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A CHAIN OBSERVER FOR NONLINEAR SYSTEMS WITH MULTIPLE TIME-VARYING MEASUREMENT DELAYS*

FILIPPO CACACE†, ALFREDO GERMANI‡, AND COSTANZO MANES§

Abstract. This paper presents a method for designing state observers with exponential error decay for nonlinear systems whose output measurements are affected by known time-varying delays. A modular approach is followed, where subobservers are connected in cascade to achieve a desired exponential convergence rate (chain observer). When the delay is small, a single-step observer is sufficient to carry out the goal. Two or more subobservers are needed in the the presence of large delays. The observer employs delay-dependent time-varying gains to achieve the desired exponential error decay. The proposed approach allows to deal with vector output measurements, where each output component can be affected by a different delay. Relationships among the error decay rate, the bound on the measurement delays, the observer gains, and the Lipschitz constants of the system are presented. The method is illustrated on the synchronization problem of continuous-time hyperchaotic systems with buffered measurements.

Key words. nonlinear systems, delay systems, state observers, chain observers

AMS subject classifications. 93A99, 93B07, 93B10

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1. Introduction.

The state estimation of dynamical systems based on delayed output measurement is an important problem in many engineering applications, for example, when the system is controlled or monitored by a remote device through a communication channel, or when the measurement process intrinsically causes a nonnegligible time delay, as in biochemical reactors. For this reason the issue of state reconstruction in the presence of time delays in the system equations and/or in the measurement process is receiving increasing attention.

State observers and observability conditions for both linear and nonlinear systems with time delays in the state equations have been studied by many authors (see, e.g., [20, 21, 24, 11, 29, 32] and the references therein). This paper considers nonlinear systems without delays in the state equations but with delayed measurements. The design of state observers that predict the current state by processing delayed output measurements is central for the design of state feedback controllers. In the case of stable linear systems, the control problem is solved by the Smith predictor [25]. An extension of the Smith approach to closed-loop control of nonlinear systems with delayed input was presented in [18, 22], where, as in the case of linear systems, the state prediction is obtained by an open-loop algorithm, so that the accuracy of the predicted state is not guaranteed for unstable systems.

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†Università Campus Bio-Medico di Roma, Via Álvaro del Portillo 21, 00128 Roma, Italy (f.cacace@unicampus.it).

‡Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica, Università degli Studi dell'Aquila, 67100 Coppito (AQ), Italy, and Università Campus Bio-Medico di Roma, Via Álvaro del Portillo 21, 00128 Roma, Italy (alfredo.germani@univaq.it).

§Corresponding author. Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica, Università degli Studi dell'Aquila, 67100 Coppito (AQ), Italy, and Istituto di Analisi dei Sistemi ed Informatica A. Ruberti, CNR, 00185 Rome, Italy (costanzo.manes@univaq.it).

The idea of achieving the convergence of the state estimate by using a cascade of two observers (elementary chain observer) was first proposed in [10], while the idea of using more than two observers in the chain to deal with large measurement delays has been proposed in [12]. In [10] it has been shown that a chain of two observers is sufficient for asymptotic state reconstruction as long as the measurement delay is below a given threshold, which depends on the Lispchitz constants of the system. When the time delay exceeds such a threshold, more links must be added to the chain [12]. Each observer in the chain is in charge of predicting the system state for a suitable fraction of the total delay. The structure of the basic observer in the chain is the one proposed in [5, 6] for undelayed measurements. A similar approach has been used in [16], where some restrictions of the chain observer in [12] have been overcome. In [3] another predictor for nonlinear systems with delayed output, based on a cascade of observers, has been proposed. Sufficient conditions for the convergence of this predictor have been derived using *linear matrix inequalities*. This predictor has been extended in [1] to triangular systems. Other recent proposals include the nonlinear observer of [21], for systems that are linearizable by additive output injection, and the constant gain observer design method proposed in [27]. In [26] the case of time-varying measurements delay has been investigated, although restricted to linear time invariant systems. In particular the stability properties of a chain observer have been investigated, under the assumption that the delay is known and piecewise constant.

Recently, the framework of high-gain observers has been used to design observers for nonlinear systems with time-varying measurement delay. In [4] a Razumikhin approach has been used to prove the asymptotic convergence to zero of the estimation error of a high-gain observer derived from [5, 6]. In [28] an observer derived from the one in [9] has been proposed, and a Liapunov–Krasovskii approach is used to derive the convergence result in the presence of time-varying delay. The exponential convergence to zero of the estimation error of this observer has been proved in [2].

This work contains two main contributions. The first is the proposal of a single-step observer for nonlinear systems with time-varying measurement delays that extends the one in [4] in two directions: it allows one to deal with vector measurements (multi-input-multi-output systems) and achieves prescribed exponential error decay, provided that the maximum delay is below a suitable bound. This result is obtained by use of time-varying delay-dependent gains in the observer. The delays are assumed uniformly bounded but not necessarily continuous functions of time. Moreover, each component of the vector output can have its own delay.

The second contribution is the proposal of a chain observer with uniform structure, which allows one to deal with the case in which the prescribed exponential convergence cannot be achieved by a single-step observer (i.e., the maximum measurement delay is *too large*). To the best of our knowledge, this is the first proposal of a chain observer for the time-varying delay case. Relationships among the exponential error decay rate, the bound on the measurement delays, the observer gains, and the Lipschitz constants of the system are investigated. In order to have a simpler and shorter exposition, only global convergence results are derived in this paper, for which we need to assume rather strong global Lipschitz and observability properties on the systems under investigation. However, local convergence results can be derived as well under weaker local Lipschitz and observability assumptions.

The paper is organized as follows. In section 2 the class of systems under investigation and the state observation problem are formulated. The single-step observer is presented in section 3, together with the convergence results. In section 4 the chain observer is described and the convergence analysis is provided. Some guidelines for

tuning the observer parameters are given in section 5, and in section 6 the approach is illustrated on the state estimation problem of a hyperchaotic system (hyperchaos synchronization) with buffered measurements. Conclusions follow. In order to get a smoother presentation, some auxiliary results are reported as appendices.

Notation. Given q *objects* H_i, for $i = 1:q$ (functions, vectors, matrices, ...), throughout this paper the symbol $\mathrm{diag}_{i=1}^{q}\{H_i\}$ denotes the block-diagonal matrix, whose diagonal blocks are the objects H_i. In the same way, if all the H_i have the same number of columns (rows) the symbol $\mathrm{col}_{i=1}^{q}\{H_i\}$ ($\mathrm{row}_{i=1}^{q}\{H_i\}$) denotes the block-column (row) matrix made with the H_i. $\mathbb{N}$ denotes the set of natural numbers (strictly positive integers). The norm of a multi-index $\bar{s} = \{s_i\}_1^q \in \mathbb{N}^q$ is $|\bar{s}| = \sum_{i=1}^{q} s_i$. For a given $p \in \mathbb{N}$, $\mathbb{1}_p \in \mathbb{R}^p$ is the column vector of ones in $\mathbb{R}^p$, while I_p is the identity matrix in $\mathbb{R}^{p\times p}$. $\Sigma_+^{p\times p} \subset \mathbb{R}^{p\times p}$ is the set of $n \times n$ symmetric positive definite (SPD) real matrices. $\mathbb{R}_+$ and $\mathbb{R}_-$ are the sets of strictly positive and strictly negative real numbers, respectively. Given $\Delta \in \mathbb{R}_+$ and $n \in \mathbb{N}$, the symbol $\mathcal{C}_\Delta^n$ denotes the space of continuous functions that map the interval $[-\Delta, 0]$ into $\mathbb{R}^n$, endowed with the sup norm. The meaning of the norm symbol $\|\cdot\|$ depends on the context: if $\phi \in \mathcal{C}_\Delta^n$, then $\|\phi(t)\| = \big(\phi^T(t)\phi(t)\big)^{1/2}$ and $\|\phi\| = \sup_{\tau\in[-\Delta,0]} \|\phi(\tau)\|$. For a given continuous function $x : \mathbb{R} \mapsto \mathbb{R}^n$, the symbol x_t denotes its restriction to the interval $[t-\Delta, t]$, i.e., $x_t \in \mathcal{C}_\Delta^n$ with $x_t(\tau) = x(t-\tau) \in \mathbb{R}^n$.

2. Preliminaries. We consider the problem of state observation for nonlinear systems with delayed vector output, in the case where each output component is measured with its own time delay. The measurement delays, possibly time-varying, are assumed to be known in real time, and are bounded by a known constant Δ. The systems considered here have the form

$$\dot{x}(t) = F\big(x(t), u(t)\big), \qquad t \geq -\Delta, \tag{1}$$

$$\bar{y}_i(t) = h_i\big(x(t-\delta_i(t))\big), \quad i = 1{:}q, \qquad t \geq 0, \tag{2}$$

$$x(-\Delta) = \bar{x} \in \mathbb{R}^n, \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^p$ is a known input, $\bar{y}(t) \in \mathbb{R}^q$ is the measured output, and $\delta_i(t) \in [0, \Delta]$ is the time-varying measurement delay of the ith output. The function $F : \mathbb{R}^n \times \mathbb{R}^p \mapsto \mathbb{R}^n$ is affine in the input, i.e.,

$$F(x,u) = f(x) + G(x)u = f(x) + \sum_{k=1}^{p} g_k(x)u_k, \tag{4}$$

where $f(x)$ and $g_k(x)$ are C^∞ vector fields. $h_i(x)$, $i = 1:q$, are C^∞ functions. Let $\bar{y}(t) \in \mathbb{R}^q$ denote the vector collecting all the delayed measurements $\bar{y}_i(t)$, i.e., $\bar{y}(t) = \mathrm{col}_{i=1}^{q}\big\{\bar{y}_i(t)\big\}$, and let $y(t) = \mathrm{col}_{i=1}^{q}\big\{h_i(x(t))\big\}$ denote the vector of undelayed measurements. The delays $\delta_i(t)$ are collected in $\bar{\delta}(t) = \{\delta_i(t)\}_{i=1}^{q}$. The component $y_i(t)$ will be available for processing at a time t_i' such that $t_i' = t + \delta_i(t_i')$.

Remark 1. We assume that the measurement delays $\delta_i(t)$ are known in real time, which means that the information available for processing at time t, with $t \geq 0$, is the pair $\big(\bar{y}(t), \bar{\delta}(t)\big) \in \mathbb{R}^q \times [0, \Delta]^q$, whose components $(\bar{y}_i(t), \delta_i(t))$ satisfy the identities (2), for $i = 1:q$. The issue of robustness with respect to uncertainties in the delay is not investigated in this paper and deserves further research work. Notice that the assumption that the delay is known is realistic in many applications. A common case is that of networked control systems [13], when the measurements are buffered and then sent over a reliable network that introduces a variable delay. In this case, the

delay is typically computed by comparing the time at which the packet is delivered with the time-stamp included in the packet at the sender side.

Although in principle it is possible to estimate $x(t-\delta)$ by exploiting the outputs $y_i(t)$ of (1)–(2) in an ordinary observer, and then to use such an estimate for estimating $x(t)$ by integrating the system equation (1) in the interval $[t-\delta, t]$, this approach is not used in the literature mainly for two reasons: the implementation of predictors containing integral terms (distributed predictors) may be computationally prohibitive for real-time applications, and the open-loop structure of the integral predictor makes it sensitive to uncertainties and modeling errors. In addition, it is not trivial to estimate $x(t-\delta)$ when the delay δ is not constant.

Thus, the integral predictor approach, although conceptually simple, may not be suited for many applications. As a consequence, predictors with state-observer structure, i.e., written in the form of measurements driven differential equations (or delay-differential equations), are generally preferred.

Throughout the paper, given a vector $w(t) = \operatorname{col}_{i=1}^{q}\{w_i(t)\}$ and a set of delays $\bar{\delta}(t) = \{\delta_i(t)\}_1^q$, the vector function $w_{\bar{\delta}}(t)$ is defined as

$$w_{\bar{\delta}}(t) = \operatorname*{col}_{i=1}^{q}\{w_i(t-\delta_i(t))\}. \tag{5}$$

Thus, we have $\bar{y}(t) = y_{\bar{\delta}}(t)$. Let $L_f^k\lambda(x)$ denote the kth Lie derivative of the scalar function $\lambda(x)$ along a vector field $f(x)$, defined as (see [15])

$$L_f^0\lambda(x) = \lambda(x), \quad L_f^k\lambda(x) = \frac{dL_f^{k-1}\lambda}{dx}f(x), \ k>0, \tag{6}$$

and let $L_G\lambda(x)$ denote the row vector $[L_{g_1}\lambda \ \cdots \ L_{g_p}\lambda]$. Following [6], we build the observability map using the Lie derivatives 0, 1, ..., s_{i-1} of each output function $h_i(x)$. For a given multi-index $\bar{s} = \{s_i\}_1^q$, let the vector functions $\Phi_i^{s_i}(x) \in \mathbb{R}^{s_i}$ and $Y_{i,s_i}(t) \in \mathbb{R}^{s_i}$ be defined as follows:

$$\Phi_i^{s_i}(x) = \operatorname*{col}_{k=1}^{s_i}\{L_f^{k-1}h_i(x)\} = [\ h_i(x) \ \ L_fh_i(x) \ \ \dots \ \ L_f^{s_i-1}h_i(x)\ ]^T, \tag{7}$$

$$Y_{i,s_i}(t) = \operatorname*{col}_{k=1}^{s_i}\{y_i^{(k-1)}(t)\} \ = [\ y_i(t) \quad y_i^{(1)}(t) \quad \dots \ y_i^{(s_i-1)}(t)\ ]^T \tag{8}$$

($y_i^{(k)}(t)$ denotes the kth derivative of the ith undelayed output), and let

$$\Phi_{\bar{s}}(x) = \operatorname*{col}_{i=1}^{q}\big\{\Phi_i^{s_i}(x)\big\}, \quad Y_{\bar{s}}(t) = \operatorname*{col}_{i=1}^{q}\big\{Y_{i,s_i}(t)\big\}. \tag{9}$$

Note that $\Phi_i^{s_i}(x)$ are maps from $\mathbb{R}^n$ to $\mathbb{R}^{s_i}$ and do not depend on time. As discussed in [6, 7], if $u(t) = 0$, then $Y_{i,s_i}(t) = \Phi_i^{s_i}(x(t))$ and $Y_{\bar{s}}(t) = \Phi_{\bar{s}}(x(t))$. If, for some $\bar{s}$, such that $|\bar{s}| = n$, the square map $z = \Phi_{\bar{s}}(x)$ is invertible, then the knowledge of $Y_{\bar{s}}(t)$ theoretically allows instantaneous exact reconstruction of the state $x(t)$. This property justifies the following definition (see [6]).

DEFINITION 1. *For a given multi-index $\bar{s}$ such that $|\bar{s}| = n$, the map $\Phi_{\bar{s}}(x)$ defined in* (9) *is said to be an* observability map *in a set $\Omega \subseteq \mathbb{R}^n$, for system* (4)–(2)*, if it is a diffeomorphism in Ω. A system that admits an observability map $\Phi_{\bar{s}}(x)$ in Ω is said to be drift-observable in Ω. A system is said to be* uniformly Lipschitz drift-observable *in Ω if it is drift-observable in Ω and the maps $\Phi_{\bar{s}}$ and $\Phi_{\bar{s}}^{-1}$ are uniformly Lipschitz*

(in Ω and $\Phi_{\bar{s}}(\Omega)$, respectively). If $\Omega = \mathbb{R}^n$ the system is said to be globally uniformly Lipschitz drift-observable *(GULDO).*

The observability property described in Definition 1 only depends on the drift component of (4) (i.e., $f(x)$). For this reason the term *drift-observability* (i.e., observability for null input) has been coined in [6].

Note that the components s_i of a multiindex $\bar{s}$ that satisfy the assumptions of Definition 1 coincide with the *observability indices* defined in [19].

If a system is drift-observable in Ω, then the Jacobian

$$Q_{\bar{s}}(x) = \frac{d\Phi_{\bar{s}}(x)}{dx} = \operatorname*{col}_{i=1}^{q}\left\{\frac{d\Phi_i^{s_i}(x)}{dx}\right\} \tag{10}$$

is nonsingular $\forall x \in \Omega$, and the inverse map $x = \Phi_{\bar{s}}^{-1}(z)$ exists in all $\Phi_{\bar{s}}(\Omega)$.

DEFINITION 2 (see [6, 7]). *The* observation relative degree *of the ith output $h_i(x)$ of system (1)–(2) in a set $\Omega \subseteq \mathbb{R}^n$ is a natural number r_i such that*

$$\forall x \in \Omega: \quad L_G L_f^k h_i(x) = 0, \quad k = 0 : r_i - 2, \tag{11}$$

$$\exists x \in \Omega: \quad L_G L_f^{r_i-1} h_i(x) \neq 0. \tag{12}$$

If $\Omega = \mathbb{R}^n$, the ith output is said to have uniform observation relative degree r_i.

Note that Definition 2 of the observation relative degree (taken from [6]) is not related to the measurement delay. (The functions in (11) and (12) are maps from $\mathbb{R}^n$ to $\mathbb{R}^p$.) Moreover, if $s_i \leq r_i$, $i = 1 : q$, we still have $Y_{\bar{s}}(t) = \Phi_{\bar{s}}\big(x(t)\big)$, as in the case of absence of input, and the drift-observability property implies the observability for any input (see [6] for further details).

Now, consider the map $\Phi_i^{s_i}(x)$ defined in (7), and assume that $s_i \leq r_i$. Let $(A_{s_i}, B_{s_i}, C_{s_i})$ denote a Brunowsky triple of size s_i (see Appendix A.1). Then

$$\frac{d\Phi_i^{s_i}}{dx}F(x,u) = \frac{d\Phi_i^{s_i}}{dx}\big(f(x) + G(x)u\big) = A_{s_i}\Phi_i^{s_i}(x) + B_{s_i}\widetilde{L}_i(x,u), \tag{13}$$

$$\text{where} \quad \widetilde{L}_i(x,u) = L_f^{s_i} h_i(x) + L_G L_f^{s_i-1} h_i(x) u. \tag{14}$$

(Note that if $s_i < r_i$, then $L_G L_f^{s_i-1} h_i(x) = 0$.) Let $z_i(t) = \Phi_i^{s_i}(x(t))$. Taking into account (13) and the identity $h_i(x) = C_{s_i}\Phi_i^{s_i}(x)$, we get

$$\dot{z}_i(t) = A_{s_i} z_i(t) + B_{s_i}\widetilde{L}_i(x(t), u(t)), \tag{15}$$

$$\bar{y}_i(t) = C_{s_i} z_i\big(t - \delta_i(t)\big), \tag{16}$$

$$z_i(\tau) = \Phi_i^{s_i}(\bar{x}). \tag{17}$$

Note that, since $s_i \leq r_i$, if follows that $z_i(t) = Y_{i,s_i}(t)$. If the system (1)–(2) is globally drift-observable and $\Phi_{\bar{s}}(x)$ is a global observability map, with $s_i \leq r_i$, $i = 1 : q$, then $z = \Phi_{\bar{s}}(x)$ defines a change of coordinates. In the new coordinates $z(t) = \operatorname{col}_{i=1}^{q}\big\{z_i(t)\big\}$, the undelayed vector measurement is $y(t) = C_{\bar{s}} z(t)$, and the system equations are

$$\dot{z}(t) = A_{\bar{s}} z(t) + B_{\bar{s}} p(z(t), u(t)), \qquad t \geq -\Delta, \tag{18}$$

$$\bar{y}_i(t) = C_{s_i} z_i(t - \delta_i(t)), \quad i = 1 : q, \qquad t \geq 0, \tag{19}$$

$$z(-\Delta) = \Phi_{\bar{s}}(\bar{x}), \tag{20}$$

where $A_{\bar{s}} \in \mathbb{R}^{n\times n}$, $B_{\bar{s}} \in \mathbb{R}^{n\times p}$, $C_{\bar{s}} \in \mathbb{R}^{q\times n}$ are

$$A_{\bar{s}} = \operatorname*{diag}_{i=1}^{q}\{A_{s_i}\}, \quad B_{\bar{s}} = \operatorname*{diag}_{i=1}^{q}\{B_{s_i}\}, \quad C_{\bar{s}} = \operatorname*{diag}_{i=1}^{q}\{C_{s_i}\}, \tag{21}$$

and the function $p(z(t), u(t)) \in \mathbb{R}^q$ is defined as

$$p(z,u) = \operatorname*{col}_{i=1}^{q}\big\{p_i(z,u)\big\}, \quad \text{where } p_i(z,u) = \widetilde{L}_i\big(\Phi_{\bar{s}}^{-1}(z), u\big), \tag{22}$$

with $\widetilde{L}_i(\cdot,\cdot)$ defined in (14). The representation (18)–(19) of system (1)–(2) will be useful in the proof of the observer convergence.

3. Single-step exponential observer. This section presents a single-step observer for system (1)–(2) and the relevant convergence analysis. The hypotheses needed are the following:

$\mathcal{H}_1$ The system (1)–(2) is GULDO, i.e., there exists a multi-index $\bar{s} = \{s_i\}_1^q$ such that the map $z = \Phi_{\bar{s}}(x)$ defined in (9) and its inverse $x = \Phi_{\bar{s}}^{-1}(z)$ are uniformly Lipschitz in $\mathbb{R}^n$.

$\mathcal{H}_2$ The function $p(z,u)$ defined in (22) is globally uniformly Lipschitz with respect to z, with the Lipschitz coefficient γ_p depending on $||u||$, i.e.,

$$\|p(z_1,u) - p(z_2,u)\| \le \gamma_p(||u||)\|z_1 - z_2\| \quad \forall z_1, z_2 \in \mathbb{R}^n. \tag{23}$$

$\mathcal{H}_3$ The components s_i of the multi-index $\bar{s}$ in $\mathcal{H}_1$ are such that $s_i \le r_i$, $i = 1\!:\!q$. (r_i is the uniform observation relative degree of each of the output functions $h_i(x)$ in (1)–(2).)

Note that the hypothesis $\mathcal{H}_2$ could be equivalently given in terms of uniformly Lipschitz assumption on $\widetilde{L}_i(x,u)$, because by definition $p_i(z,u) = \widetilde{L}_i\big(\Phi_{\bar{s}}^{-1}(z), u\big)$, where $\Phi_{\bar{s}}^{-1}(z)$ is uniformly Lipschitz by hypothesis $\mathcal{H}_1$.

The proposed observer for the system (1)–(2) is the following delay system:

$$\begin{aligned} &\dot{\hat{x}}(t) = F\big(\hat{x}(t), u(t)\big) + \big(Q_{\bar{s}}(\hat{x}(t))\big)^{-1} K_{\bar{s}}(t)\nu(t), \quad t \ge 0,\\ &\hat{x}(\tau) = \phi(\tau), \qquad \tau \in [-\Delta,\ 0], \end{aligned} \tag{24}$$

where

$$K_{\bar{s}}(t) = \operatorname*{diag}_{i=1}^{q}\big\{e^{-\eta\delta_i(t)}k_i\big\} \quad \text{with } k_i \in \mathbb{R}^{s_i},\ \eta \in \mathbb{R}_+, \tag{25}$$

$$\nu(t) = \operatorname*{col}_{i=1}^{q}\big\{\nu_i(t)\big\} = \operatorname*{col}_{i=1}^{q}\big\{\bar{y}_i(t) - h_i\big(\hat{x}(t-\delta_i(t))\big)\big\}. \tag{26}$$

The matrix $Q_{\bar{s}}(\hat{x})$ is the Jacobian of $\Phi_{\bar{s}}(\hat{x})$, defined in (10). The gain vectors k_i and the constant η are the only design parameters for the observer. In particular, η in (25) is a desired exponential decay rate for the observation error. The function $\phi \in \mathcal{C}_\Delta^n$ is used for the observer initialization.

DEFINITION 3 (global η-exponential convergence). *For a given $\eta \in \mathbb{R}_+$ a system of the type (24)–(26) is said to be a global η-exponential observer for system (1)–(2) if, for any given $\phi \in \mathcal{C}_\Delta^n$ and initial state $x(-\Delta) \in \mathbb{R}^n$, there exists $c \in \mathbb{R}_+$ such that*

$$\|x(t) - \hat{x}(t)\| \le c\, e^{-\eta t} \quad \forall t \ge 0. \tag{27}$$

The observer equations (24) can be written in the z-coordinates by defining $\hat{z}_i(t) = \Phi_i^{s_i}(\hat{x}(t))$ and $\hat{z}(t) = \mathrm{col}_{i=1}^{q}\{\hat{z}_i(t)\} = \Phi_{\bar{s}}\big(x(t)\big)$. Differentiating and using (13)–(14) we get

$$\begin{aligned} \dot{\hat{z}}(t) &= A_{\bar{s}}\hat{z}(t) + B_{\bar{s}}p(\hat{z}(t), u(t)) + K_{\bar{s}}(t)\nu(t), \quad t \geq 0, \\ \hat{z}(\tau) &= \Phi_{\bar{s}}(\phi(\tau)), \qquad \tau \in [-\Delta,\ 0]. \end{aligned} \tag{28}$$

Let $\tilde{z}_i(t) = z_i(t) - \hat{z}_i(t)$ be the components of the observation error in z-coordinates. Using (16), the ith block of $\nu(t)$ in (26) can be written as

$$\nu_i(t) = \bar{y}_i(t) - C_{s_i}\hat{z}_i(t - \delta_i(t)) = C_{s_i}\tilde{z}_i(t - \delta_i(t)). \tag{29}$$

The error $\tilde{z}(t) = \mathrm{col}_{i=1}^{q}\{\tilde{z}_i(t)\}$ is also $\tilde{z}(t) = z(t) - \hat{z}(t) = \Phi_{\bar{s}}\big(x(t)\big) - \Phi_{\bar{s}}\big(\hat{x}(t)\big)$.

Subtracting equations (18) and (28) and defining

$$\tilde{p}\big(z(t), u(t), \tilde{z}\big) = p\big(z(t), u(t)\big) - p\big(z(t) - \tilde{z}, u(t)\big), \tag{30}$$

where $p(\cdot,\cdot)$ is defined in (22), we obtain the following differential equation for $\tilde{z}$:

$$\begin{aligned} \dot{\tilde{z}}(t) &= A_{\bar{s}}\tilde{z}(t) + B_{\bar{s}}\tilde{p}\big(z(t), u(t), \tilde{z}(t)\big) - K_{\bar{s}}(t)C_{\bar{s}}\tilde{z}_{\bar{\delta}}(t), \quad t \geq 0, \\ \tilde{z}(\tau) &= z(\tau) - \Phi_{\bar{s}}\big(\phi(\tau)\big), \qquad \tau \in [-\Delta,\ 0], \end{aligned} \tag{31}$$

where $\tilde{z}_{\bar{\delta}}(t) = z_{\bar{\delta}}(t) - \hat{z}_{\bar{\delta}}(t) = \mathrm{col}_{i=1}^{q}\{\tilde{z}_i(t - \delta_i(t))\}$. By the Lipschitz assumption $\mathcal{H}_2$

$$\|\tilde{p}\big(z(t), u(t), \tilde{z}\big)\| \leq \gamma_p(\|u(t)\|)\,\|\tilde{z}\|. \tag{32}$$

Remark 2. Under assumption $\mathcal{H}_1$, the inequality (27) that defines the global η-exponential convergence in Definition 3 is equivalent to

$$\lim_{t\to\infty} e^{\eta t}\|\tilde{z}(t)\| = 0 \quad \forall \phi \in \mathcal{C}_{\Delta}^{n},\ \forall x(-\Delta) \in \mathbb{R}^n. \tag{33}$$

Before giving the main convergence theorem, we need the following lemma. (The proof is in Appendix A.2.)

LEMMA 4. *For a given multi-index $\bar{s} = \{s_i\}_1^q$ such that $|\bar{s}| = n$, consider the q Brunowsky triples $(A_{s_i}, B_{s_i}, C_{s_i})$, $i = 1:q$. Then, for any given $a > 0$ and $b > 0$, there exist q vectors $k_i \in \mathbb{R}^{s_i}$ and q matrices $P_i \in \Sigma_+^{s_i \times s_i}$ such that the following q inequalities hold for $i = 1:q$:*

$$(A_{s_i} - k_i C_{s_i})^T P_i + P_i(A_{s_i} - k_i C_{s_i}) + a\,P_i + b\,I_{s_i}\sum_{i=1}^{q}(B_{s_i}^T P_i B_{s_i}) \leq 0. \tag{34}$$

Moreover, given q vectors $v_i \in \mathbb{R}_-^{s_i}$, $i = 1:q$, with distinct and negative components, the pairs $\big(\bar{k}_i(\rho), \overline{P}_i(\rho)\big)$ are defined, for $i = 1:q$, as follows:

$$\bar{k}_i(\rho) = -\operatorname*{diag}_{h=1}^{s_i}\{\rho^h\}V^{-1}(v_i)v_i^{(s_i)}, \quad \overline{P}_i(\rho) = \operatorname*{diag}_{h=1}^{s_i}\{\rho^{-h}\}V^T(v_i)V(v_i)\operatorname*{diag}_{h=1}^{s_i}\{\rho^{-h}\}\rho^{2s_i}, \tag{35}$$

where $V(v_i)$ is the Vandermonde matrix associated to v_i, and $v_i^{(s_i)}$ is the vector of componentwise s_ith powers of v_i (see definitions (100) *in Appendix* A.1*), are solution pairs $\big(\bar{k}_i(\rho), \overline{P}_i(\rho)\big)$ of* (34) *if*

$$\rho > \max\left\{1, \max_{i=1}^{q}\left\{\frac{a + b\,n\,\|V^{-1}(v_i)\|^2}{2w_i}\right\}\right\}, \tag{36}$$

where $w_i = -\max_{h=1}^{p}\{(v_i)_h\}$ (i.e., w_i is the smallest component of $-v_i$).

Now the main convergence theorem can be given.

THEOREM 5. *Consider the system* (1)–(2)*, with* $\delta_i(t) \in [0, \Delta]$*,* $i = 1:q$*, under assumptions* $\mathcal{H}_1$*,* $\mathcal{H}_2$*,* $\mathcal{H}_3$*, and assume that* $\|u(t)\| \le \bar{u}\ \forall t \ge -\Delta$*, for some* $\bar{u} > 0$*.*

Then, for any assigned $\eta > 0$*, there exist* q *gain vectors* $k_i \in \mathbb{R}^{s_i}$*,* $i = 1:q$*, and a positive* $\overline{\Delta}$ *such that if* $\Delta < \overline{\Delta}$*, then* (24)–(26) *is a global* η*-exponential observer for system* (1)–(2)*.*

In particular, the q *gains* k_i *can be chosen to satisfy, together with* q *matrices* P_i*, the inequalities* (34)*, with* $a = 2\eta + \alpha + 1$*, where* $\alpha > 0$ *is arbitrarily chosen, and* $b = \gamma_p^2(\bar{u})$*. With this choice, the* η*-exponential convergence is ensured if* $\Delta < \overline{\Delta}$*, with*

$$\overline{\Delta} = \frac{\alpha}{2+\beta}, \quad \textit{where} \quad \beta = \sum_{i=1}^{q} (k_i^T P_i k_i) \|P_i^{-1}\| \left(1 + \eta^2 + (k_i)_1^2\right). \tag{37}$$

($(k_i)_1$*,* $i = 1:q$*, are the first components of the vectors* k_i*, i.e.,* $(k_i)_1 = C_{s_i} k_i$*.)*

Proof. Note first that the uniformly Lipschitz assumptions in $\mathcal{H}_1$ and $\mathcal{H}_2$ guarantee the existence of solutions in $[0, \infty)$ of both the system and the observer. By assumption, the observer gains k_i are chosen such to satisfy the q inequalities

$$(A_{s_i} - k_i C_{s_i})^T P_i + P_i (A_{s_i} - k_i C_{s_i}) + (2\eta + \alpha + 1) P_i + \gamma_p^2(\bar{u})\, I_{s_i} \sum_{i=1}^{q} (B_{s_i}^T P_i B_{s_i}) \le 0, \tag{38}$$

where the matrices P_i are symmetric and positive definite. In order to get more compact formulas, let us define the matrices

$$\bar{A}_{s_i} = A_{s_i} + \eta I_{s_i}, \quad \bar{A}_i = A_{s_i} - k_i C_{s_i} + \eta I_{s_i} \tag{39}$$

(i.e., $\bar{A}_i = \bar{A}_{s_i} - k_i C_{s_i}$) so that inequalities (38) can be rewritten as

$$\bar{A}_i^T P_i + P_i \bar{A}_i + P_i + \gamma_p^2(\bar{u})\, I_{s_i} \sum_{i=1}^{q} (B_{s_i}^T P_i B_{s_i}) \le -\alpha P_i. \tag{40}$$

As discussed in Remark 2, the assumption $\mathcal{H}_1$ allows us to prove the exponential convergence by showing that (33) holds. By defining the variables

$$\epsilon_i(t) = e^{\eta t} \tilde{z}_i(t), \quad \epsilon(t) = \operatorname*{col}_{i=1}^{q} \{\epsilon_i(t)\} = e^{\eta t} \tilde{z}(t), \tag{41}$$

the condition (33) becomes $\lim_{t \to \infty} \|\epsilon(t)\| = 0$.

Let $K_0 = \operatorname{diag}_{i=1}^{q} \{k_i\}$, so that $K_{\bar{s}}(t) = K_0 \operatorname{diag}_{i=1}^{q} \{e^{-\eta \delta_i(t)}\}$. Then, the feedback term in (31) is

$$K_{\bar{s}}(t) C_{\bar{s}} \tilde{z}_{\bar{\delta}}(t) = K_0 C_{\bar{s}} \operatorname*{col}_{i=1}^{q} \{e^{-\eta \delta_i(t)} \tilde{z}_i(t - \delta_i(t))\}. \tag{42}$$

By definitions (41) we have $e^{-\eta \delta_i(t)} \tilde{z}_i(t - \delta_i(t)) = e^{-\eta t} \epsilon_i(t - \delta_i(t))$, so that (42) can be written as

$$K_{\bar{s}}(t) C_{\bar{s}} \tilde{z}_{\bar{\delta}}(t) = K_0 C_{\bar{s}} \operatorname*{col}_{i=1}^{q} \{e^{-\eta t} \epsilon_i(t - \delta_i(t))\} = e^{-\eta t} K_0 C_{\bar{s}}\, \epsilon_{\bar{\delta}}(t), \tag{43}$$

where $\epsilon_{\bar{\delta}}(t) = \operatorname{col}_{i=1}^{q} \{\epsilon_i(t - \delta_i(t))\}$. The computation of the time derivative of $\epsilon(t)$ gives

$$\dot{\epsilon}(t) = \eta \epsilon(t) + e^{\eta t} \dot{\tilde{z}}(t). \tag{44}$$

Replacing $\dot{\tilde{z}}(t)$ in (44) with (31), and using (43) we get the system

$$
\begin{aligned}
(45)\qquad & \dot\epsilon(t) = (A_{\bar s} + \eta I_n)\epsilon(t) + B_{\bar s}\varphi\big(t,\epsilon\big) - K_0 C_{\bar s}\epsilon_{\bar\delta}(t), \quad t \ge 0,\\
& \epsilon(\tau) = e^{\eta\, t}\tilde z(\tau) = z(\tau) - \phi(\tau), \qquad\qquad \tau \in [-\Delta, 0],\\
& \text{where}\ \ \varphi(t,\epsilon) = e^{\eta t}\tilde p\big(z(t), u(t), e^{-\eta t}\epsilon\big) = \operatorname*{col}_{i=1}^{q}\{e^{\eta t}\tilde p_i(z(t), u(t), e^{-\eta t}\epsilon)\},
\end{aligned}
$$

where $\tilde p(\cdot,\cdot,\cdot)$ has been defined in (30). From (32), under the assumption $||u(t)|| \le \bar u$, it follows that

$$
(46)\qquad \|\varphi(t,\epsilon)\| \le e^{\eta t}\gamma_p(\bar u)\|e^{-\eta t}\epsilon\| = \bar\gamma_p\|\epsilon\| \quad \forall t \ge \Delta,\ \forall \epsilon \in \mathbb{R}^n,
$$

where $\bar\gamma_p = \gamma_p(\bar u)$. Now, to prove the theorem it is sufficient to prove the global asymptotic stability of the solution $\epsilon \equiv 0$ of the system (45), because this ensures the limit (33), which in turn implies η-exponential convergence of the observer.

Let $\tilde\epsilon_i(t) = \epsilon_i(t) - \epsilon_i(t - \delta_i(t))$ and $\tilde\epsilon(t) = \operatorname{col}_{i=1}^{q}\{\tilde\epsilon_i(t)\}$, so that

$$
(47)\qquad \tilde\epsilon(t) = \epsilon(t) - \epsilon_{\bar\delta}(t).
$$

Then, adding to and subtracting from (45) the term $K_0 C_{\bar s}\,\epsilon(t)$ we get

$$
(48)\qquad \dot\epsilon(t) = (A_{\bar s} + \eta I_n - K_0 C_{\bar s})\epsilon(t) + B_{\bar s}\varphi\big(t,\epsilon\big) + K_0 C_{\bar s}\tilde\epsilon(t),
$$

so that (45) can be rewritten as

$$
\begin{aligned}
(49)\qquad & \dot\epsilon(t) = \bar A\epsilon(t) + B_{\bar s}\varphi(t,\epsilon) + K_0 C_{\bar s}\tilde\epsilon(t), \quad t \ge 0,\\
& \epsilon(\tau) = e^{\eta\tau}\tilde z(\tau), \qquad\qquad \tau \in [-\Delta, 0]
\end{aligned}
$$

with $\bar A = A_{\bar s} - K_0 C_{\bar s} + \eta I_n = \operatorname{diag}_{i=1}^{q}\{\bar A_i\}$. For $t \ge \Delta$, each $\tilde\epsilon_i(t) \in \mathbb{R}^{s_i}$ satisfies the integral equation

$$
\begin{aligned}
(50)\qquad \tilde\epsilon_i(t) &= \epsilon_i(t) - \epsilon_i(t - \delta_i(t)) = \int_{-\delta_i(t)}^{0} \dot\epsilon_i(t+\theta)d\theta\\
&= \int_{-\delta_i(t)}^{0}\Big((A_{s_i} + \eta I_{s_i})\epsilon_{t,i}(\theta) - k_i C_{s_i}\epsilon_{t,i}(\theta - \delta_i(t+\theta)) + B_{s_i}\varphi_i\big(t+\theta, \epsilon_t(\theta)\big)\Big)d\theta
\end{aligned}
$$

with $\epsilon_t(\theta) = \epsilon(t+\theta)$, $\epsilon_{t,i}(\theta) = \epsilon_i(t+\theta)$, and $\varphi_i(t,\epsilon) = e^{\eta t}\tilde p_i(z(t), u(t), e^{-\eta t}\epsilon)$.

Substitution into (49), taking into account that $k_i C_{s_i} B_{s_i}\varphi_i(t,\epsilon) = 0$, provides the following functional differential equation for $\epsilon(t)$ that holds for $t \ge \Delta$:

$$
(51)\qquad \dot\epsilon(t) = \bar A\epsilon(t) + B_{\bar s}\varphi(t,\epsilon(t)) + K_0 C_{\bar s}\begin{bmatrix} \displaystyle\int_{-\delta_1(t)}^{0} \bar A_{s_1}\epsilon_{t,1}(\theta) - k_1 C_{s_1}\epsilon_{t,1}(\theta - \delta_1(t+\theta))d\theta \\ \cdots \\ \displaystyle\int_{-\delta_q(t)}^{0} \bar A_{s_q}\epsilon_{t,q}(\theta) - k_q C_{s_q}\epsilon_{t,q}(\theta - \delta_q(t+\theta))d\theta \end{bmatrix},
$$

where $\bar A_{s_i} = A_{s_i} + \eta I_{s_i}$.

In order to prove the asymptotic stability of $\epsilon \equiv 0$, we consider the auxiliary time-delay system, defined for $t \geq \Delta$,

(52)
$$\dot{\xi}(t) = \bar{A}\xi(t) + B_{\bar{s}}\varphi(t,\xi(t)) + K_0 C_{\bar{s}} \begin{bmatrix} \displaystyle\int_{-\delta_1(t)}^{0} \bar{A}_{s_1}\xi_{t,1}(\theta) - k_1 C_{s_1}\xi_{t,1}(\theta - \delta_1(t+\theta))d\theta \\ \cdots \\ \displaystyle\int_{-\delta_q(t)}^{0} \bar{A}_{s_q}\xi_{t,q}(\theta) - k_q C_{s_q}\xi_{t,q}(\theta - \delta_q(t+\theta))d\theta \end{bmatrix},$$
$$\xi(\theta) = \psi(\theta), \quad \theta \in [-\Delta,\ \Delta],$$

where $\psi(\cdot) \in \mathcal{C}([-\Delta,\Delta],\mathbb{R}^n)$ is the initial condition. Clearly, the dynamics of (52) include the dynamics of (51) when $\xi(\theta) = \epsilon(\theta)$, $\theta \in [-\Delta,\ \Delta]$. (Note that $\xi(t)$ is not required to obey a differential equation of the type (51) in the interval $[0,\Delta]$.) As a consequence, asymptotic stability of (52) implies the asymptotic stability of (51). Let us rewrite (52) as

$$\dot{\xi}_i(t) = \bar{A}_i\xi_i(t) + B_{s_i}\varphi_i(t,\xi(t)) + k_i C_{s_i}\mu_i(\xi_t), \quad t \geq \Delta, \quad i = 1\!:\!q, \tag{53}$$

$$\mu_i(\xi_t) = \int_{-\delta_i(t)}^{0} \bar{A}_{s_i}\xi_{t,i}(\theta) - k_i C_{s_i}\xi_{t,i}(\theta - \delta_i(t+\theta))d\theta. \tag{54}$$

Now we can apply the Razumikhin method (see, e.g., [17]) to prove the asymptotic stability of system (53)–(54). Consider the Liapunov–Razumikhin function candidate

$$\mathcal{V}(\xi) = \sum_{i=1}^{q} \mathcal{V}_i(\xi) \quad \text{with } \mathcal{V}_i(\xi) = \xi_i^T P_i \xi_i, \tag{55}$$

where the q matrices P_i, symmetric and positive definite, satisfy (40). The computation of the derivative of $\dot{\mathcal{V}}\big(\xi(t)\big)$ gives

$$\begin{aligned} \dot{\mathcal{V}}\big(\xi_t\big) = &\sum_{i=1}^{q} \xi_i^T(t)\left(\bar{A}_i^T P_i + P_i\bar{A}_i\right)\xi_i(t) \\ &+ \sum_{i=1}^{q} 2\xi_i^T(t)P_i k_i C_{s_i}\mu_i\big(\xi_t\big) + 2\xi_i^T(t)P_i B_{s_i}\varphi_i\big(t,\xi(t)\big). \end{aligned} \tag{56}$$

In light of Razumikhin theorem it is sufficient to show that if the inequality $\mathcal{V}(\xi_t(\theta)) \leq \kappa\mathcal{V}(\xi(t))$ holds $\forall\theta \in [-2\Delta, 0]$, for some $\kappa > 1$, then $\dot{\mathcal{V}}(\xi_t)) \leq -\alpha_\kappa\mathcal{V}\big(\xi(t)\big)$, for some $\alpha_\kappa > 0$.

Let us compute upper bounds for each term in the right-hand side of (56). Consider the rightmost term, which contains both $\xi_i(t)$ and $\xi(t)$. By applying the inequality $|2x^T Pb| \leq x^T Px + b^T Pb$ and the Lipschitz condition (46) we have

$$|2\xi_i^T P_i B_{s_i}\varphi_i| \leq \xi_i^T P_i\xi_i + B_{s_i}^T P_i B_{s_i}|\varphi_i|^2 \leq \xi_i^T P_i\xi_i + \bar{\gamma}_p^2(B_{s_i}^T P_i B_{s_i})\xi^T\xi. \tag{57}$$

Substitution in (56) gives

$$\begin{aligned} \dot{\mathcal{V}}\big(\xi_t\big) \leq &\sum_{i=1}^{q} \xi_i^T(t)\left(\bar{A}_i^T P_i + P_i\bar{A}_i + P_i\right)\xi_i(t) + \left(\sum_{i=1}^{q} \bar{\gamma}_p^2(\bar{u})(B_{s_i}^T P_i B_{s_i})\right)\xi^T(t)\xi(t) \\ &+ \sum_{i=1}^{q} 2\xi_i^T(t)P_i k_i C_{s_i}\mu_i\big(\xi_t\big). \end{aligned} \tag{58}$$

Taking into account that $\xi^T\xi = \sum_{i=1}^q \xi_i^T\xi_i$ and thanks to (40) we get

$$\begin{aligned}\dot{\mathcal{V}}(\xi_t) &\le \sum_{i=1}^q \xi_i^T(t)\left(\bar{A}_i^T P_i + P_i\bar{A}_i + P_i + \bar{\gamma}_p^2(\bar{u})I_{s_i}\sum_{i=1}^q (B_{s_i}^T P_i B_{s_i})\right)\xi_i(t)\\ &\quad + \sum_{i=1}^q 2\xi_i^T(t)P_i k_i C_{s_i}\mu_i(\xi_t)\\ &\le -\sum_{i=1}^q \alpha\xi_i^T(t)P_i\xi_i(t) + \sum_{i=1}^q 2\xi_i^T(t)P_i k_i C_{s_i}\mu_i(\xi_t).\end{aligned} \tag{59}$$

From this

$$\dot{\mathcal{V}}(\xi_t) \le -\alpha\mathcal{V}(\xi(t)) + \sum_{i=1}^q 2\xi_i^T(t)P_i k_i C_{s_i}\mu_i(\xi_t). \tag{60}$$

Consider now the terms in the summation in (60). Using (54) we have

$$\begin{aligned}2\xi_i^T P_i k_i C_{s_i}\mu_i &\le \int_{-\delta_i(t)}^0 \left|2\xi_i^T(t)P_i k_i C_{s_i}\bar{A}_{s_i}\xi_{t,i}(\theta)\right|d\theta\\ &\quad + \int_{-\delta_i(t)}^0 \left|2\xi_i^T(t)P_i k_i C_{s_i}k_i C_{s_i}\xi_{t,i}(\theta-\delta_i(t+\theta))\right|d\theta.\end{aligned} \tag{61}$$

Using again $|2x^TPb| \le x^TPx + b^TPb$, we have for the first integrand

$$|2\xi_i^T(t)P_i k_i C_{s_i}\bar{A}_{s_i}\xi_{t,i}(\theta)| \le \xi_i^T(t)P_i\xi_i(t) + \left(k_i{}^T P_i k_i\right)\|C_{s_i}(A_{s_i}+\eta I_{s_i})\|^2\|\xi_{t,i}(\theta)\|^2. \tag{62}$$

Noting that $\|C_{s_i}(A_{s_i}+\eta I_{s_i})\|^2 \le \|C_{s_i}(A_{s_i}\|^2 + \eta^2\|C_{s_i})\|^2 = 1+\eta^2$, due to the Brunowsky structure of the pair (A_{s_i}, C_{s_i}), and that for any $P>0$ it is $\|x\|^2 \le \|P^{-1}\|x^TPx$, the following bound holds for the first integrand:

$$\begin{aligned}|2\xi_i^T(t)P_i k_i C_{s_i}\bar{A}_{s_i}\xi_{t,i}(\theta)| &\le \xi_i^T(t)P_i\xi_i(t) + \left(k_i{}^T P_i k_i\right)(1+\eta^2)\|P_i^{-1}\|\xi_{t,i}^T(\theta)P_i\xi_{t,i}^T(\theta)\\ &\le \mathcal{V}_i(\xi_i(t)) + \left(k_i^T P_i k_i\right)(1+\eta^2)\|P_i^{-1}\|\mathcal{V}_i(\xi_{t,i}(\theta)).\end{aligned} \tag{63}$$

Thus, under the Razumikhin hypothesis $\mathcal{V}(\xi_t(\theta)) \le \kappa\mathcal{V}(\xi(t))$ at time t, $\kappa>1$, which obviously implies $\mathcal{V}_i(\xi_{t,i}(\theta)) \le \kappa\mathcal{V}(\xi))$, for $i=1:q$, we get the following bound for the first integral of (61):

$$\begin{aligned}&\int_{-\delta_i(t)}^0 \left|2\xi_i^T(t)P_i k_i C_{s_i}\bar{A}_{s_i}\xi_{t,i}(\theta)\right|d\theta\\ &\quad \le \delta_i(t)\left(\mathcal{V}_i(\xi_i(t)) + \kappa(1+\eta^2)\left(k_i^T P_i k_i\right)\|P_i^{-1}\|\mathcal{V}(\xi(t))\right).\end{aligned} \tag{64}$$

Similarly, for the second integrand of (61) we have

$$\begin{aligned}&\left|2\xi_i^T(t)P_i k_i C_{s_i}k_i C_{s_i}\xi_{t,i}(\theta-\delta_i(t+\theta))\right|\\ &\quad \le\ \xi_i^T(t)P_i\xi_i(t) + \left(k_i^T P_i k_i\right)\|C_{s_i}k_i\|^2\|C_{s_i}\xi_{t,i}(\theta-\delta_i(t+\theta))\|^2\\ &\quad \le\ \mathcal{V}_i(\xi_i(t))) + \left(k_i^T P_i k_i\right)k_{i,1}^2\|P_i^{-1}\|\mathcal{V}_i(\xi_{t,i}(\theta-\delta_i(t+\theta))),\end{aligned} \tag{65}$$

where $k_{i,1} = C_{s_i} k_i$ is the first element of the vector k_i. Under the Razumikhin condition we have the following bound for the integral:

$$\int_{-\delta_i(t)}^{0} \left|2\xi_i^T(t) P_i k_i C_{s_i} k_i C_{s_i} \xi_{t,i}(\theta - \delta_i(t+\theta))\right| d\theta \leq \delta_i(t) \left(\mathcal{V}_i\big(\xi_i(t)\big) + \kappa\, k_{i,1}^2 \big(k_i^T P_i k_i\big) \|P_i^{-1}\| \mathcal{V}\big(\xi(t)\big)\right). \tag{66}$$

Putting together (64) and (66), and taking into account the bound $\delta_i(t) \leq \Delta$, from (61) the following inequality is obtained, which holds in all times $t \geq \Delta$, where the Razumikhin condition holds:

$$2\xi_i^T(t) P_i k_i C_{s_i} \mu_i(\xi_t) \leq \Delta \left(2\mathcal{V}_i\big(\xi_i(t)\big) + \kappa\,(1 + \eta^2 + k_{i,1}^2)\big(k_i^T P_i k_i\big) \|P_i^{-1}\| \mathcal{V}\big(\xi(t)\big)\right). \tag{67}$$

The substitution of the bound (67) into (60), after simple manipulations, gives

$$\dot{\mathcal{V}}\big(\xi(t)\big) \leq \left(-\alpha + \Delta\,(2 + \kappa\,\beta)\right) \mathcal{V}\big(\xi(t)\big), \tag{68}$$

where $\beta = \sum_{i=1}^{q} (1 + \eta^2 + k_{i,1}^2)(k_i^T P_i k_i) \|P_i^{-1}\|$ has been defined in (37).

Recall that if at time t the Razumikhin condition is verified for some $\kappa > 1$ (i.e., $\mathcal{V}\big(\xi_t(\theta)\big) \leq -\kappa \mathcal{V}\big(\xi(t)\big)$, for $\theta \in [-2\Delta, 0]$), then the inequality (68) holds true. It remains to show that if $\Delta < \overline{\Delta} = \alpha/(2+\beta)$ (see (37)), then there exists $\kappa_\Delta > 1$ and $\alpha_{\kappa_\Delta} > 0$ such that in all $t \geq \Delta$ where the Razumikhin condition is verified, the inequality $\dot{\mathcal{V}}\big(\xi(t)\big) \leq -\alpha_{\kappa_\Delta} \mathcal{V}\big(\xi(t)\big)$ holds. To this aim, consider the function

$$\pi(\kappa) = \frac{(2-\kappa)\alpha}{2 + \kappa\,\beta}, \tag{69}$$

which is such that $\pi(1) = \overline{\Delta}$ and $\pi(2) = 0$. Being $\pi(\kappa)$ monotonically decreasing in the interval $[1, 2]$, it follows that for any given $\Delta \in (0, \overline{\Delta})$ there exists $\kappa_\Delta \in (1, 2)$ such that $\pi(\kappa_\Delta) = \Delta$. The pair (Δ, κ_Δ) is such that $(2 + \kappa_\Delta \beta)\Delta = (2 - \kappa_\Delta)\alpha$, and its substitution into (68) gives

$$\dot{\mathcal{V}}\big(\xi(t)\big) \leq \left(-\alpha + (2 - \kappa_\Delta)\,\alpha\right)\big) \mathcal{V}\big(\xi(t)\big) = (1 - \kappa_\Delta)\alpha\, \mathcal{V}\big(\xi(t)\big) = -\alpha_\kappa\, \mathcal{V}\big(\xi(t)\big), \tag{70}$$

where $\alpha_\kappa = (\kappa_\Delta - 1)\alpha > 0$ (recall that $\kappa_\Delta \in (1, 2)$). This proves that $\overline{\Delta}$ given in (37) is such that for any $\Delta < \overline{\Delta}$ there exists $\kappa_\Delta > 1$ that satisfies the conditions of the Razumikhin theorem for the equilibrium $\xi_t = 0$ of system (52). As previously discussed, this implies $\epsilon(t) \to 0$, and this in turn implies the η-exponential convergence to zero of the observation error, in both x- and z-coordinates. □

The following theorem provides a more explicit criterion for the choice of the observer gains k_i and for the associated bound $\overline{\Delta}$ on the maximum delay.

THEOREM 6. *Consider system* (1)–(2) *under the same assumptions as Theorem* 5. *Let* $v_i \in \mathbb{R}_-^{s_i}$, $i = 1:q$, *be* q *vectors with distinct and negative components, and let* $w_i = -\max_{h=1}^{s_i}\{(v_i)_h\}$. *Consider the family of observer gains* $\bar{k}_i(\rho)$, *with* $\rho \in \mathbb{R}_+$, *defined as*

$$\bar{k}_i(\rho) = -\operatorname*{diag}_{h=1}^{s_i}\{\rho^h\} V^{-1}(v_i) v_i^{(s_i)}, \quad i = 1{:}q. \tag{71}$$

Then, given a desired exponential decay rate $\eta > 0$, *and given an arbitrary* $\alpha > 0$, *the system* (24)–(26) *is a global* η*-exponential observer for system* (1)–(2) *for any* ρ *satisfying*

$$\rho > \max\left\{1, \max_{i=1}^{q}\left\{\frac{2\eta+\alpha+1+\bar{\gamma}_p^2(\bar{u})\,n\,\|V^{-1}(v_i)\|^2}{2w_i}\right\}\right\}, \tag{72}$$

provided that $\Delta \in [0, \overline{\Delta})$*, where*

$$\overline{\Delta} = \frac{\alpha}{2+\beta(\rho)} \quad \textit{with} \quad \beta(\rho) = \sum_{i=1}^{q} \rho^{2s_i}\|v_i^{(s_i)}\|^2\|V^{-1}(v_i)\|^2\big(1+\eta^2+\rho^2(\mathbb{1}_{s_i}^T v_i)^2\big). \tag{73}$$

Proof. When ρ satisfies (36), the gains $\bar{k}_i(\rho)$ in (71) together with the SPD matrices $\overline{P}_i(\rho)$ defined in (35) are solution pairs of the inequalities (34). Since $\rho > 1$, we have $\big\|\big(\overline{P}_i(\rho)\big)^{-1}\big\| \le \|V^{-1}(v_i)\|^2$. Moreover, $\bar{k}_i^T(\rho)\overline{P}_i(\rho)\bar{k}_i(\rho) = \|v_i^{(s_i)}\|^2\rho^{2s_i}$, and $\big(\bar{k}_i(\rho)\big)_1 = \rho(\mathbb{1}_{s_i}^T v_i)$. Using these in (37), the formula (73) for $\overline{\Delta}$ is easily obtained. ∎

Remark 3. The global convergence results of Theorems 5 and 6 have been obtained under the global Lipschitz and observability assumptions $\mathcal{H}_1$ and $\mathcal{H}_2$. However, weaker local results can be obtained if local Lipschitz and observability assumptions are adopted instead. For those systems that admit compact invariant subsets of the state space, the assumptions $\mathcal{H}_1$ and $\mathcal{H}_2$ need to be satisfied only in such sets. The convergence of the observer (24)–(26) in such invariant sets can be proved following the same lines of the proof of Corollary 1 in [5].

4. Chain observer. It may happen that the maximum measurement delay Δ in system (1)–(2) is *too large* for a single-step observer of the type (24)–(26) (e.g., for any $\eta > 0$ the observer gains k_j that achieve η-exponential convergence may not satisfy the condition $\Delta < \overline{\Delta}$ of Theorem 5, with $\overline{\Delta}$ given by (37)). In this case we can resort to a *chain observer*, in which, roughly speaking, the delay Δ is split into smaller subdelays in order to satisfy the convergence conditions. For a precise description of the operations of a chain observer the following definition is useful.

Definition 7. *Given a delay* $\Delta > 0$ *and an integer* $m > 1$*, an* m-partition *of* Δ *is a strictly increasing sequence* $\bar{\sigma} = \{\sigma_j\}_{j=0}^m$*, such that* $\sigma_0 = 0$ *and* $\sigma_m = \Delta$*, so that* $\Delta = \sum_{j=1}^m \tilde{\sigma}_j$*, where* $\tilde{\sigma}_j = \sigma_j - \sigma_{j-1}$.

As a general statement, given an integer $m > 1$ and an m-partition $\bar{\sigma}$ of the maximum delay Δ, a chain observer is a set of (m or $m+1$) interconnected observers, each one devoted to the observation of the state at time $t - \sigma_j$. Previous proposals of chain observers (e.g., [12, 16, 2]) considered only the case of a single constant measurement delay, i.e., $\delta_i(t) \equiv \Delta$, and used a uniform m-partition of Δ. In this paper we extend this framework to multiple and time-dependent delays, using a nonnecessarily uniform m-partition. In the proposed approach, we consider a cascade of m observers, numbered with $j = 1:m$, where the output of the jth observer, denoted $\hat{x}_j(t)$, is aimed at estimating $x(t-\sigma_{j-1})$. The output of the first observer, $\hat{x}_1(t)$, is devoted to the observation of the *current* state $x(t)$ (i.e., $x(t-\sigma_0)$) and is the output of the chain observer. As we will see below, the jth observer is driven by the available measurement pairs $\big\{\bar{y}_i(t), \delta_i(t)\big\}_1^q$ or by the output $\hat{x}_{j+1}(t)$ of the *previous* observer of the chain, depending on whether $\delta_i(t) \le \sigma_j$ or $\delta_i(t) > \sigma_j$.

Given the system (1)–(2), where $x(t)$ is defined for $t \ge -\Delta$, and given $\bar{\sigma}$, an m-partition of Δ, let us define the delayed variables $x_j(t) = x(t-\sigma_{j-1})$, defined for $t \ge -\Delta+\sigma_{j-1}$, which obey the m equations $\dot{x}_j(t) = F\big(x_j(t), u(t-\sigma_{j-1})\big)$, $j = 1:m$. In order to achieve a correct overall behavior, each observer in the chain must be driven by a suitable transformation of the measurement pairs $\big\{\bar{y}_i(t), \delta_i(t)\big\}_1^q$, as described below. The proposed chain of m observers is as follows:

$$\dot{\hat{x}}_j(t) = F\big(\hat{x}_j(t), u(t-\sigma_{j-1})\big) + \big(Q_{\bar{s}}(\hat{x}_j(t))\big)^{-1} K_j(t) \operatorname*{col}_{i=1}^{q}\{\nu_{j,i}(t)\}, \quad t \geq 0, \tag{74}$$
$$\hat{x}_j(\tau) = \phi(\tau - \sigma_{j-1}), \quad \tau \in [-\tilde{\sigma}_j, 0], \quad (\tilde{\sigma}_j = \sigma_j - \sigma_{j-1}), \quad j = 1\!:\!m,$$

$$K_j(t) = \operatorname*{diag}_{i=1}^{q}\{e^{-\eta_j \delta_{j,i}(t)} K_{j,i}\}, \quad K_{j,i} \in \mathbb{R}^{s_i}, \quad \eta_j > 0, \tag{75}$$

$$\nu_{j,i}(t) = \bar{y}_{j,i}(t) - h_i\big(\hat{x}_j(t - \delta_{j,i}(t))\big), \tag{76}$$

where the transformed measurements $\big(\bar{y}_{j,i}(t), \delta_{j,i}(t)\big)$ are defined, for $i\!:\!q$, as

(77)
$$\begin{array}{lll}
\bar{y}_{j,i}(t) = y_i(t-\sigma_{j-1}), \ \delta_{j,i}(t) = 0 & \text{if } \delta_i(t) \in [0, \sigma_{j-1}), & j = 2\!:\!m, \\
\bar{y}_{j,i}(t), = \bar{y}_i\big(t\big), \quad \delta_{j,i}(t) = \delta_i(t) - \sigma_{j-1} & \text{if } \delta_i(t) \in [\sigma_{j-1}, \sigma_j], & j = 1\!:\!m, \\
\bar{y}_{j,i}(t) = h_i(\hat{x}_{j+1}(t)), \quad \delta_{j,i}(t) = \tilde{\sigma}_j & \text{if } \delta_i(t) \in [\sigma_j, \Delta], & j = 1\!:\!m-1.
\end{array}$$

Note that when $\delta_i(t) \in [0, \sigma_{j-1})$, the measured output $\bar{y}_i(t)$ is further delayed: $\bar{y}_i(t^*)$ is used in the place of $\bar{y}_i(t)$, with an additional delay $\sigma_{j-1} - \delta(t^*)$, where $t^* = t + \delta(t^*) - \sigma_{-1}$. Overall, we have $\bar{y}_{j,i}(t) = y_i(t - \sigma_{j-1})$, so that the jth observer behaves as a delayless observer ($\delta_{j,i}(t) = 0$) aimed at estimating $x_j(t) = x(t - \sigma_{j-1})$. When $\delta_i(t) \in [\sigma_{j-1}, \sigma_j]$ the measurement is not modified, but from the viewpoint of the jth observer the delay is $\delta_{j,i}(t) = \delta_i(t) - \sigma_{j-1}$. When $\delta_i(t) \in (\sigma_j, \Delta]$, the measured output is replaced by $h_i(\hat{x}_{j+1}(t))$, that is, the estimate of $y_i(t - \sigma_j)$ coming from the $(j+1)$th observer of the chain. In this case the delay with respect to $x(t - \sigma_{j-1})$ is $\delta_{j,i}(t) = \tilde{\sigma}_j$. Using definitions (77) the modified measurements $\bar{y}_{j,i}(t)$ and the output error terms $\nu_{j,i}(t) = \bar{y}_{j,i}(t) - h_i\big(\hat{x}_j(t - \delta_{j,i}(t))\big)$ in (74) are as follows:

(78)
$$\left.\begin{array}{l} \delta_i(t) \in [0, \sigma_{j-1}) \\ j = 2 : m \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \bar{y}_{j,i}(t) = y_i(t - \sigma_{j-1}), \qquad \delta_{j,i}(t) = 0, \\ \quad \Rightarrow \quad \nu_{j,i}(t) = y_i(t - \sigma_{j-1}) - h_i\big(\hat{x}_j(t)\big), \end{array}\right.$$
$$\left.\begin{array}{l} \delta_i(t) \in [\sigma_{j-1}, \sigma_j] \\ j = 1 : m \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \bar{y}_{j,i}(t) = y_i(t - \delta_i(t)), \qquad \delta_{j,i}(t) = \delta_i(t) - \sigma_{j-1}, \\ \quad \Rightarrow \quad \nu_{j,i}(t) = y_i(t - \delta_i(t)) - h_i\big(\hat{x}_j(t - \delta_{j,i}(t))\big), \end{array}\right.$$
$$\left.\begin{array}{l} \delta_i(t) \in (\sigma_j, \Delta] \\ j = 1 : m-1 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \bar{y}_{j,i}(t) = h_i\big(\hat{x}_{j+1}(t)\big), \qquad \delta_{j,i}(t) = \sigma_j - \sigma_{j-1} = \tilde{\sigma}_j, \\ \quad \Rightarrow \quad \nu_{j,i}(t) = h_i\big(\hat{x}_{j+1}(t)\big) - h_i\big(\hat{x}_j(t - \tilde{\sigma}_j)\big). \end{array}\right.$$

Figure 1 shows the chain configuration in the case of a scalar measurement, $q = 1$ and $y = h(x(t - \delta(t)))$. The chain has four observers, and the case illustrates the situation when $\sigma_2 > \delta(t) > \sigma_1$. Here the current output is used for the observer $\hat{x}_2(t)$, whereas the previous observers $\hat{x}_3(t)$, $\hat{x}_4(t)$ use past measurements. The first observer, $\hat{x}_1(t)$, that provides the estimate of $x(t)$ uses the estimated output provided by $\hat{x}_2(t)$.

Remark 4. The new delays $\delta_{j,i}(t)$ are defined in (77) in such a way that $\delta_{j,i}(t) \in [0, \tilde{\sigma}_j]$, where $\tilde{\sigma}_j = \sigma_j - \sigma_{j-1} < \Delta$, and $\delta_i(t) = \sum_{j=1}^{m} \delta_{j,i}(t)$ $\forall t \geq 0$ (delay decomposition). Note that in order to artificially introduce the delay $\sigma_{j-1} - \delta_i(t)$ in the measurement $\bar{y}_i(t)$, when $\delta_i(t) \in [0, \sigma_{j-1})$, the delays $\delta_i(t)$ must be continuous functions of $t \geq 0$.

Now, consider, as in section 3, the change of coordinates $z_j = \Phi_{\bar{s}}(x_j)$, and let $\zeta_{j,i}(t) = \Phi_i^{s_i}\big(x_j(t)\big)$, so that $z_j(t) = \operatorname{col}_{i=1}^{q}\{\zeta_{j,i}(t)\}$ $(z_j(t) \in \mathbb{R}^n)$. The dynamics of

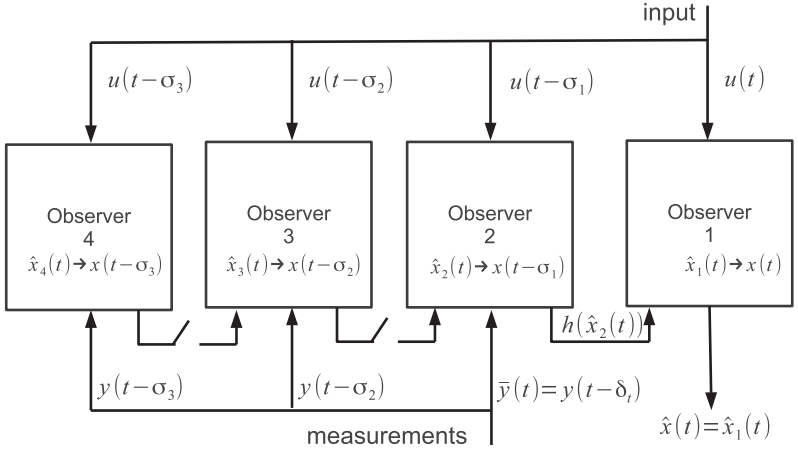

FIG. 1. *Configuration of a chain of four observers, scalar measurement, when* $\sigma_2 > \delta(t) > \sigma_1$.

$z_j(t) = \Phi_{\bar{s}}\big(x_j(t)\big)$ is $\dot{z}_j(t) = A_{\bar{s}} z_j(t) + B_{\bar{s}} p\big(z_j(t), u(t-\sigma_{j-1})\big)$, $t \geq -\Delta + \sigma_{j-1}$, while the dynamics of $\hat{z}_j(t) = \Phi_{\bar{s}}\big(\hat{x}_j(t)\big)$ is

$$\begin{aligned} \dot{\hat{z}}_j(t) &= A_{\bar{s}}\hat{z}_j(t) + B_{\bar{s}} p(\hat{z}_j(t), u(t-\sigma_{j-1})) + K_j(t) \operatorname*{col}_{i=1}^{q}\{\nu_{j,i}(t)\}, \qquad t \geq 0, \\ \hat{z}_j(\tau) &= \Phi_{\bar{s}}\big(\phi(\tau-\sigma_{j-1})\big), \quad \tau \in [-\tilde{\sigma}_j, 0]. \end{aligned} \tag{79}$$

In the z-coordinates, the modified outputs $\bar{y}_{j,i}(t)$ defined in (77) can be written as

$$\bar{y}_{j,i}(t) = \begin{cases} C_{s_i}\zeta_{j,i}(t-\delta_{j,i}(t)) & \text{if } \delta_i(t) \in [0, \sigma_j], \\ C_{s_i}\hat{z}_{j+1,i}(t) & \text{if } \delta_i(t) \in (\sigma_j, \Delta]. \end{cases} \tag{80}$$

Remark 5. The measurement transformations (77) can be considerably simplified if known lower and upper bounds Δ_L and Δ_U exist for the measurement delays, i.e., if $\delta_i(t) \in [\Delta_L, \Delta_U]$ $\forall t \geq 0$, $i = 1\!:\!q$. In this case, if an m-partition $\bar{\sigma}$ is chosen such that $\sigma_{m-1} = \Delta_L$ and $\sigma_m = \Delta_U$, then the modified measurement pairs $\big\{\bar{y}_{j,i}(t), \delta_{j,i}(t)\big\}_{j=1}^{m}$ defined in (77) are as follows:

$$\begin{aligned} \bar{y}_{j,i}(t) &= h_i\big(\hat{x}_{j+1,i}(t)\big), \quad & \delta_{j,i}(t) &= \tilde{\sigma}_j, \qquad & j &= 1\!:\!m-1, \\ \bar{y}_{m,i}(t) &= \bar{y}_i(t), & \delta_{m,i}(t) &= \delta_i(t) - \sigma_{m-1}. \end{aligned} \tag{81}$$

Note that in this case the delays $\delta_i(t)$ do not need to be continuous. In the particular case of constant delays $\delta_i(t) = \Delta$, $i = 1\!:\!q$, then the second of (81) is replaced by

$$\bar{y}_{m,i}(t) = \bar{y}_i(t), \qquad \delta_{m,i}(t) = \tilde{\sigma}_m. \tag{82}$$

4.1. Convergence analysis. Before giving the main convergence result for the chain observer (74), a preliminary lemma is needed. (The proof is in Appendix A.3.)

LEMMA 8. *Consider a delay system of the type*

$$\dot{\xi}(t) = b(t, \xi_t) + \mu(t), \quad t \geq 0, \qquad \textit{with} \quad \xi_0(\tau) = \phi(\tau), \quad \tau \in [-\Delta, 0], \tag{83}$$

where ξ_t *is the system state, with initial value* $\phi \in \mathcal{C}^n_\Delta$. $\mu(t) \in \mathbb{R}^n$ *is an input function, and* $b : \mathbb{R}_+ \times \mathcal{C}^n_\Delta \mapsto \mathbb{R}^n$ *is such that, for some* $\gamma_b > 0$,

$$\|b(t, \phi_1) - b(t, \phi_2)\| \leq \gamma_b \|\phi_1 - \phi_2\| \quad \forall \phi_1, \phi_2 \in \mathcal{C}^n_\Delta, \quad \forall t \geq 0. \tag{84}$$

Assume that, for a given $\bar\eta > 0$, the solution $\xi_t = 0$ is globally $\bar\eta$-exponentially stable:

$$\mu(t) = 0\ \forall t \ge 0 \quad \Rightarrow \quad \forall \xi_0 \in \mathcal{C}^n_\Delta,\ \exists \bar c > 0 :\ \|\xi(t)\| \le e^{-\bar\eta t}\bar c\ \ \forall t \ge 0. \tag{85}$$

Then, given arbitrary $\bar\mu > 0$ and $\eta \in (0, \bar\eta)$, we have

$$\|\mu(t)\| \le e^{-\bar\eta t}\bar\mu\ \forall t \ge 0 \quad \Rightarrow \quad \forall \xi_0 \in \mathcal{C}^n_\Delta,\ \exists c > 0 :\ \|\xi(t)\| \le e^{-\eta t}c\ \ \forall t \ge 0. \tag{86}$$

THEOREM 9. *Consider system (1)–(2) with $\delta_i : \mathbb{R}_+ \mapsto [0, \Delta]$ continuous functions, $i = 1\!:\!q$. Let conditions $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ be satisfied, and assume that $\|u(t)\| \le \bar u$ $\forall t \ge -\Delta$ for some $\bar u > 0$. For a given integer $m > 1$, consider an m-partition $\bar\sigma$ of Δ. Let $\eta > 0$ be a given desired error decay rate. Consider a strictly increasing sequence of m positive numbers $\{\eta_j\}_0^m$ with $\eta_0 = \eta$. Consider $m\cdot q$ pairs $(K_{j,i}, P_{j,i}) \in \mathbb{R}^{s_i} \times \Sigma_+^{s_i \times s_i}$, for $(i, j) \in (1\!:\!q) \times (1\!:\!m)$, that satisfy $m\cdot q$ inequalities of the type (34), with $a = 2\eta_j + \alpha + 1$, with arbitrary $\alpha > 0$. For $j = 1\!:\!m$ let*

$$\overline{\Delta}_j = \frac{\alpha}{2 + \beta_j} \quad \text{with } \beta_j = \sum_{i=1}^{q} (K_{j,i}^T P_{j,i} K_{j,i}) \|P_{j,i}^{-1}\| \left(1 + \eta_j^2 + (K_{j,i})_1^2\right). \tag{87}$$

Then, if $\tilde\sigma_j < \overline{\Delta}_j$, for $j = 1\!:\!m$, the system (74)–(76) is a global η-exponential chain observer for system (1)–(2).

Proof. Consider the error equation of the jth observer in z-coordinates. Exploiting the variables $\zeta_{j,i}(t)$ and $\hat z_{j,i}(t)$, and using (80), the forcing terms $\nu_{j,i}(t)$ of the observer, defined as $\nu_{j,i}(t) = \bar y_{j,i}(t) - h_i\big(\hat x_j(t - \delta_{j,i}(t))\big)$, can be rewritten as

$$\nu_{j,i}(t) = \begin{cases} C_{s_i}\big(\zeta_{j,i}(t - \delta_{j,i}(t)) - \hat z_{j,i}(t - \delta_{j,i}(t))\big) & \text{if } \delta_i(t) \in [0, \sigma_j], \\ C_{s_i}\big(\hat z_{j+1,i}(t) - \hat z_{j,i}(t - \tilde\sigma_j)\big) & \text{if } \delta_i(t) \in (\sigma_j, \Delta]. \end{cases} \tag{88}$$

Let us define the error components $\tilde z_{j,i}(t) = \zeta_{j,i}(t) - \hat z_{j,i}(t)$. Notice that, by definition, $\zeta_{j+1,i}(t) = \zeta_{j,i}(t - \tilde\sigma_j)$, so that in (88) we have

$$\begin{aligned} \hat z_{j+1,i}(t) - \hat z_{j,i}(t - \tilde\sigma_j) &= \hat z_{j+1,i}(t) - \hat z_{j,i}(t - \tilde\sigma_j) - \zeta_{j+1,i}(t) + \zeta_{j,i}(t - \tilde\sigma_j) \\ &= \tilde z_{j,i}(t - \tilde\sigma_j) - \tilde z_{j+1,i}(t), \end{aligned} \tag{89}$$

and from this, recalling that $\delta_{j,i}(t) = \tilde\sigma_j$ when $\delta_i(t) \in (\sigma_j, \Delta]$,

$$\nu_{j,i}(t) = \begin{cases} C_{s_i}\tilde z_{j,i}\big(t - \delta_{j,i}(t)\big) & \text{if } \delta_i(t) \in [0, \sigma_j], \\ C_{s_i}\tilde z_{j,i}\big(t - \delta_{j,i}(t)\big) - C_{s_i}\tilde z_{j+1,i}(t) & \text{if } \delta_i(t) \in (\sigma_j, \Delta]. \end{cases} \tag{90}$$

Thus, we can write in short

$$\nu_{j,i}(t) = C_{s_i}\tilde z_{j,i}(t - \delta_{j,i}(t)) - C_{s_i}\chi_j\big(\delta_i(t)\big)\tilde z_{j+1,i}(t), \quad \delta_i(t) \in [0, \Delta], \tag{91}$$

where $\chi_j : [0, \Delta] \mapsto \{0, 1\}$ is the characteristic function of the interval $(\sigma_j, \Delta]$ (i.e., $\chi_j(\delta) = 0$ if $\delta \in [0, \sigma_j)$, $\chi_j(\delta) = 1$ if $\delta \in (\sigma_j, \Delta]$). Using (91) we can rewrite (79) as

$$\begin{aligned} &\dot{\tilde z}_j(t) = A_{\bar s}\tilde z_j(t) + B_{\bar s}\tilde p_j\big(z(t), u(t), \tilde z_j(t)\big) + K_j(t)C_{\bar s}\tilde z_{j,\bar\delta_j}(t) - K_j(t)C_{\bar s}\tilde n_j(t, \bar\delta(t)),\ \ t \ge 0, \\ &\tilde z_j(\tau) = z(\tau - \sigma_{j-1}) - \Phi_{\bar s}\big(\phi(\tau - \sigma_{j-1})\big), \qquad \tau \in [-\tilde\sigma_j, 0], \\ &\text{where} \quad \tilde z_{j,\bar\delta_j}(t) = \operatorname*{col}_{i=1}^{q}\big\{\tilde z_{j,i}\big(t - \delta_{j,i}(t)\big)\big\}, \quad \tilde n_j(t, \bar\delta(t)) = \operatorname*{col}_{i=1}^{q}\big\{\chi_j(\delta_i(t))\tilde z_{j+1,i}(t)\big\}. \end{aligned} \tag{92}$$

Note that for $j = m$ this equation coincides with the single-step observer equation of Theorem 5, so that, under the given conditions for $\tilde{\sigma}_m$ (i.e., $\tilde{\sigma}_m < \overline{\Delta}_m$ given in (87)) and for the gains $K_{m,i}$ (i.e., satisfying (34)), we have that the mth observer of the chain is a global η_m-exponential observer for $x_m(t) = x(t - \sigma_{m-1})$, because by construction $\delta_{m,i}(t) \in [0, \tilde{\sigma}_m]$. Thus, for any $\phi \in \mathcal{C}^n_\Delta$ there exists $c_m > 0$ such that $\|\hat{z}_m(t)\| \le e^{-\eta_m t} c_m$. Recalling that by assumption $\eta_m > \eta_{m-1}$, we have also

$$\|\hat{z}_m(t)\| \le e^{-\eta_{m-1} t} c_m. \tag{93}$$

Note that equations (92), for $j = 1\!:\!m$, are of the form

$$\dot{\tilde{z}}_j(t) = b_j(t, \tilde{z}_{j,t}) + \mu_j\big(t, \tilde{z}_{j+1}(t)\big), \quad t \ge 0, \qquad \tilde{z}_{j,0} \in \mathcal{C}^n_{\tilde{\sigma}_j}, \tag{94}$$

where the term $\mu_j\big(t, \tilde{z}_{j+1}(t)\big) = -K_j(t) C_{\bar{s}} \tilde{n}_j(t, \bar{\delta}(t))$ can be regarded as an external input to the jth observer. Thus, (92) is a system of the type (83) of Lemma 8. If $\tilde{z}_{j+1}(t) \equiv 0$, the given assumptions on $\tilde{\sigma}_j$ and $K_{j,i}$ ensure, thanks to Theorem 5, that the equilibria $\tilde{z}_{j,t} = 0$ are η_j-exponentially stable. Thus, thanks to Lemma 8, the following implication holds true for any $j = 1\!:\!m-1$:

$$\|\tilde{z}_{j+1}(t)\| \le e^{-\eta_j t} c_{j+1}, \text{ for some } c_{j+1} > 0 \quad \Rightarrow \quad \exists c_j : \ \|\tilde{z}_j(t)\| \le e^{-\eta_{j-1} t} c_j. \tag{95}$$

Thus, by finite induction, we have that inequality (93) implies that $\|\tilde{z}_1(t)\| \le e^{-\eta_0 t} c_1$. Being $\eta_0 = \eta$, the η-exponential observation error decay is proved. ∎

Now we can give the main theorem that ensures the existence of a global η-exponential chain observer for any value of the maximum measurement delay Δ.

THEOREM 10. *Under the same assumptions of Theorem* 9 *on system* (1)–(2)*, for any given desired error decay rate* $\eta > 0$*, and for any given maximum delay* Δ*, there exist* $m \in \mathbb{N}$*, a* m*-partition* $\bar{\sigma}$ *of* Δ *and gains* $K_{j,i} \in \mathbb{R}^{s_i}$*,* $(i,j) \in (1\!:\!q) \times (1\!:\!m)$*, such that* (74)–(76) *is a global* η*-exponential chain observer for system* (1)–(2)*.*

Proof. The proof is achieved by showing how to choose $m \in \mathbb{N}$, an m-partition $\bar{\sigma}$, a sequence $\{\eta_j\}_0^m$, and gains $K_{j,i}$, together with SPD matrices $P_{j,i}$, such to satisfy the assumptions of Theorem 9. Choose arbitrary $\eta' > \eta$ and $\alpha > 0$, and consider a set of solution pairs $(k_i, P_i) \in \mathbb{R}^{s_i} \times \Sigma_+^{s_i \times s_i}$ for the q inequalities (34), for $i = 1\!:\!q$, with $a = 2\eta' + \alpha + 1$. Set $K_{j,i} = k_i$, $P_{j,i} = P_i$, for $j = 1\!:\!m$. Let $\overline{\Delta}'$ and β' be given by (37) with η replaced by η' in the summation. Choose an arbitrary $d_{\eta'} < \overline{\Delta}'$, let $m = \lceil \Delta / d_{\eta'} \rceil$, and let $\sigma_j = \frac{j}{m}\Delta$, $j = 1\!:\!m$. Note that if $m = 1$ the assumptions of Theorem 5 are satisfied, so we get a single-step global η-exponential observer. Thus, we consider the general case $m > 1$. Note that with the chosen $\bar{\sigma}$ we have $\tilde{\sigma}_j = \frac{\Delta}{m} \le d_{\eta'} < \overline{\Delta}'$, $j = 1\!:\!m$. Now choose an arbitrary strictly increasing sequence $\{\eta_j\}_0^m$ such that $\eta_0 = \eta$ and $\eta_m = \eta'$. Being $\eta_j < \eta'$, it can be easily seen that the chosen $(K_{j,i}, P_{j,i})$ are solution pairs of the $m{\cdot}q$ inequalities (34) with $a = 2\eta_j + \alpha + 1$. Let $\overline{\Delta}_j$ be computed as in (87). Being $\eta_j < \eta'$, we have $\beta_j < \beta'$ and $\overline{\Delta}_j > \overline{\Delta}'$ and, as a consequence, $\tilde{\sigma}_j < \overline{\Delta}_j$, so that all the assumptions of Theorem 9 are satisfied. This proves that with the given gains $K_{j,i}$ and m-partition $\bar{\sigma}$, the system (74)–(76) is a global η-exponential chain observer for (1)–(2). ∎

5. Guidelines for the tuning of a chain observer. Given a nonlinear system of the type (1)–(2) and a multi-index $\bar{s} = \{s_i\}_1^q$, with $|\bar{s}| = n$, that defines an invertible observability map $z = \Phi_{\bar{s}}(x)$, and given a maximum measurement delay Δ, the chain observer of order m defined in (74) is characterized by the time-varying gains $K_j(t)$, defined in (75), and by a delay partition $\bar{\sigma} = \{\sigma_j\}_0^m$ (see Definition 7). When choosing

the time-dependent gains $K_j(t)$ of the jth observer of the chain, the design parameters are the constant gains $K_{j,i} \in \mathbb{R}^{s_i}$, for $(j,i) \in \{1\!:\!m\} \times \{1\!:\!q\}$, and a set of m positive reals $\bar{\eta} = \{\eta_j\}_1^m$, a total of $m(n+1)$ parameters. In practical applications the use of the convergence conditions of Theorem 9 or of Theorem 10 may be inconvenient for two main reasons: the Lipschitz constant γ_p in the inequalities is difficult to compute, and the convergence conditions are likely to be too conservative and restrictive. Thus, a trial and error tuning procedure, based on computer simulations, is preferred.

In order to simplify the process of tuning the observer gains, these can be chosen with the structure suggested in Theorem 6 for the single-step observer, $K_{j,i} = \bar{k}_i(\rho_j)$, where $\bar{k}_i(\cdot)$ is defined as in (71), by choosing once for all the m vectors $v_i \in \mathbb{R}^{s_i}_-$, and only tuning the scalar parameter $\rho_j \in \mathbb{R}_+$. In this way, each observer of the chain is characterized by the pair of scalar parameters (ρ_j, η_j), and the chain observer of order m is characterized by the two sets of parameters $\bar{\rho} = \{\rho_j\}_1^m$ and $\bar{\eta} = \{\eta_j\}_1^m$, a total of $2m$ parameters.

A tuning procedure can roughly proceed as follows. The preliminary step consists in choosing the q vectors $v_i \in \mathbb{R}^{s_i}_-$, $i = 1\!:\!q$. (These are the eigenvalues of the matrix $A_b - \bar{k}_i(1)C_n$.) Then, chain observers of increasing orders are tuned to achieve a satisfactory observer error convergence in the presence of increasing measurement delays, until the given maximum delay Δ is reached. The set of parameters $\bar{\rho}$ and $\bar{\eta}$ and the delay partitions $\bar{\sigma}$ tuned for a chain observer of a given order m can be used as a starting point for the tuning of the observer of order $m+1$. The next section provides an exemplification of this tuning procedure.

6. Example: Hyperchaos synchronization with buffered measurements. As an example, the chain observer is applied to a problem of hyperchaos synchronization when the measurements are stored in data packets before to be sent to the processing unit (buffered measurements). Hyperchaotic systems find application in the field of secure communications [30, 23]. In [8, 14] the following hyperchaotic modification of the classical (chaotic) Lorenz system has been proposed:

$$\begin{aligned} \dot{x}_1(t) &= \alpha\big(x_2(t) - x_1(t)\big), \\ \dot{x}_2(t) &= \beta x_1(t) + x_2(t) - x_1(t)x_3(t) - x_4(t), \\ \dot{x}_3(t) &= x_1(t)x_2(t) - \gamma x_3(t), \\ \dot{x}_4(t) &= \theta x_2(t)x_3(t), \end{aligned} \tag{96}$$

where for $\alpha = 10$, $\beta = 28$, $\gamma = 8/3$, and $\theta = 0.1$ exhibits a hyperchaotic behavior. The following measurements on system (96) are assumed available:

$$\bar{y}(t) = \begin{bmatrix} h_1\big(x(t-\delta_1(t))\big) \\ h_2\big(x(t-\delta_2(t))\big) \end{bmatrix}, \qquad \text{where} \quad \begin{cases} h_1(x) = x_1, \\ h_2(x) = x_2 + x_3. \end{cases} \tag{97}$$

We assume that the measurements are taken over regular time intervals of the type $[(k-1)T_c, kT_c)$ for $k = 0, 1, \ldots$, and are supplied to the processing unit at a high rate during the time interval $[kT_c, kT_c + T_a)$, where $T_a < T_c$. The first packet, available to the observer at time $t = 0$, is made of the measurements over the interval $[-T_c, 0]$ (thus, the maximum delay Δ is T_c). The resulting delay functions are as follows:

$$\delta_i(t) = \begin{cases} T_c - \frac{T_c - T_a}{T_a}(t - kT_c) & \text{if } t \in [kT_c, kT_c + T_a), \\ t - kT_c & \text{if } t \in [kT_c + T_a, (k+1)T_c), \end{cases} \qquad k = 0, 1, \ldots. \tag{98}$$

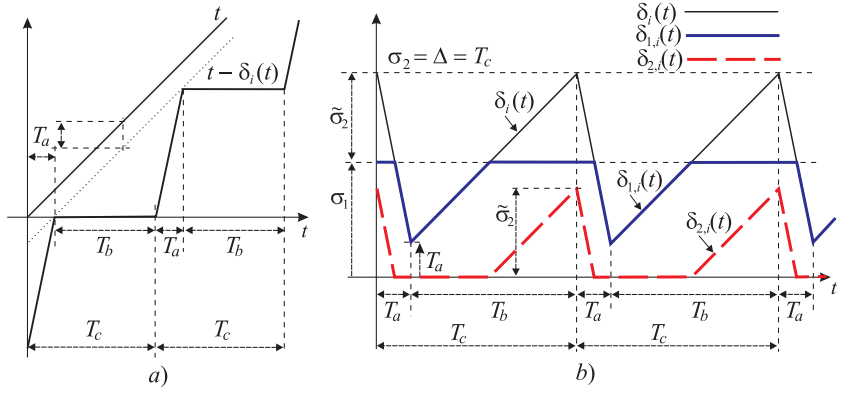

FIG. 2. *Decomposition of the buffer delay function* $\delta_i(t)$ *for* $m = 2$. (a) *Plot of* $t - \delta_i(t)$. (b) *Plot of* $\delta_i(t)$, $\delta_{1,i}(t)$, $\delta_{2,i}(t)$ *(nonuniform 2-partition* $\bar{\sigma} = \{\sigma_1, \sigma_2\}$*:* $\Delta = \sigma_1 + \tilde{\sigma}_2$*).*

Figure 2 graphically represents the delay in the process of buffering and transmitting measured data. In Figure 2(a) we see that, within a period T_c, the delivery data rate is high for a short duration T_a, and is 0 for a duration $T_b = T_c - T_a$ Figure 2(b) shows the delay function $\delta_i(t)$ and its decomposition $\delta_{1,i}(t) + \delta_{2,i}(t)$ associated to a 2-partition.

For simplicity, in the following the same delay has been assumed for both measurements (97) (i.e., $\delta_1(t) = \delta_2(t) = \delta(t)$). For the observer construction we consider the multi-index $\bar{s} = \{2, 2\}$. The computation of the observability map $z = \Phi_{\bar{s}}(x)$ gives

$$\Phi_{\bar{s}}(x) = \begin{bmatrix} h_1(x) \\ L_f h_1(x) \\ h_2(x) \\ L_f h_2(x) \end{bmatrix} = \begin{bmatrix} x_1 \\ \alpha(x_2 - x_1) \\ x_2 + x_3 \\ \beta x_1 + x_2 - x_1 x_3 - x_4 + x_1 x_2 - \gamma x_3 \end{bmatrix}, \tag{99}$$

which is invertible in all $\mathbb{R}^4$.

Note that the Lorenz system (96) and the observability map (99) are locally Lipschitz but not uniformly Lipschitz in all $\mathbb{R}^n$. However, it is known that there exists an invariant compact set for the system trajectories, and in this set the Lipschitz assumptions $\mathcal{H}_1$ and $\mathcal{H}_2$ are satisfied (see Remark 3).

Chain observers for $m = 1, 2, 3$ have been designed for system (96)–(97) for different values of the maximum delay Δ (recall that $\Delta = T_c$). In the buffered measurement model we consider the time T_a negligible with respect to T_c. The guidelines given in section 5 have been followed for tuning of the observers. The choice $K_{j,i} = \bar{k}_i(\rho_j)$, $i = 1, 2$, where $\bar{k}_i(\cdot)$ is defined as in (71), and $v_1 = v_2 = [-1.00\ -1.05]^T$, has been done. We tuned the single-step observer first ($m = 1$), and with $(\rho_1, \eta_1) = (18, 36)$ we had convergence for any $\Delta \leq 0.140$. Then we considered a chain observer with $m = 2$, where we set $(\rho_2, \eta_2) = (18, 36)$ and then tuned the two parameters (ρ_1, η_1) only, achieving satisfactory convergence with $(\rho_1, \eta_1) = (16, 16)$ and $\Delta = 0.240$. The chain observer with $m = 3$ was designed by setting $(\rho_3, \eta_3) = (18, 36)$, $(\rho_2, \eta_2) = (16, 16)$ and then tuning the parameters (ρ_1, η_1). The results are summarized in Table 1 and clearly show that larger measurement delays can be handled increasing the order m of the chain observer.

The true variable $x_4(t)$ and the observed one $\hat{x}_{1,4}(t)$ (the fourth component of the output $\hat{x}_1(t)$ of the first observer in a chain with $m = 3$) are plotted in Figure 3,

TABLE 1
Parameters used in the chain observers and maximum delays achieved.

	$m=1$	$m=2$	$m=3$
max Δ	0.140	0.240	0.340
m-partition	$\bar{\sigma}=\{0, 0.140\}$	$\bar{\sigma}=\{0, 0.05, 0.240\}$	$\bar{\sigma}=\{0, 0.05, 0.120, 0.340\}$
$(\rho_j,\, \eta_j)$	$(18, 36)_1$	$(16, 16)_1\ (18, 36)_2$	$(13, 12)_1\ (16, 16)_2\ (18, 36)_3$

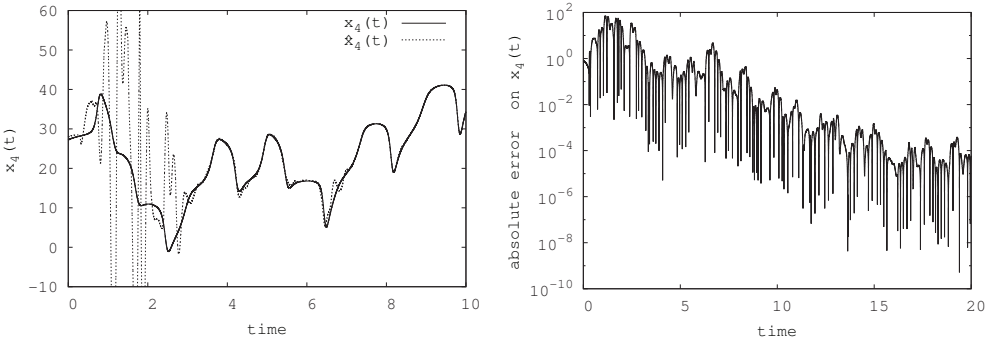

FIG. 3. *True and observed variable x_4 using the chain observer with $m=3$ and output buffering interval $T_c=0.3$ (left), and the logarithm of the observation error (right).*

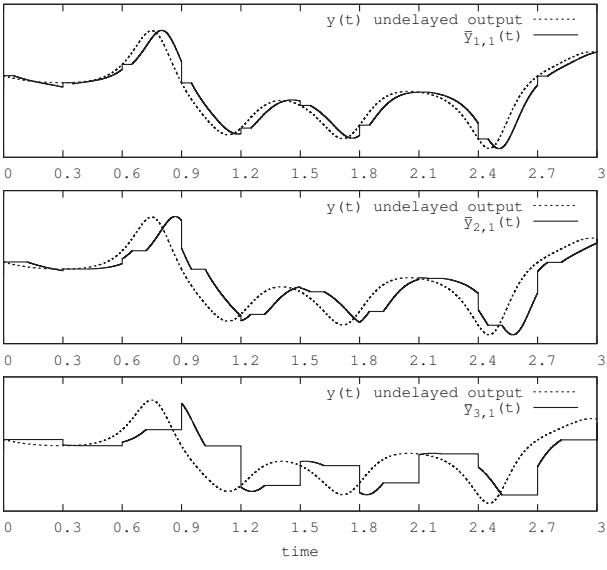

FIG. 4. *True undelayed output $y_1(t)$ of system* (96)–(97) *and transformed outputs $\bar{y}_{1,1}(t)$, $\bar{y}_{2,1}(t)$, $\bar{y}_{3,1}(t)$ for the chain observer of order $m=3$ under buffered measurements with $T_c=0.3$.*

together with the logarithm of the absolute observation error in the case $T_c=0.300$. The delay partition used in this simulation is $\bar{\sigma}=\{0, 0.5, 0.12, 0.3\}$, and the observer gain parameters are $\bar{\rho}=\{16,\ 18,\ 20\}$, $\bar{\eta}=\{20,\ 21,\ 22\}$. The initial conditions are $x(0)=[4.0,\ 2.4,\ 24.6,\ 27.2]^T$ for the system and $\hat{x}(\tau)=[4.0,\ 2.0,\ 25.0,\ 28.0]^T$, $\tau\in[-\Delta, 0]$ for the observer. In the log-plot of the error the exponential decay of the observation error is evident. In Figure 4 the true undelayed output $y_1(t)$ is plotted,

together with the transformed outputs $\bar{y}_{i,1}(t)$, $j = 1, 2, 3$. Recall that $\bar{y}_{i,1}(t)$ is the input to the ith observer in the chain. Note that $\bar{y}_{3,1}(t)$ coincides with the measured output $\bar{y}_1(t) = y_1\big(t - \delta(t)\big)$ when $\delta(t) \in [\sigma_2, T_c]$, while for $\delta(t) \in [0, \sigma_2)$ $\bar{y}_{3,1}(t)$ is the artificially delayed output $y_1(t - \sigma_2)$. $\bar{y}_{1,1}(t)$, the input to the first observer, coincides with $\bar{y}_1(t)$ when $\delta(t) \in [0, \sigma_1]$. When $\delta(t) \in (\sigma_1, T_c]$ $\bar{y}_{1,1}(t)$ is the estimate of the output y_1 at time $t - \sigma_1$ provided by the second observer, i.e., $\bar{y}_{1,1}(t) = h_1\big(\hat{x}_2(t)\big)$.

7. Conclusions. An approach for the chain-observer design in the case of nonlinear systems with vector output and time-varying measurement delays has been presented in this paper. When the maximum delay is sufficiently small, then a single-step observer can achieve a prescribed exponential observation error decay. A cascade of observers is needed to deal with larger delays. With respect to previous proposals of cascades of observers we have introduced a more flexible design that allows nonuniform delay intervals for each observer in the chain. Although no continuity assumption for the time-varying delay is needed for the implementation of the single-step observer, the continuity is required for the chain-observer implementation. The case of buffered measurements has been considered as an example. Future research will be aimed at investigating alternative chain structures that do not need the continuity hypothesis on the delay function.

Appendix. A.

A.1. Some facts on the Vandermonde matrix. Let (A_p, B_p, C_p) denote a Brunowsky triple of order p, defined as $[A_p]_{i,j} = 1$ if $i = j - 1$ and $[A_p]_{i,j} = 0$ elsewhere, $[B_p]_p = 1$ and $[B_p]_i = 0$ for $i < p$, and $[C_p]_1 = 1$ and $[C_p]_i = 0$ for $i > 1$. Let $\mathbb{1}_p \in \mathbb{R}^p$ denote a vector of ones of dimension p. For a given $v \in \mathbb{C}^p$, let $v_1, \dots, v_p$ denote its components, and let $v^{(k)}$ denote the componentwise kth power of v and let $V(v)$ denote the Vandermonde matrix associated to v, defined as

$$v^{(k)} = \begin{bmatrix} v_1^k \\ \vdots \\ v_p^k \end{bmatrix}, \quad V(v) = \begin{bmatrix} v^{(p-1)} & \cdots & v & \mathbb{1}_p \end{bmatrix} = \begin{bmatrix} v_1^{p-1} & \cdots & v_1 & 1 \\ \vdots & \vdots\vdots & \vdots & \vdots \\ v_p^{p-1} & \cdots & v_p & 1 \end{bmatrix}. \tag{100}$$

It is known that $V(v)$ is nonsingular if and only if the components of v are all distinct and nonzero. Let

$$\Lambda(v) = \operatorname*{diag}_{i=1}^{p}\{v_i\} \quad \text{and} \quad k(v) = -V^{-1}(v)\, v^{(p)}. \tag{101}$$

It is easily seen that $\Lambda(v)V(v) = V(v)A_p + v^{(p)}C_p$, and therefore

$$V(v)\Big(A_p - k(v)C_p\Big) = \Lambda(v)V(v) \quad \text{and} \quad V(v)\Big(A_p - k(v)C_p\Big)V^{-1}(v) = \Lambda(v), \tag{102}$$

which means that $k(v)$ assigns the eigenvalues $v_1, \dots, v_p$ to the matrix $A_p - k(v)C_p$. It follows that for any given $\rho \in \mathbb{R}$, we have $V(\rho\, v)\big(A_p - k(\rho, v)C_p\big)V^{-1}(\rho\, v) = \Lambda(\rho\, v) = \rho\Lambda(v)$. It is not difficult to see that

$$k(\rho\, v) = \operatorname*{diag}_{j=1}^{p}\{\rho^j\}k(v), \qquad \begin{aligned} V(\rho\, v) &= V(v) \operatorname*{diag}_{j=1}^{p}\{\rho^{-j}\}\rho^p, \\ V^{-1}(\rho\, v) &= \operatorname*{diag}_{j=1}^{p}\{\rho^j\}V^{-1}(v)\rho^{-p}. \end{aligned} \tag{103}$$

LEMMA 11. *For any pair $a > 0$ and $b > 0$, there always exists a solution pair $(k, P) \in \mathbb{R}^p \times \Sigma_+^{p\times p}$ to the inequality*

$$(A_p - k\,C_p)^T P + P(A_p - k\,C_p) + a\,P + b\,I_{s_i}(B_p^T P B_p) \le 0. \tag{104}$$

Moreover, for any given $v \in \mathbb{R}^p$, with distinct and negative components, the pairs $\big(\bar{k}(\rho), \overline{P}(\rho)\big)$ defined as

$$\bar{k}(\rho) = k(\rho\,v), \quad \overline{P}(\rho) = V^T(\rho\,v)V(\rho\,v) \tag{105}$$

are solution pairs if

$$\rho > \max\left\{1, \frac{a + b\,p\,\|V^{-1}(\,v)\|^2}{2w}\right\}, \tag{106}$$

where $w = -\max_{i=1}^p(v_i)$.

Proof. Note first that, by assumption, v has all distinct components, and therefore $V(\rho\,v)$ is nonsingular, and therefore $P(\rho) \in \Sigma_+^{p\times p}$. From (102) we have

$$\overline{P}(\rho)\big(A_p - \bar{k}(\rho)C_p\big) = \big(A_p - \bar{k}(\rho)C_p\big)^T\overline{P}(\rho) = V^T(\rho\,v)\Lambda(\rho\,v)V(\rho\,v). \tag{107}$$

Thus, inequality (104) becomes

$$2\,V^T(\rho\,v)\Lambda(\rho\,v)V(\rho\,v) + a\,V^T(\rho\,v)V(\rho\,v) + b\,I_{s_i}(B_p^T V^T(\rho\,v)V(\rho\,v)B_p) \le 0. \tag{108}$$

Note that $V(\rho\,v)B_p = \mathbb{1}_p$, so that $B_p^T V^T(\rho\,v)V(\rho\,v)B_p = \mathbb{1}_p^T\mathbb{1}_p = p$. Premultiplying by $\big(V(\rho\,v)\big)^{-T}$ and postmultiplying by $V^{-1}(\rho\,v)$ the inequality (108), we get

$$2\,\Lambda(\rho\,v) + a\,I_p + b\,p\,\big(V(\rho\,v)\big)^{-T}V^{-1}(\rho\,v) \le 0. \tag{109}$$

Since $\Lambda(\rho\,v) \le -\rho\,w\,I_p$, and $V^{-T}V^{-1} \le \|V^{-1}\|^2 I_p$ and, for $\rho > 1$, since $\|V^{-1}(\rho\,v)\|^2 < \|V^{-1}(v)\|^2$, we have

$$2\,\Lambda(\rho\,v) + a\,I_p + b\,p\,V^{-T}(\rho\,v)V^{-1}(\rho\,v) \le -2\rho\,w\,I_p + a\,I_p + b\,p\,\|V^{-1}(v)\|^2 I_p. \tag{110}$$

Thus, the inequality (109) holds for any $\rho > 1$ such that

$$-2\rho\,w + a + b\,p\,\|V^{-1}(v)\|^2 \le 0. \tag{111}$$

It is clear that condition (106) implies (111), and this in turn implies (109), and the lemma is proved. □

A.2. Proof of Lemma 4.

Proof. Note first that the matrices $\bar{k}_i(\rho)$ and $\overline{P}_i(\rho)$ defined in (35) can be written as

$$\bar{k}_i(\rho) = k(\rho\,v_i), \quad \overline{P}_i(\rho) = V^T(\rho\,v_i)V(\rho\,v_i), \quad i = 1\!:\!q, \tag{112}$$

where $k(\cdot)$ is defined in (101). Thus, with the choice (35) we have $V(\rho\,v_i)B_{s_i} = \mathbb{1}_{s_i}$ and $(B_{s_i}^T\overline{P}_i(\rho)B_{s_i}) = \mathbb{1}_{s_i}^T\mathbb{1}_{s_i} = s_i$. Being $|\bar{s}| = \sum_{i=1}^q = n$ by assumption, the summation in (34) is n. Thus, the q inequalities (34) can be rewritten as

$$\big(A_{s_i} - \bar{k}_i(\rho)C_{s_i}\big)^T\overline{P}_i(\rho) + \overline{P}_i(\rho)\big(A_{s_i} - \bar{k}_i(\rho)C_{s_i}\big) + a\,\overline{P}_i(\rho) + b\,n\,I_{s_i} \le 0. \tag{113}$$

The condition (36) on ρ is easily obtained following the steps of Lemma 11 for each of the q inequalities (113). □

A.3. Proof of Lemma 8.

Proof. For the given $\eta \in (0, \bar{\eta})$, let $\epsilon(t) = e^{\eta\, t}\xi(t)$, and let $\tilde{\eta} = \bar{\eta} - \eta$. From this definition, ϵ_t is the state of a delay system whose trivial solution $\epsilon_t = 0$ is globally $\tilde{\eta}$-exponentially stable. In order to derive the system equations for ϵ_t, let us consider the operator $L_\eta : \mathbb{R}_+ \times \mathcal{C}^n_\Delta \mapsto \mathcal{C}^n_\Delta$, defined as $L_\eta(t, \phi)(\tau) = e^{-\eta(t+\tau)}\phi(\tau)$, $\tau \in [-\Delta, 0]$. Thus, by definition, $\xi_t = L_\eta(t, \epsilon_t)$. The computation of the time-derivative of $\epsilon(t)$ gives $\dot{\epsilon}(t) = \eta\, \epsilon(t) + e^{\eta\, t} b(t, \xi_t) + e^{\eta\, t}\mu(t)$, and from this we get the system

$$\begin{aligned} \dot{\epsilon}(t) &= b_\eta(t, \epsilon_t) + e^{\eta\, t}\mu(t), && t \geq 0, \\ \epsilon_0(\tau) &= e^{\eta\, t}\phi(\tau), && \tau \in [-\Delta, 0], \end{aligned} \tag{114}$$

where $b_\eta(t, \epsilon_t) = \eta\epsilon(t) + e^{\eta\, t} g\big(t, L_\eta(t, \epsilon_t)\big)$, and is such that

$$\begin{aligned} \|b_\eta(t, \phi_1) - b_\eta(t, \phi_2)\| &\leq \eta\|\phi_1 - \phi_2\| + e^{\eta\, t}\|b\big(t, L_\eta(\phi_1)\big) - b\big(t, L_\eta(\phi_2)\big)\| \\ &\leq \eta\|\phi_1 - \phi_2\| + e^{\eta\, t}\gamma_b\|L_\eta(\phi_1) - L_\eta(\phi_2)\| \end{aligned} \tag{115}$$

thanks to assumption (84). Since $\|L_\eta(t, \phi_1) - \|L_\eta(t, \phi_2)\| \leq e^{-\eta(t+\Delta)}\|\phi_1 - \phi_2\|$, the following holds true:

$$\|b_\eta(t, \phi_1) - b_\eta(t, \phi_2)\| \leq \big(\eta + \gamma_b e^{\eta\Delta}\big)\|\phi_1 - \phi_2\| \quad \forall \phi_1, \phi_2 \in \mathcal{C}^n_\Delta, \quad \forall t \geq 0. \tag{116}$$

It follows that system (114) satisfies the assumptions of Theorem 3.2 in [31] and therefore is input-state-stable i.e.,

$$\|e^{\eta\, t}\mu(t)\| \leq \bar{p}\ \forall t \geq 0 \qquad \Rightarrow \qquad \forall \epsilon_0 \in \mathcal{C}^n_\Delta,\ \exists c > 0 :\ \|\epsilon(t)\| \leq c\ \forall t \geq 0. \tag{117}$$

From this, being $\|e^{\eta\, t}\mu(t)\| \leq \|e^{\bar{\eta}\, t}\mu(t)\|$, the thesis (86) easily follows. ∎

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