue associated to a matrix \( M \), and \( \eta_h = \{ \eta_1, \ldots, \eta_{m_h} \} \) be the multi-index associated to \( \lambda_i \), with \( m_h \) the number of chains of generalized eigenvectors associated to \( \lambda_i \), and \( \eta_h \) is the length of the \( h \)-th chain, \( 1 \leq h \leq m_h \), related to \( \lambda_i \). The modulus \( |\eta_h| \) of the multi-index \( \eta_h \), defined as \( |\eta_h| = \sum_{h=1}^{m_h} \eta_{h_i} \), gives back the algebraic multiplicity of \( \lambda_i \). Moreover, we denote with \( J_{\eta_h}(\lambda_i) \) the Jordan block of order \( \eta_h \) associated to \( \lambda_i \), and with:

\[
J_{\eta_h}(\lambda_i) = \text{diag}\{J_{\eta_1}(\lambda_i), \ldots, J_{\eta_{m_h}}(\lambda_i)\},
\]

the complex Jordan block-diagonal matrix of order \( |\eta_h| \) associated to \( \lambda_i \). The following Lemma provides a coordinate transformation for a class of matrices, which will be useful in the sequel.

Lemma 3: Let \( M \) be a square matrix of order \( n \) with \( \alpha \) of complex-conjugate eigenvalues, \( \lambda_1 = \alpha + \beta j \) and \( \lambda_2 = \alpha - \beta j \), both of them associated to the multi-index \( \tilde{\eta} = \{ \eta_1, \ldots, \eta_m \} \), with \( n = 2|\tilde{\eta}| \). Then, it is known that \( M \) is similar to the following Jordan block-diagonal form, [24]:

\[
J = \text{diag}\{J_{\eta_1}(\lambda_1), J_{\eta_2}(\lambda_2)\},
\]

with \( J_{\eta_k}(\lambda_k) \) the Jordan block-diagonal matrix associated to \( \lambda_k \), \( k = 1, 2 \). Then, there exists a coordinate transformation matrix \( T \in \mathbb{R}^{m \times n} \) such that the Jordan block-diagonal matrix \( M \) is transformed into the following block matrix (Real Jordan Form):

\[
J(\alpha, \beta) = T JT^{-1} = \begin{bmatrix}
J_{\eta_1}(\alpha) & \beta I_{|\eta_1|} \\
-\beta I_{|\eta_2|} & J_{\eta_2}(\alpha)
\end{bmatrix}.
\]

Proof: Consider the following coordinate transformation:

\[
T = \begin{bmatrix}
I_{|\eta_1|} & J_{\eta_1}(\alpha) \\
-J_{\eta_2}(\beta) & I_{|\eta_2|}
\end{bmatrix},
\]

\[
T^{-1} = \frac{1}{2} \begin{bmatrix}
I_{|\eta_1|} & -J_{\eta_1}(\alpha) \\
-J_{\eta_2}(\beta) & I_{|\eta_2|}
\end{bmatrix}.
\]

Then matrix \( T JT^{-1} \) is given by:

\[
\frac{1}{2} \begin{bmatrix}
J_{\eta_1}(\lambda_1) + J_{\eta_2}(\lambda_2) & j(J_{\eta_1}(\lambda_2) - J_{\eta_2}(\lambda_1)) \\
-J(J_{\eta_2}(\lambda_2) - J_{\eta_1}(\lambda_1)) & J_{\eta_1}(\lambda_1) + J_{\eta_2}(\lambda_2)
\end{bmatrix}.
\]

Note that, as \( J_{\eta_i}(\lambda_i) \) is the Jordan block-diagonal matrix associated to \( \lambda_i \), it comes that:

\[
J_{\eta_1}(\lambda_1) + J_{\eta_2}(\lambda_2) = 2J_{\eta_1}(\alpha),
\]

\[
j(J_{\eta_2}(\lambda_2) - J_{\eta_1}(\lambda_1)) = 2J_{\eta_1}(\beta).
\]

Then, matrix (23) is obtained.

Consider a linear system \( S \), let \( \nu \) be the number of real eigenvalues of \( A \) and \( \mu \) the number of pairs of complex eigenvalues of \( A \). According to Lemma 3, assume that matrix \( A \) is in the following Real Jordan Form:

\[
A = \text{diag}\{J_{\eta_1}, \ldots, J_{\eta_\nu}, J_{\sigma_1}, \ldots, J_{\sigma_\mu}\},
\]

where \( J_{\eta_i} \) and \( J_{\sigma_k} \) are the Jordan block-diagonal matrices associated to the real eigenvalues \( \lambda_i \) with the multi-indeces \( \tilde{\eta_i} = \{ \eta_{1_i}, \ldots, \eta_{m_i} \} \), and \( J_{\sigma_k} \) are the block-matrices of order \( 2|\sigma_k| \) of the type of (23), associated to complex pairs of eigenvalues \( \alpha_k \pm j \beta_k \) with the multi-index \( \tilde{\sigma_k} = \{ \sigma_{1_k}, \ldots, \sigma_{m_k} \} \), \( k = 1, \ldots, \mu \). Then, \( \nu + \mu \) subsystems in parallel can be singled out, and the following representation of system (2) can be written

\[
x_i(t+1) = J_{\eta_i}(x_i(t)) + B_i u(t), \quad i = 1, \ldots, \nu,
\]

\[
x_{\sigma+k}(t+1) = J_{\sigma_k}(x_{\sigma+k}(t)) + B_{\sigma+k} u(t), \quad k = 1, \ldots, \mu,
\]

\[
y(t) = \sum_{i=1}^{\nu} C_i x_i(t) + \sum_{k=1}^{\mu} C_{\sigma+k} x_{\sigma+k}(t) + Du(t),
\]

with \( x_i(t) \in \mathbb{R}^{|\eta_i|}, \quad i = 1, \ldots, \nu \) and \( x_{\sigma+k}(t) \in \mathbb{R}^{2|\sigma_k|}, \quad k = 1, \ldots, \mu \). If for some \( i \) the Jordan block \( J_{\eta_i} \) is associated to nonnegative real eigenvalues, and if the corresponding input/state and state/output matrices \( B_i \) and \( C_i \) are nonnegative, then the subsystem \( \{ J_{\eta_i}, B_i, C_i, 0, \mathbb{R}^+, \mathbb{R}^{|\eta_i|}, \mathbb{R}^+ \} \) is a positive system, and therefore there is no reason for computing an IPR. Assume that, without loss of generality, the first \( h \) quadruples \( \{ J_{\eta_i}, B_i, C_i, 0 \}, \quad i = 1, \ldots, h \leq \nu \), are made of
nonnegative matrices. Then, the whole state can be partitioned as:

\[
\begin{align*}
\xi(t) &= \begin{bmatrix} \xi_1^T(t) & \xi_2^T(t) \end{bmatrix}^T \in \mathbb{R}^n, \\
\xi_1 &= \begin{bmatrix} x_1^T & \cdots & x_{\beta}^T \end{bmatrix}^T \in \mathbb{R}^{n_1}, \\
\xi_2 &= \begin{bmatrix} x_{\beta+1}^T & \cdots & x_{\gamma}^T & x_{\gamma+1}^T & \cdots & x_{\gamma+\mu}^T \end{bmatrix}^T \in \mathbb{R}^{n_2},
\end{align*}
\]

where \( n_1 = \sum_{i=1}^{\beta} |\bar{p}_i| \), and \( n_2 = n - n_1 \), and the system can be decomposed into the parallel of the following two subsystems

\[
S_1 : \begin{cases} \\
\xi_1(t + 1) &= A_1 \xi_1(t) + B_1 u(t), \\
y_1(t) &= C_1 \xi_1(t),
\end{cases}
\]

\[
S_2 : \begin{cases} \\
\xi_2(t + 1) &= \bar{A}_2 \xi_2(t) + \bar{B}_2 u(t), \\
y_2(t) &= \bar{C}_2 \xi_2(t) + D u(t),
\end{cases}
\]  

where

\[
A_1 = \text{diag}\{J_{\bar{p}_1}, \ldots, J_{\bar{p}_\beta}\}, \\
B_1 = \begin{bmatrix} B_{\bar{p}_1}^T & \cdots & B_{\bar{p}_\beta}^T \end{bmatrix}^T, \\
C_1 = [C_1 \cdots C_\gamma],
\]

\[
\bar{A}_2 = \text{diag}\{J_{\bar{p}_{\beta+1}}, \ldots, J_{\bar{p}_\gamma}, J_{\bar{q}_1}, \ldots, J_{\bar{q}_\nu}\}, \\
\bar{B}_2 = \begin{bmatrix} B_{\bar{p}_{\beta+1}}^T & \cdots & B_{\bar{p}_\gamma}^T & B_{\bar{q}_1}^T & \cdots & B_{\bar{q}_\nu}^T & \cdots & B_{\bar{q}_{\mu}}^T \end{bmatrix}^T, \\
\bar{C}_2 = [C_{\bar{p}_{\beta+1}} \cdots C_{\bar{p}_\gamma} \cdots C_{\bar{q}_1} \cdots C_{\bar{q}_\nu} \cdots C_{\bar{q}_{\mu}}].
\]

The output of the original system \( y(t) \) is computed as

\[
y(t) = y_1(t) + y_2(t),
\]

where \( y_1(t) \) is the output of the positive subsystem \( S_1 \), of dimension \( n_1 \), while \( y_2(t) \) is the output of the IPR of the subsystem \( S_2 \), of dimension \( 2(n - n_1) \), realized according to Theorem 1. Therefore, the complete IPR (see Fig. 3) has dimension \( n_1 + 2(n - n_1) = 2n - n_1 \). This number will be used in Section 6, where some aspects of the proposed approach are compared with the existing approaches for Positive Realization of digital filters.

A more general realization theorem for systems with nonnegative inputs is the following:

**Theorem 4:** Consider a linear system, with nonnegative input space, \( S = \{A, B, C, D, \mathbb{R}^{n_1}_{+} \times \mathbb{R}^{n_2}, \mathbb{R}^{p}, \mathbb{R}^{q}\} \), such that the system matrices have the following block structure

\[
A = \begin{bmatrix} A_1 & 0 \\
0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\
B_2 \end{bmatrix},
\]

\[
C = [C_1 \cdots C_2],
\]

where \( A_1 \in \mathbb{R}^{n_1}_{+} \times \mathbb{R}^{n_1}_{+}, B_1 \in \mathbb{R}^{n_1}_{+} \times \mathbb{R}^{p} \), and \( C_1 \in \mathbb{R}^{q \times n_1} \), while no sign restriction is given on \( A_2 \in \mathbb{R}^{n_2} \times \mathbb{R}^{n_2}_{+}, B_2 \in \mathbb{R}^{n_2} \times \mathbb{R}^{p} \), \( C_2 \in \mathbb{R}^{q \times n_2} \), and \( D \in \mathbb{R}^{q \times q} \).

Then, the positive system \( S = \{A, B, C, D, \mathbb{R}^{n_1}_{+} \times \mathbb{R}^{n_2}_{+}, \mathbb{R}^{p}, \mathbb{R}^{q}\} \) with matrices:

\[
\begin{align*}
S_1 \quad \text{dim: } n_1 \\
\text{IPR of } S_2 \quad \text{dim: } 2(n - n_1)
\end{align*}
\]

\[
\begin{array}{c}
\text{Fig. 3. Reduced order positive realization, when } u(t) \in \mathbb{R}_{+}.
\end{array}
\]
it has been shown that the positive realization can be always reduced in a way that the dominant eigenvalue coincides with a pole of $H(z)$.

In the following, the symbol $\sigma(M)$ is used to denote the spectrum of a matrix $M$, i.e. the set of the eigenvalues of $M$. The symbol $\Gamma$ is used to denote the open unit disk centered at the origin in the complex plane. Moreover, following the notation in [6], let $P_j$ denote the interior of the square inscribed in $\Gamma$, with the four corners in the points $1, j, -1, -j$.

**Lemma 5:** Given a system $S = \{A, B, C, D; X, U, Y\}$, the construction of Theorem 1 provides a stable IPR if and only if both $\sigma(A)$ and $\sigma(|A|)$ belong to $\Gamma$.

**Proof:** Note that the IPR of Theorem 1 is stable if and only if $\sigma(A) \subset \Gamma$, where $A$ is the matrix defined in (11). Consider the nonsingular matrix

$$T = \begin{bmatrix} I_n & -I_n \\ I_n & I_n \end{bmatrix}. \quad (44)$$

It is such that

$$TAT^{-1} = \begin{bmatrix} A^+ - A^- & O_{n \times n} \\ O_{n \times n} & A^+ + A^- \end{bmatrix}, \quad (45)$$

From the invariance of the spectrum under similarity transformations it follows

$$\sigma(A) = \sigma(TAT^{-1}) = \sigma(A) \cup \sigma(|A|), \quad (46)$$

from which it follows that $\sigma(A) \in \Gamma$ if and only if both $\sigma(A) \in \Gamma$ and $\sigma(|A|) \in \Gamma$.

**Remark 7:** It must be stressed that, given a square matrix $A$ and a nonsingular matrix $Q$, in general

$$\sigma(|A|) \neq \sigma(|QAQ^{-1}|). \quad (47)$$

It follows that the choice of a particular representation of a system plays a crucial role in the stability of the IPR.

For a stability analysis of the IPR of Theorem 1 it is useful to consider systems represented in a form in which the matrix $A$ has the Real Jordan block-diagonal structure (27). This representation allows to give a necessary and sufficient condition for the stability of the IPR.

**Definition 3:** A $J$-class IPR is the IPR provided by the construction of Theorem 1, when the original representation of the system is in the Real Jordan form (27).
Theorem 6: Assume to have a stable realization of the type (27). Then, the J-class IPR provided by eq.'s (11) is stable if and only if the eigenvalues of matrix $A$ belong to $P_A$.

Proof: Let $\lambda_i, i = 1, \ldots, n$, denote the eigenvalues of $A$. The assumption $\lambda_i \in P_A$ can also be written as $|R(\lambda_i)| + |I(\lambda_i)| < 1$. Let $\lambda_h, h = 1, \ldots, \nu$, denote the real eigenvalues and $\alpha_k \pm j\beta_k, k = 1, \ldots, \mu$ denote the pairs of complex conjugate eigenvalues, so that the condition $\lambda_i \in P_A$ can be written as

$$|\lambda_h| < 1, \quad h = 1, \ldots, \nu \quad |\alpha_k + j\beta_k| < 1, \quad k = 1, \ldots, \mu.$$  \hfill (48)

Given the block diagonal structure of (27), it follows:

$$\sigma (|A|) = \left( \bigcup_{h=1}^{\nu} \sigma (|J_{\lambda_h}|) \right) \cup \left( \bigcup_{k=1}^{\mu} \sigma (|J_{\alpha_k}|) \right) .$$  \hfill (49)

First, consider the matrices $|J_{\lambda_h}|$. Thanks to the triangular structure of $|J_{\lambda_h}|$, see (21), with terms $|\lambda_h|$ on the diagonal, it follows that $\sigma (|J_{\lambda_h}|) = \{ |\lambda_h| \}$. Therefore, the matrices $|J_{\lambda_h}|$ are stable if and only if $|\lambda_h| < 1$, for $h = 1, \ldots, \nu$. Now consider the matrices $|J_{\alpha_k}|$, associated to complex eigenvalues of the matrix $A$. According to (23):

$$|J_{\alpha_k}| = \begin{bmatrix} |J_{\alpha_k}| & |J_{\alpha_k}| \\ |J_{\alpha_k}| & |J_{\alpha_k}| \end{bmatrix} .$$  \hfill (50)

The eigenvalues of $|J_{\alpha_k}|$ can be computed applying the following formula [12]:

$$\det \begin{bmatrix} U & V \\ W & Z \end{bmatrix} = \det(U) \cdot \det(Z - MU^{-1}V) .$$  \hfill (51)

in the case where $U = Z = \lambda I_{|\alpha_k|} - |J_{\alpha_k}|$ and $V = W = -|J_{\alpha_k}|$. It follows:

that means, the eigenvalues of $|J_{\alpha_k}|$ are equal to

$$\det (\lambda I_{|\alpha_k|} - |J_{\alpha_k}|) = \det (\lambda I_{|\alpha_k|} - |J_{\alpha_k}|) \cdot \det \left( (\lambda I_{|\alpha_k|} - |J_{\alpha_k}|) - \beta_k^2 I_{|\alpha_k|} \right) \left( (\lambda I_{|\alpha_k|} - |J_{\alpha_k}|) - \beta_k^2 I_{|\alpha_k|} \right)^{-1} \hfill (52)$$

$|\alpha_k| \pm j\beta_k$, each one with algebraic multiplicity $|\alpha_k|$. A necessary and sufficient condition for $|\alpha_k| \pm j\beta_k$ to be in the unit circle is that $\alpha_k \pm j\beta_k \in P_A$.

A consequence of the previous theorem is that the dominant eigenvalue of $A$ is the largest among the $n$ sums $|R(\lambda_i)| + |I(\lambda_i)|$. It can be proven that, in case of a pair of complex poles, the modes associated to the (possibly unstable) dominant eigenvalue of a J-class IPR, are observable, and therefore no stabilizing reduction is possible. For simplicity, without loss of generality, the proof is reported only in the case of a unique pair of complex eigenvalues.

Example 1: Consider a pair $(A, C)$, where $A \in \mathbb{R}^{2 \times 2}$, $C \in \mathbb{R}^{2 \times 2}$, where

$$A = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}, \quad \alpha, \beta > 0 , \hfill (53)$$

and $C \neq 0$. Consider the pair $(A, C)$ given in (11). Then, the dominant eigenvalue of $A$ is $\lambda = \alpha + \beta$. Moreover, the natural mode associated to $\lambda$ is C-observable.

Proof: The eigenvalues of matrix $A$ are $\alpha \pm j\beta$ and, as shown in Theorem 6, the eigenvalues of $|A|$ are $\alpha \pm j\beta$. Thus $\sigma (A) = \sigma (A) \cup \sigma (|A|) = \{ \alpha \pm j\beta, \alpha \pm j\beta \}$, so that $\lambda = \alpha + \beta$ is the dominant eigenvalue. The proof that the mode associated to $\lambda$ is C-observable can be done by showing that $Cu \neq 0$, where $u$ is the eigenvector associated to $\lambda$. Note that the only assumption on the pair $(A, C)$ is that $C \neq 0$ (non trivial C). This implies that $C \neq 0$. The explicit form of matrix $A$, as given in (11), is

$$A = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & \beta & 0 \\ 0 & 0 & \alpha & \beta \\ \beta & 0 & 0 & \alpha \end{bmatrix} , \hfill (54)$$

while $\tilde{\lambda} I_4 - A$, with $\tilde{\lambda} = \alpha + \beta$, is

$$\tilde{\lambda} I_4 - A = \begin{bmatrix} \beta & -\beta & 0 & 0 \\ 0 & \beta & -\beta & 0 \\ 0 & 0 & \beta & -\beta \\ -\beta & 0 & 0 & \beta \end{bmatrix} . \hfill (55)$$

It is easy to verify that the eigenvector $u$ associated to $\lambda$ is $u = [1 \ 1 \ 1 \ 1]^T$. As $C$ is nonnegative by construction, then necessarily $C \neq 0$ implies $Cu \neq 0$. \hfill \blacksquare

Remark 8: Note that the stability analysis of IPRs worked out in this section is limited only to the case of stationary systems (it is well-known that eigenvalues inside the unit circle do not imply stability in the case of time-varying systems).

Remark 9: The stability result of Theorem 6 provides guidelines for the design of filters that must be implemented
by means of IPR: the poles of the filter must be inside $P_A$ to guarantee a stable $J$-class Internal Positive Realization. In the case of filter design in continuous-time, the designer must be sure that after discretization the poles are in $P_A$.

5. Design and IPR of a State Observer

In this section, a procedure is given for the design and implementation as an IPR of a state observer for a linear system $S = \{A, B, C, D; \mathbb{R}^n, \mathbb{R}^p, \mathbb{R}^q\}$. The only assumption needed is that the pair $(A, C)$ is observable. A strictly causal observer takes the form

$$
\chi(t + 1) = (A - KC)\chi(t) + (B - KD)u(t) + Ky(t),
$$

while a nonstrictly causal observer has the following structure

$$
\chi(t + 1) = (I_n - KC)A\chi(t) + (I_n - KC)Bu(t)
- KDu(t + 1) + Ky(t + 1)
\hat{x}(t) = \chi(t). 
$$

In the following, only strictly causal observers (56) are considered, for brevity. The first step for the construction of a stable IPR of an observer is to choose the observer gain $K$ such that all eigenvalues are inside $P_A$, and to compute $T \in \mathbb{R}^{n \times n}$ such that the observer transition matrix is in the Real Jordan form, denoted $J$:

$$
\lambda(A - KC) \in P_A, \quad J = T(A - KC)T^{-1}. 
$$

Then, consider the change of coordinates $(t) = T\chi(t)$, and write the observer in the new coordinates

$$
\tilde{\chi}(t + 1) = \tilde{J}\tilde{\chi}(t) + \tilde{B}u(t) + \tilde{K}y(t)
\tilde{x}(t) = \tilde{C}\tilde{\chi}(t),
$$

where

$$
\tilde{J} = T(A - KC)T^{-1}, \quad \tilde{B} = T(B - KD),
\tilde{K} = TK, \quad \tilde{C} = T^{-1}. 
$$

The IPR construction of Theorem 1 applied to the representation (59) provides a $J$-class IPR of the observer, which is stable, as stated by Theorem 6, because all eigenvalues are in $P_A$.

6. Examples of Engineering Applications

In this section, the proposed method of IPR construction is demonstrated on two engineering applications: a fourth-order Chebyshev filter and a Kalman tracking filter in state–space form.

6.1. IPR of a Chebyshev Filter

The example worked out in this section is aimed to compare the proposed methodology of positive realization with the existing ones [4, 6], limited to the realization of SISO filters forced by nonnegative input sequences.

The example is taken from [6]: $H(z)$ is a fourth-order low-pass digital Chebyshev filter with 0.5 dB ripple in the passband and with cut-off frequency 0.5-times half the sample rate:

$$
H(z) = \frac{0.06728(z + 1)^4}{(z^2 + 0.526z + 0.7095)(z^2 - 0.5843z + 0.2314)} 
$$

There are two pairs of complex-conjugate poles: $\lambda_1/2 = 0.29215 \pm j0.38216$ and $\lambda_3/4 = -0.0263 \pm j0.84191$. Both pairs are in $P_A$, but the pair $\lambda_1/2$ is also in $P_\Sigma$, the equilateral triangle inscribed in $\Gamma$, with one vertex in $1 + j0$ (see [6]). The partial fraction decomposition gives

$$
H(z) = \frac{-0.2831z + 0.0992}{z^2 + 0.526z + 0.7095}
+ \frac{0.5880z + 0.0469}{z^2 - 0.5843z + 0.2314} + 0.06728, 
$$

from which the block-diagonal representation of the type (27) can be computed:

$$
A = \begin{bmatrix}
-0.0263 & 0.84191 & 0 & 0 \\
-0.84191 & -0.0263 & 0 & 0 \\
0 & 0 & 0.29215 & 0.38216 \\
0 & 0 & -0.38216 & 0.29215
\end{bmatrix}, 
$$

$$
B = \begin{bmatrix}
-0.2049 \\
-0.0782 \\
0.0079 \\
0.5810
\end{bmatrix},
C = [1 \ 1 \ 1 \ 1], 
D = 0.06728. 
$$

Using the construction of Theorem 1, the IPR is readily computed by separating the positive and negative parts of all system matrices.
The dimension of the state–space of the IPR is 8, the input transformation defined in (14) is not needed be computed using equations (11). Note that the matrices $A^+$, $A^-$, $B^+$, $B^-$, $C^+$, $C^-$, $D^+$, and $D^-$ are obtained using the technique presented in [6]. How-

\[
A^+ = \begin{bmatrix}
0 & 0.84191 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0.29215 & 0.38216 \\
0 & 0 & 0 & 0.29215 \\
0.0263 & 0 & 0 & 0 \\
0.84191 & 0.0263 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0.38216 & 0
\end{bmatrix},
\]

(64)

\[
A^- = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0.5810 & 0 & 0 & 0
\end{bmatrix}; \quad B^+ = \begin{bmatrix} 0.0079 \\ 0.2049 \\ 0.0782 \\ 0 \end{bmatrix},
\]

(65)

From these, the matrices $(A, B, C, D)$ of the IPR can be computed using equations (11). Note that the input transformation defined in (14) is not needed when only positive input sequences are considered. The dimension of the state–space of the IPR is 8, the double of the order of $H(z)$, the same dimension obtained using the technique presented in [6]. However, note that by considering also the input transformation $T_k$ in (14), the IPR constructed with the proposed method is a good IPR also when the input signals do not have any sign limitation, whereas the approach in [4, 6] is limited to nonnegative input sequences. More in general, consider the Positive Realization in [4, 6], where $H(z)$ is decomposed as $H(z) = H_1(z) - H_2(z)$ (see the discussion in Remark 2), in the case of $\nu$ real poles, $\mu_1$ pairs of conjugate complex poles in $\mathcal{P}_3 \cap \mathcal{P}_4$ and $\mu_2$ pairs of conjugate complex poles in $\mathcal{P}_4 \setminus \mathcal{P}_3$. Let $n = \nu + 2\mu_1 + 2\mu_2$ be the total number of poles of $H(z)$ and let $n_1 \leq \nu$ be the number of nonnegative real poles with nonnegative residuals. In this case, the dimensions $s_1$ and $s_2$ of the positive realizations of $H_1(z)$ and $H_2(z)$ are $s_1 = n - n_1 + \mu_1 + 2\mu_2$ and $s_2 = 1$ (see [6]). Therefore, the order of the IPR made of $H_1$ and $H_2$ is:

\[
s_1 + s_2 = n - n_1 + \mu_1 + 2\mu_2 + 1.
\]

(66)

On the other hand, the dimension of the IPR of Corollary 2, after the reduction described at the end of Section 2, (feasible only when positive input sequences are considered), is $2n - n_1$. The following expression is useful for comparing $2n - n_1$ with $s_1 + s_2$:

\[
2n - n_1 = n + (n - n_1) = n + \nu - n_1 + 2\mu_1 + 2\mu_2.
\]

(67)

The comparison of (67) with (66), allows to conclude that: if $\mu_1 = 0$ the construction of Theorem 1 provides an IPR of lower order; when $\mu_1 \geq 2$ the approach in [4, 6] provides the IPR of lower order; when $\mu_1 = 1$ the orders of the IPRs of the two approaches are the same.

The comparison is limited to the case of poles in $\mathcal{P}_4$, because only in this case the presented approach provides stable IPRs of transfer functions.

### 6.2. IPR of a Kalman Tracking Filter

A tracking filter is aimed to the estimation of a continuous time signal and of some of its derivatives, through real-time processing of noisy samples of the signal. A tracking filter can be constructed as a Kalman filter designed on the basis of a simple state–space model that formalize the assumption of continuity of the signal and of some of its derivatives.

Let $z(t)$, $t \in \mathbb{R}$, be a continuous time signal and let $z_n(k)$, $k \in \mathbb{Z}$ denote the noisy measurements sampled at constant intervals $\Delta$:

\[
z_n(k) = z(k\Delta) + w_k, \quad k \in \mathbb{Z},
\]

(68)

where $w_k$ denotes a white measurement noise, with known variance $R$. Consider the problem to estimate the signal $\tilde{z}(t)$ together with its first two derivatives $\dot{\tilde{z}}(t)$ and $\ddot{\tilde{z}}(t)$ at the sampling instants $k\Delta$ (signal tracking problem). A state–space model suitable for the filter design is constructed as follows. Let $\tilde{x}(t) \in \mathbb{R}^3$ be a state vector defined as

\[
\tilde{x}(t) = [z(t) \ \dot{z}(t) \ \ddot{z}(t)]^T, \quad t \in \mathbb{R}.
\]

By definition, the state components are such that $\dot{\tilde{x}}(t) = \dot{z}(t)$ and $\ddot{\tilde{x}}(t) = \ddot{z}(t)$. The third derivative $\dddot{z}(t)$ (i.e. $\dddot{x}(t)$) is modeled as white noise. The resulting continuous-time state–space model with discrete-time measurements takes the form

\[
\dot{\tilde{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tilde{x}(t) + \begin{bmatrix} 0 \\ 0 \\ \sigma \end{bmatrix} n(t),
\]

\[
z_n(k) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tilde{x}(k\Delta) + w_k,
\]

(69)

where $n(t)$ is a standard white noise. The parameter $\sigma$ (standard deviation of the state noise) models the band amplitude of the signal $z(t)$. A standard stochastic discretization procedure provides the following discrete-time model, where $x(k) = \tilde{x}(k\Delta)$:
\[
x(k + 1) = Ax(k) + v_k, \quad \text{cov}(v_k) = Q\sigma^2,
\]
\[
z_m(k) = Cx(k) + w_k, \quad \text{var}(w_k) = R,
\]
(70)

where
\[
A = \begin{bmatrix}
1 & \Delta & \frac{\Delta^3}{8} \\
0 & 1 & \Delta \\
0 & 0 & 1
\end{bmatrix}, \quad Q = \begin{bmatrix}
\frac{\Delta^3}{20} & \frac{\Delta^4}{8} & \frac{\Delta^5}{6} \\
\frac{\Delta^4}{8} & \frac{\Delta^5}{2} & \Delta \\
\frac{\Delta^5}{6} & \frac{\Delta^6}{2} & \Delta
\end{bmatrix}
\]
\[
C = [1 \ 0 \ 0].
\]
(71)

The steady-state Kalman filter for the system (70) takes the form
\[
\begin{aligned}
\chi(k + 1) &= (I_3 - KC)A\chi(k) + Kz_m(k + 1) \\
\dot{x}(k) &= \chi(k),
\end{aligned}
\]
(72)

where \( I_3 \) is the 3 \( \times \) 3 identity matrix. The Kalman gain is computed as
\[
K = \frac{1}{PC^T + R}PC^T,
\]
(73)

where \( P \) is the solution of the algebraic Riccati equation
\[
P = APA^T - \frac{APC^T CPA^T}{PC^T + R} + Q\sigma^2.
\]
(74)

Setting \( \Delta = 1, \ \sigma = 1 \) and \( R = 1 \) in the system equations and solving for \( K \) the equations (73) and (74), one has
\[
K = [0.8647 \ 0.7976 \ 0.3679]^T.
\]
(75)

The eigenvalues of the filter matrix \( A_f = (I_3 - KC)A \) are
\[
\lambda = (0.3679, \ 0.3930 \pm 0.4620j).
\]
(76)

Although the filter matrix is stable, it can be easily verified that the IPR constructed in the original coordinates is unstable (has a real positive eigenvalue \( > 1 \)). However, as it is easily checked, the eigenvalues of \( A_f \) are in \( P_k \) (0.3679 \( < \) 1 and \( 0.3930 \pm 0.4620 \) \( < \) 1) and therefore, according to Theorem 6 a stable \( J \)-class IPR can be constructed. The matrix \( T \) that puts the filter in the Real Jordan block-diagonal form (27), and its inverse, are
\[
T = \begin{bmatrix}
1.4340 & -0.8346 & 0.9030 \\
-0.8154 & 1.7553 & -0.5134 \\
-2.1668 & -0.5540 & 1.6944
\end{bmatrix},
\]
(77)
\[
T^{-1} = \begin{bmatrix}
0.4788 & 0.1627 & -0.2059 \\
0.4440 & 0.7808 & 0.0000 \\
0.7574 & 0.4633 & 0.3269
\end{bmatrix},
\]
(78)

After the change of coordinates \( \tilde{x}(t) = T\chi(t) \) the filter is written as
\[
\begin{aligned}
\tilde{x}(k + 1) &= J\tilde{x}(k) + \tilde{K}z_m(k) \\
\dot{\chi}(k) &= T^{-1}\tilde{\chi}(k),
\end{aligned}
\]
(79)

where
\[
\tilde{J} = \begin{bmatrix}
0.3679 & 0 & 0 \\
0 & 0.3930 & 0.4620 \\
0 & -0.4620 & 0.3930
\end{bmatrix},
\]
(80)
\[
\tilde{K} = \begin{bmatrix}
0.9065 \\
0.5061 \\
-1.6921
\end{bmatrix}.
\]
(81)

Note that the real eigenvalue of \( \tilde{J} \) is positive, and therefore \( J_{1,1} \) is positive, and the first row of \( \tilde{K} \) and the first column of \( T^{-1} \) are positive too. Thus, if the sensor is such to provide a nonnegative sequence \( z_m(k) \), and the component \( \chi_1 \) of the initial state of the filter is positive, then the reduced order realization algorithm described in Theorem 4 can be applied (in this case \( A_1 = \tilde{J}_{1,1} = 0.369, \ B_1 = \tilde{K}_1 = 0.9065 \) and \( C_1 \) is the first column of \( T^{-1} \)). In this example, it is \( n_1 = 1 \) and \( n_2 = 2 \), so that the order of the IPR is \( m_1 + 2n_2 = 5 \). The matrices and the state of the IPR are the following
\[
A = \begin{bmatrix}
0.3679 & 0 & 0 & 0 & 0 \\
0 & 0.3930 & 0.4620 & 0 & 0 \\
0 & 0 & 0.3930 & 0.4620 & 0 \\
0 & 0 & 0 & 0 & 0.3930
\end{bmatrix},
\]
(81)
\[
B = \begin{bmatrix}
0.9065 \\
0.5061 \\
1.6921
\end{bmatrix}, \quad \chi(k) = \begin{bmatrix}
\chi_1(k) \\
\chi_2(k) \\
\chi_3(k)
\end{bmatrix}.$
transformation function would lead to the construction of three realization methodologies [4, 6, 20], to each transfer function. The application of the existing filter (72) can be described by three third-order scalar equations:

\[
Y(k) = \begin{bmatrix}
\dot{x}_1(k) \\
\dot{x}_2(k) \\
\dot{x}_3(k) \\
\dot{x}_4(k) \\
\end{bmatrix}
\]

(82)

together with \( D = 0_{6,0} \) and \( d(k) = z_m(k) \). The input transformation \( T_y \) is the identity, because it is assumed that \( z_m(k) \geq 0 \), while the output transformation is the usual \( T_y = [I_3 - I_3] \). In tracking filters, if no a priori information is available, the first measurement provides the initial estimate as follows:

\[
\hat{x}(k_0) = \chi(k_0) = [z_m(k_0) \ 0 \ 0]^T, \text{ and then }
\]

\[
\hat{\chi}(k_0) = T\chi(k_0) = T_{1,1}z_m(k_0), \text{ where } T_{1,1} \text{ denotes the first column of matrix } T. \text{ Then, the initial state of the IPR is computed as }
\]

\[
\chi(k_0) = [1.4340 \ 0 \ 0 \ 0.8154 \ 2.1668]^T z_m(k_0).
\]

(83)

Note that the input-output behavior of the tracking filter (72) can be described by three third-order scalar transfer functions. The application of the existing realization methodologies [4, 6, 20], to each transfer function would lead to the construction of three decoupled realizations, with obvious redundancy.

7. Conclusions

The concept of IPR of discrete-time linear systems has been introduced in this paper, and a methodology for the construction of IPRs is presented. The notion of IPR of systems generalizes the idea of filter realization through combination of positive systems presented in [6], and applies both to systems in state-space form and to transfer-matrices. The technique here proposed for the construction of IPRs naturally applies to MIMO systems without any sign restriction on the input sequence, while all existing approaches of positive realization of digital filters have been conceived for SISO filters driven by nonnegative input signals. The method proposed here is very easy and, differently form existing approaches, does not involve the numerical solution of optimization problems. The dimension of the IPR is the double of the dimension of the state-space of the system to be realized, regardless to the positions of the eigenvalues/poles in the complex plane. An interesting class of IPR is also defined in this paper (J-class IPR), for which necessary and sufficient conditions are provided that ensure the stability of the IPR. Two engineering applications have been developed to illustrate the method: an IPR of a tracking filter (Single-Input-Multi-Output signal processing scheme), and an IPR of a fourth-order Chebyshev low-pass filter (SISO).

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References

13. Germani A, Manes C, Palumbo P. State representation of a class of MIMO systems via positive systems. In: