Impact, bouncing and motility

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SUMMARY. The impact of a soft contractile body on a hard support is described by fields of short range forces. Besides repulsion these forces are able to describe also friction, damping and adhesion allowing the body to have complex motions which look rather realistic. The contractility is used to make the body look like a living body with some basic locomotion capabilities. The simulated motion, like jumping or crawling, is driven either by a contraction or by the corresponding force. Although only affine motions are allowed, the model arises from a general theory of remodeling in finite elasticity and shows also creep and plastic deformations. The body is made of a viscous incompressible Mooney-Rivlin material.

1 INTRODUCTION

Contractility is the ability of bodies, like muscle cells and fibres, to contract or to extend in order to apply forces. More precisely, contractility can be defined as the ability to modify the zero stress, or relaxed, configuration. This property can also give an elementary body the capability to move over a substrate. In the form of a short tutorial this paper shows how a simple model can be defined within the context of finite elasticity, including all the details needed to get a working toy model. This allows fully physics-based simulations exhibiting some funny and hopefully realistic motions, with bouncing and rolling and even crawling over a flat rigid support. Based upon methods which have been employed to cope with constraints, the Lagrangian multiplier method [1] and the penalty method [2], a model for the contact between a contractile body and a rigid flat support is defined by constitutive laws for the contact forces. The contact traction on the body boundary is given by four short range force fields of different kinds: i) a repulsive force field; ii) an adhesive force field, both described by a Lennard-Jones–like potential; iii) a damping force field, describing the impact dissipation and depending both on the normal velocity and on the distance; iv) a friction force field, depending both on the sliding velocity and on the distance. Because of the nature of these force fields the body will never *touch* the support. Instead, the true *contact distance* will depend on the motion. We follow here the theory of material remodeling as set up in [3, 4], applied and further expounded in [5, 6] and in endless discussions with their authors, also about some projects still in progress. While restricting it to *affine motions* a summary of that theory is given in sections 2 and 3. Contact forces are defined and illustrated in section 4, as they were in [7]. Some simulations are shown in the next sections.

2 AFFINE CONTRACTILE BODY

The motion of a body $\mathcal{B}$ is described at each time t by a *transplacement* $\boldsymbol{p}$ defined on the *reference shape* $\mathcal{D}$:

$$\boldsymbol{p} : \mathcal{D} \times \mathcal{I} \rightarrow \mathcal{E} \tag{1}$$

where $\mathcal{E}$ is a three-dimensional Euclidean space. An *affine* or *homogeneous* motion is characterized by the following representation:

$$\boldsymbol{p}(\boldsymbol{x},t)=\boldsymbol{p}_0(t)+\nabla\boldsymbol{p}(t)(\boldsymbol{x}-\boldsymbol{x}_0)\,, \tag{2}$$

where $\boldsymbol{x}_0$ is a given point of $\mathcal{D}$ and the *transplacement gradient*:

$$\nabla\boldsymbol{p}(t):\mathcal{V}\rightarrow\mathcal{V} \tag{3}$$

is a tensor, i.e. an endomorphism of the translation space of $\mathcal{E}$, such that $\det\nabla\boldsymbol{p}(t)>0$. An affine velocity field $\boldsymbol{v}$ at time t has the representation:

$$\boldsymbol{v}(\boldsymbol{x})=\boldsymbol{v}_0+\nabla\boldsymbol{v}(\boldsymbol{x}-\boldsymbol{x}_0)\,, \tag{4}$$

where $\nabla\boldsymbol{v}$ is the velocity gradient. Along the motion (2) at time t

$$\boldsymbol{v}_0=\dot{\boldsymbol{p}}_0(t),\ \nabla\boldsymbol{v}=\nabla\dot{\boldsymbol{p}}(t)\,. \tag{5}$$

where $\dot{\boldsymbol{p}}_0(t):=d\,\boldsymbol{p}_0(t)/dt$ and $\nabla\dot{\boldsymbol{p}}(t):=d\,\nabla\boldsymbol{p}(t)/dt$.

In order to describe *contractility* we introduce a new tensor $\boldsymbol{G}(t)$, such that $\det\boldsymbol{G}(t)>0$, transforming the reference shape $\mathcal{D}$ into a *relaxed shape* at time t and will assume that the strain energy is a function of the *warp* defined by the Kröner-Lee decomposition:

$$\boldsymbol{F}(t):=\nabla\boldsymbol{p}(t)\,\boldsymbol{G}(t)^{-1}\,. \tag{6}$$

Let us denote by $\boldsymbol{V}$ the corresponding velocity which takes the value

$$\boldsymbol{V}=\dot{\boldsymbol{G}}(t)\,\boldsymbol{G}(t)^{-1} \tag{7}$$

at time t along a motion described by $(\boldsymbol{p},\boldsymbol{G})$. By assuming, as *the balance principle*, that at any time t

$$\int_{\mathcal{D}}\boldsymbol{b}(\boldsymbol{x},t)\cdot\boldsymbol{v}\,dV+\int_{\partial\mathcal{D}}\boldsymbol{q}(\boldsymbol{x},t)\cdot\boldsymbol{v}\,dA-\boldsymbol{S}(t)\cdot\nabla\boldsymbol{v}\,\mathrm{vol}\,\mathcal{D}+\big(\boldsymbol{Q}(t)\cdot\boldsymbol{V}-\boldsymbol{A}(t)\cdot\boldsymbol{V}\big)\,\mathrm{vol}\,\mathcal{D}=0 \tag{8}$$

for any test velocity field $(\boldsymbol{v},\boldsymbol{V})$, we get the following balance equations:

$$-m\,\ddot{\boldsymbol{p}}_0(t)-m\,\boldsymbol{g}+\boldsymbol{f}(t)=0\,, \tag{9}$$

$$-\nabla\ddot{\boldsymbol{p}}(t)\,\boldsymbol{J}+\boldsymbol{M}(t)-\boldsymbol{S}(t)\,\mathrm{vol}(\mathcal{D})=0\,, \tag{10}$$

$$\boldsymbol{Q}(t)-\boldsymbol{A}(t)=0\,. \tag{11}$$

where $\nabla\ddot{\boldsymbol{p}}(t):=d^2\,\nabla\boldsymbol{p}(t)/dt^2$, $\boldsymbol{S}(t)$ is the Piola Kirchhoff stress, and the bulk density force has been assumed to be composed of the inertial force and the gravity force densities:

$$\boldsymbol{b}(\boldsymbol{x},t):=-\rho\left(\ddot{\boldsymbol{p}}(\boldsymbol{x},t)+\boldsymbol{g}\right)\,. \tag{12}$$

The Euler tensor has been denoted by:

$$\boldsymbol{J}:=\int_{\mathcal{D}}\rho(\boldsymbol{x}-\boldsymbol{x}_0)\otimes(\boldsymbol{x}-\boldsymbol{x}_0)dV\,, \tag{13}$$

the total mass by $m := \int_{\mathcal{D}} \rho dV$ and $\boldsymbol{x}_0$ has been chosen to be the barycenter of $\mathcal{D}$.

The traction $\boldsymbol{q}$ on the boundary is assumed to be the sum of different contact force fields $\boldsymbol{q}_j$ giving rise to the total force:

$$\boldsymbol{f}(t) := \sum_j \int_{\partial\mathcal{D}} \boldsymbol{q}_j(\boldsymbol{x}, t) dA \tag{14}$$

and to the moment tensor:

$$\boldsymbol{M}(t) := \sum_j \int_{\partial\mathcal{D}} (\boldsymbol{x} - \boldsymbol{x}_0) \otimes \boldsymbol{q}_j(\boldsymbol{x}, t) dA \,. \tag{15}$$

Tensors $\boldsymbol{A}(t)$ and $\boldsymbol{Q}(t)$ are called *inner* and *outer accretive couples* per unit reference volume.

3 DISSIPATION INEQUALITY AND MATERIAL CHARACTERIZATION

We assume (as in [3, 4]) that along any motion at any time t:

$$\boldsymbol{A} \cdot \dot{\boldsymbol{G}}\boldsymbol{G}^{-1} + \boldsymbol{S} \cdot \nabla\dot{\boldsymbol{p}} - \frac{d}{dt}\big(\varphi(\boldsymbol{F}) \det \boldsymbol{G}\big) \geq 0 \,, \tag{16}$$

where φ is the *strain energy density* per unit relaxed volume. By replacing $\nabla\dot{\boldsymbol{p}}$ with the time derivative of the Kröner-Lee decomposition of $\nabla\boldsymbol{p}$ defined in (6), we get

$$\boldsymbol{A} \cdot \dot{\boldsymbol{G}}\boldsymbol{G}^{-1} + \boldsymbol{S}\boldsymbol{G}^{\mathsf{T}} \cdot \dot{\boldsymbol{F}} + \boldsymbol{F}^{\mathsf{T}}\boldsymbol{S}\boldsymbol{G}^{\mathsf{T}} \cdot \dot{\boldsymbol{G}}\boldsymbol{G}^{-1} - (\det \boldsymbol{G})\frac{d}{dt}\varphi(\boldsymbol{F}) - (\det \boldsymbol{G})\varphi(\boldsymbol{F})\boldsymbol{I} \cdot \dot{\boldsymbol{G}}\boldsymbol{G}^{-1} \geq 0 \,, \tag{17}$$

Since $d\,\varphi(\boldsymbol{F})/dt$ is linear in $\dot{\boldsymbol{F}}$, we can define the *elastic response function* $\widehat{\boldsymbol{S}}$ as the function $\mathrm{Lin}(\mathcal{V}) \to \mathrm{Lin}(\mathcal{V})$ such that in any motion:

$$\widehat{\boldsymbol{S}}(\boldsymbol{F})\boldsymbol{G}^{\mathsf{T}} \cdot \dot{\boldsymbol{F}} = (\det \boldsymbol{G})\frac{d}{dt}\varphi(\boldsymbol{F}) \,. \tag{18}$$

By requiring φ to be frame-indifferent it turns out that the Cauchy stress

$$\boldsymbol{T} = \boldsymbol{S}\,\nabla\boldsymbol{p}^{\mathsf{T}}(\det \nabla\boldsymbol{p})^{-1} \tag{19}$$

is a symmetic tensor. These property will be reflected on $\boldsymbol{S}$ and its response function $\widehat{\boldsymbol{S}}$.

Going back to (17), after substituting (18) we can collect terms thus getting:

$$\big(\boldsymbol{S} - \widehat{\boldsymbol{S}}(\boldsymbol{F})\big)\nabla\boldsymbol{p}^{\mathsf{T}} \cdot \dot{\boldsymbol{F}}\boldsymbol{F}^{-1} + \big(\boldsymbol{A} + \boldsymbol{F}^{\mathsf{T}}\boldsymbol{S}\boldsymbol{G}^{\mathsf{T}} - (\det \boldsymbol{G})\varphi(\boldsymbol{F})\boldsymbol{I}\big) \cdot \dot{\boldsymbol{G}}\boldsymbol{G}^{-1} \geq 0 \tag{20}$$

or equivalently

$$\boldsymbol{S}^{+}\nabla\boldsymbol{p}^{\mathsf{T}} \cdot \dot{\boldsymbol{F}}\boldsymbol{F}^{-1} + \boldsymbol{A}^{+} \cdot \dot{\boldsymbol{G}}\boldsymbol{G}^{-1} \geq 0 \,, \tag{21}$$

where we set

$$\begin{aligned} \boldsymbol{S}^{+} &:= \boldsymbol{S} - \widehat{\boldsymbol{S}}(\boldsymbol{F}) \,, \\ \boldsymbol{A}^{+} &:= \boldsymbol{A} + \boldsymbol{F}^{\mathsf{T}}\boldsymbol{S}\boldsymbol{G}^{\mathsf{T}} - (\det \boldsymbol{G})\varphi(\boldsymbol{F})\boldsymbol{I} \,. \end{aligned} \tag{22}$$

The simplest way for satisfying *a-priori* the dissipation inequality (21) is to assume:

$$\begin{aligned} \boldsymbol{S}^{+}\nabla\boldsymbol{p}^{\mathsf{T}} &= \mu \,\mathrm{sym}\,(\dot{\boldsymbol{F}}\boldsymbol{F}^{-1}) \,, \\ \boldsymbol{A}^{+} &= \mu_\gamma \,\dot{\boldsymbol{G}}\boldsymbol{G}^{-1} \,, \end{aligned} \tag{23}$$

with positive scalars μ_γ and μ. Hence $\boldsymbol{A}$ and $\boldsymbol{S}$ are constitutively characterized by the expressions:

$$\boldsymbol{S} = \boldsymbol{S}^+ + \widehat{\boldsymbol{S}}(\boldsymbol{F}) = \mu \,\mathrm{sym}\,(\dot{\boldsymbol{F}}\boldsymbol{F}^{-1})(\nabla \boldsymbol{p}^\mathsf{T})^{-1} + \widehat{\boldsymbol{S}}(\boldsymbol{F})\,, \tag{24}$$

$$\boldsymbol{A} = \boldsymbol{A}^+ - \big(\boldsymbol{F}^\mathsf{T}\boldsymbol{S}\boldsymbol{G}^\mathsf{T} - (\det \boldsymbol{G})\varphi(\boldsymbol{F})\boldsymbol{I}\big) = \mu_\gamma\, \dot{\boldsymbol{G}}\boldsymbol{G}^{-1} - \big(\boldsymbol{F}^\mathsf{T}\boldsymbol{S}\boldsymbol{G}^\mathsf{T} - (\det \boldsymbol{G})\varphi(\boldsymbol{F})\boldsymbol{I}\big)\,. \tag{25}$$

The balance equations take more clearly the form of equations of motion:

$$-m\,\ddot{\boldsymbol{p}}_0 - m\,\boldsymbol{g} + \boldsymbol{f} = 0\,, \tag{26}$$

$$-\nabla\ddot{\boldsymbol{p}}\,\boldsymbol{J} + \boldsymbol{M} - \boldsymbol{S}\,\mathrm{vol}(\mathcal{D}) = 0\,, \tag{27}$$

$$\mu_\gamma\, \dot{\boldsymbol{G}}\boldsymbol{G}^{-1} = \boldsymbol{F}^\mathsf{T}\boldsymbol{S}\boldsymbol{G}^\mathsf{T} - (\det \boldsymbol{G})\varphi(\boldsymbol{F})\boldsymbol{I} + \boldsymbol{Q}\,. \tag{28}$$

where $\boldsymbol{S}$ is meant to be given by (24). The outer accretive couple $\boldsymbol{Q}$ is the driving force, which could be related to some other quantity like a voltage or a biochemical signal. In some of the preliminary simulations shown in this paper the motion is driven by $\boldsymbol{G}$ instead. Thus $\boldsymbol{Q}$ is just a reactive force given by (28).

We will consider an incompressible Mooney-Rivlin material defined by the strain energy function:

$$\varphi(\boldsymbol{F}) := c_1(\imath_1(\boldsymbol{C}) - 3) + c_2(\imath_2(\boldsymbol{C}) - 3)\,. \tag{29}$$

where c_1 and c_2 are elastic moduli and $\imath_1(\boldsymbol{C})$ and $\imath_2(\boldsymbol{C})$ are the principal invariants of the Cauchy–Green tensor $\boldsymbol{C} := \boldsymbol{F}^\mathsf{T}\boldsymbol{F}$:

$$\imath_1(\boldsymbol{C}) := \mathrm{tr}\,(\boldsymbol{C}), \quad \imath_2(\boldsymbol{C}) := \frac{1}{2}\big(\,\mathrm{tr}(\boldsymbol{C})^2 - \mathrm{tr}(\boldsymbol{C}^2)\big)\,. \tag{30}$$

Because of the incompressibility constraint $\det \boldsymbol{F} = 1$, the velocity fields turn out to be isochoric, i.e. such that $\mathrm{tr}\,\dot{\boldsymbol{F}}\boldsymbol{F}^{-1} = 0$. Thus there exists a *reactive* part $\boldsymbol{S}_r$ of the stress $\boldsymbol{S}$ whose power in any isochoric velocity field is zero. Hence $\boldsymbol{S}_r\nabla\boldsymbol{p}^\mathsf{T}$ is a spherical tensor, which can be denoted by $-\pi\boldsymbol{I}$. For isochoric motions the inequality (20) characterizes only the deviatoric part $\boldsymbol{S}_0$ of the stress, allowing an arbitrary spherical part. Thus (24) will be replaced by:

$$\boldsymbol{S} = \mu\ \mathrm{sym}\,(\dot{\boldsymbol{F}}\boldsymbol{F}^{-1})(\nabla\boldsymbol{p}^\mathsf{T})^{-1} + \widehat{\boldsymbol{S}}_0(\boldsymbol{F}) - \pi\,(\nabla\boldsymbol{p}^\mathsf{T})^{-1} \tag{31}$$

4 CONTACT FORCE CONSTITUTIVE LAWS

Denoting by $\boldsymbol{o}$ any place on the flat surface $\mathcal{S}$ of the rigid support and by $\boldsymbol{n}$ the outward unit normal vector to that surface, let us define the distance of a point $\boldsymbol{x}$ on $\partial\mathcal{D}$ from the flat surface $\mathcal{S}$ at time t:

$$d(\boldsymbol{x}, t) := (\boldsymbol{p}(\boldsymbol{x}, t) - \boldsymbol{o})\cdot\boldsymbol{n} \tag{32}$$

where $\boldsymbol{p}(\boldsymbol{x}, t)$ is the position occupied by $\boldsymbol{x}$ at time t. The distance of the body from the surface $\mathcal{S}$ is defined as the minimum value of $d(\boldsymbol{x}, t)$ over the boundary $\partial\mathcal{D}$. Because of the large deformations a soft body can undergo, when defining tractions per unit area of $\partial\mathcal{D}$ we have to take into account the area change factor:

$$k(\boldsymbol{x}, t) := \|\nabla\boldsymbol{p}^\star(\boldsymbol{x}, t)\,\boldsymbol{m}(\boldsymbol{x})\| \tag{33}$$

where $\boldsymbol{m}(\boldsymbol{x})$ is the outward unit normal vector to $\partial\mathcal{D}$ and $\nabla\boldsymbol{p}^\star(\boldsymbol{x}, t) := (\nabla\boldsymbol{p}(\boldsymbol{x}, t)^\mathsf{T})^{-1}\det\nabla\boldsymbol{p}(\boldsymbol{x}, t)$ is the cofactor of $\nabla\boldsymbol{p}(\boldsymbol{x}, t)$. The *repulsive* traction field is assumed to be described by the following constitutive law:

$$\boldsymbol{q}_\mathfrak{r}(\boldsymbol{x}, t) = k(\boldsymbol{x}, t)\ \alpha_\mathfrak{r}\, d(\boldsymbol{x}, t)^{-\nu_\mathfrak{r}}\ \boldsymbol{n} \tag{34}$$

where the coefficient $\alpha_{\mathfrak{r}}$ is a positive real number and the exponent $\nu_{\mathfrak{r}}$ is a positive integer number. A value for $\alpha_{\mathfrak{r}}$ can be obtained by requiring that the repulsive forces balance the gravity forces when the body stays at rest at a distance d_0 from a horizontal surface. In other words, $\alpha_{\mathfrak{r}}$ is determined by choosing a characteristic distance d_0 and an equilibrium configuration. The impact dissipation can be described by the following *damping* traction on $\partial\mathcal{D}$:

$$\boldsymbol{q}_{\mathfrak{d}}(\boldsymbol{x},t) = -k(\boldsymbol{x},t)\;\beta_{\mathfrak{d}}\,d(\boldsymbol{x},t)^{-\nu_{\mathfrak{d}}}\,(\boldsymbol{n}\otimes\boldsymbol{n})\,\dot{\boldsymbol{p}}(\boldsymbol{x},t) \tag{35}$$

where the damping factor $\beta_{\mathfrak{d}}$ is a positive real number and $\nu_{\mathfrak{d}}$ a positive integer number. The tensor $(\boldsymbol{n}\otimes\boldsymbol{n})$ is the projector onto the direction orthogonal to $\mathcal{S}$. The *friction* traction field is given the constitutive law:

$$\boldsymbol{q}_{\mathfrak{f}}(\boldsymbol{x},t) = -k(\boldsymbol{x},t)\;\beta_{\mathfrak{f}}\,d(\boldsymbol{x},t)^{-\nu_{\mathfrak{f}}}\,(\boldsymbol{I}-\boldsymbol{n}\otimes\boldsymbol{n})\,\dot{\boldsymbol{p}}(\boldsymbol{x},t) \tag{36}$$

where the friction coefficient $\beta_{\mathfrak{f}}$ is a positive real number and $\nu_{\mathfrak{f}}$ is a positive integer number. Differently from the previous traction fields, both normal to the surface $\mathcal{S}$, the friction traction field is tangent to the surface $\mathcal{S}$.

Although it has not been used in the shown simulations, an adhesive force could be introduced and given the following law:

$$\boldsymbol{q}_{\mathfrak{a}}(\boldsymbol{x},t) = -k(\boldsymbol{x},t)\;\beta_{\mathfrak{a}}\left(d(\boldsymbol{x},t)^{-\nu_{\mathfrak{aa}}} - d(\boldsymbol{x},t)^{-\nu_{\mathfrak{ar}}}\right)\boldsymbol{n} \tag{37}$$

where the coefficient $\beta_{\mathfrak{a}}$ is a positive real number and $\nu_{\mathfrak{aa}}$ and $\nu_{\mathfrak{ar}}$ are positive integer numbers such that $\nu_{\mathfrak{ar}} < \nu_{\mathfrak{r}}$ and $\nu_{\mathfrak{aa}} = \nu_{\mathfrak{ar}}/2$.

5 NUMERICAL SIMULATIONS

Several numerical simulations have been performed using different constitutive parameters and starting from different initial conditions. The whole boundary of the body was supposed to be able to interact with the support surface with uniform values of the coefficients $\alpha_{\mathfrak{r}}$, $\beta_{\mathfrak{d}}$, $\beta_{\mathfrak{f}}$, $\beta_{\mathfrak{a}}$. All the simulations aimed at challenging both the body model and the contact model to exhibit a somewhat realistic behavior, at least at first sight. Calibration of the parameters was done to this end. Whether or not they are realistic is still to be investigated.

Some selected simulations are illustrated here. In all of them the body has a nonzero uniform mass density and there is a downward gravity field while the support is flat and horizontal.

5.1 *Creep*

If $\boldsymbol{Q}(t) = 0$, and waiting long enough, one can observe the body flattening because of gravity.

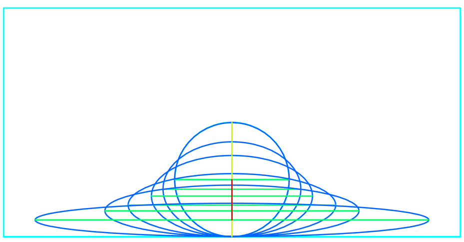

Figure 1: The body flattens.

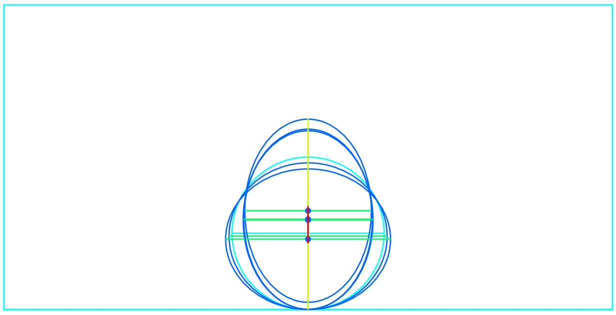

Figure 2: The body jumps upward driven by an oscillating accretive couple.

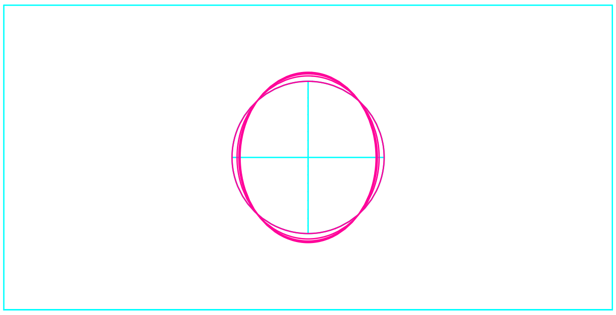

Figure 3: Evolution of the relaxed shape generated by an oscillating accretive couple with fixed principal axes.

5.2 $\boldsymbol{Q}$ *driven motion*

The motion is allowed to be vertical only. Thus there is no friction. An oscillating signal represented by the outer accretive couple $\boldsymbol{Q}$ can make the body oscillate and even jump upward, as the graph in Fig. 4 shows. The relaxed shape changes as shown in Fig. 3 as a consequence of the applied accretive couple.

5.3 $\boldsymbol{G}$ *driven motion*

The motion is two-dimensional and is driven directly by an oscillating tensor $\boldsymbol{G}$ with $\det \boldsymbol{G}(t) = 1$. Denoting its two principal values by $1/\gamma$ and γ, the contraction γ is given a periodic law while the principal axes are given a constant spin. The relaxed shapes are shown in Fig. 7. The body starts oscillating and soon moves forward rolling, almost crawling, and even jumping a little. The motion is influenced less by the amplitude of the oscillation of γ than by its frequency and the spin of the principal axes. Without friction the body cannot move forward any more while the center follows a vertical trajectory (Fig. 6).

6 CONCLUSIONS

The simulated motions are difficult to display in a still picture. But looking at the animations generated after integrating the equation of motion is amazing and sometimes it is a very surprising and fresh experience. All the computations, both symbolic and numerical, have been performed by using *Mathematica*®, starting from the very basic expressions in sections 2, 3 and 4.

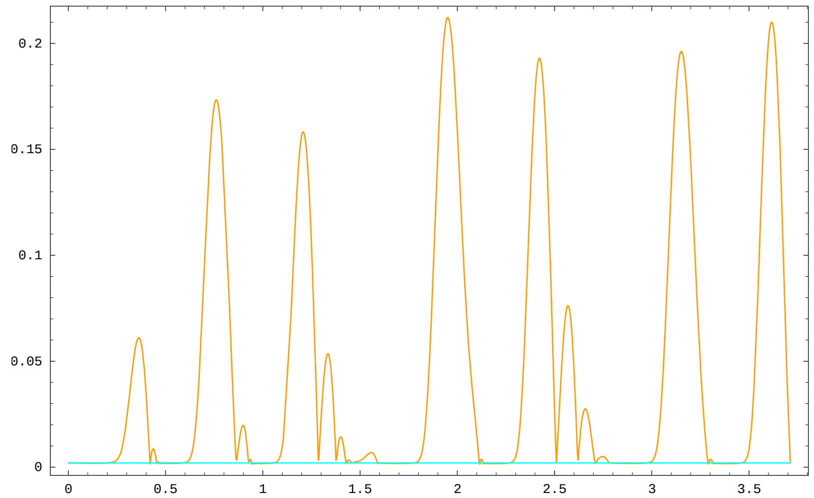

Figure 4: Bottom jumps in the motion in Fig. 2.

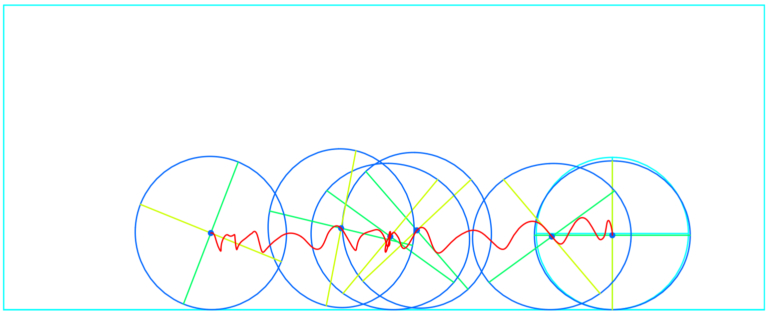

Figure 5: Trajectory of the center and some frames in a motion from right to left, driven by an oscillating contraction with rotating principal axes.

Figure 6: Without friction the body oscillates but does not move forward while the center moves on a vertical line.

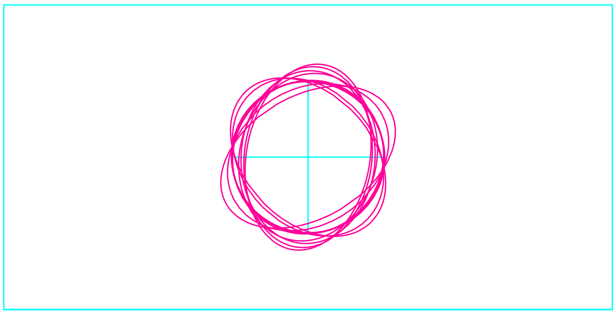

Figure 7: Evolution of the relaxed shape generated by an oscillating contraction with rotating principal axes.

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