Growing shells

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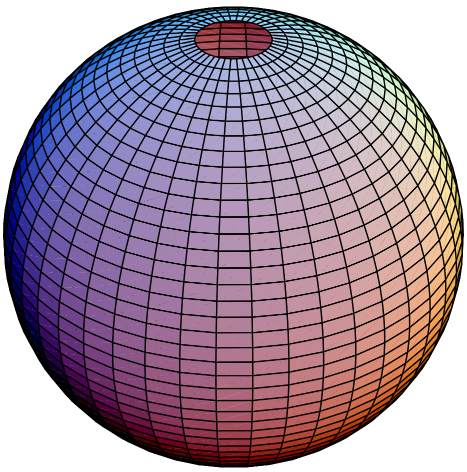

[Draft: A.T., February 4, 2006 – October 5, 2007 (17:15)]

Abstract

Our study focusses precisely on the two-way coupling between growth and stress, which we model within the theoretical framework set forth in [1], [2]. In this dynamical theory, bulk growth is governed by a novel balance law (the balance of remodelling couples). We develop and implement a layered shell theory, in order to eschew unduly restrictive hypotheses on growth distribution across the thickness.

Contents

Chapter 1

Introduction

Soft shell-like structures are ubiquitous in living organisms, ranging from organelles and cell membranes to lymph and blood vessels, the alimentary canal and respiratory ducts, the urinary tract, and the uterus. The mechanical response of all of these structures—a key feature of their physiological and pathological functioning—is subtle and elusive. Another critical issue is their ability to grow and remodel, in a way which is both biochemically controlled and strongly coupled with the prevailing mechanical conditions. While the characterization of the mechanical response of soft tissue is progressing at a reasonably fast pace nowadays, we find that growth mechanics is definitely the weakest link in the modelling chain. Our study focusses precisely on the two-way coupling between growth and stress, which we model within the theoretical framework set forth in [1], [2]. In this dynamical theory, bulk growth is governed by a novel balance law (the balance of remodelling couples). We develop and implement a layered shell theory, in order to eschew unduly restrictive hypotheses on growth distribution across the thickness. As a first application, we consider toy problems inspired by the evolution of saccular aneurisms [3] and by the enlargement of the uterus during pregnancy [8].

Chapter 2

Growing body (3D)

Following [1] and [2], we define a body as a smooth manifold $\mathcal{B}$ with boundary $\partial\mathcal{B}$ and call **complete placement** any smooth embedding

$$p : \mathcal{B} \to \mathcal{E} \tag{2.1}$$

together with

$$\mathbb{P} : \mathsf{T}\mathcal{B} \to \mathsf{V}\mathcal{E} \tag{2.2}$$

such that for any **body point** $b \in \mathcal{B}$, $p(b)$ is a place in the three-dimensional Euclidean space $\mathcal{E}$. Here $\mathsf{V}\mathcal{E}$ is the translation vector space of $\mathcal{E}$.

A **complete motion** is a family of complete placements smoothly parametrized by the *time line* $\mathbb{R}$. In such a motion the velocity is described by a base velocity

$$\mathrm{v}|_b := \dot{p}|_b \tag{2.3}$$

and a remodelling velocity

$$\mathbb{V}|_b := \dot{\mathbb{P}}|_b \mathbb{P}|_b^{-1} \tag{2.4}$$

Denoting by a tilde any test velocity field (any field belonging to the corresponding space of realizable velocities), we assume the total working be zero

$$\int_{\mathcal{B}} \left(\mathrm{b} \cdot \tilde{\mathrm{v}} + \mathbb{B} \cdot \tilde{\mathbb{V}}\right) + \int_{\partial\mathcal{B}} \mathrm{t} \cdot \tilde{\mathrm{v}} + \int_{\mathcal{B}} -\left(\mathrm{s} \cdot \tilde{\mathrm{v}} + \mathbb{C} \cdot \tilde{\mathbb{V}} + \langle \mathrm{S}, \nabla \tilde{\mathrm{v}} \rangle\right) = 0. \tag{2.5}$$

Let us assume[1]

$$\int_{\mathcal{B}} \mathbb{C} \cdot \tilde{\mathbb{V}} = \int_{\mathcal{D}} \mathbb{C} \cdot \tilde{\mathbb{V}} \tag{2.6}$$

$$\int_{\mathcal{B}} \langle \mathrm{S}, \nabla \tilde{\mathrm{v}} \rangle = \int_{\mathcal{D}} \langle \mathrm{S}, \nabla \tilde{\mathrm{v}} \rangle \tag{2.7}$$

Different test velocity gradients give rise to different definitions of stress

$$\int_{\mathcal{D}} \langle \mathrm{S}, \nabla \tilde{\mathrm{v}} \rangle = \int_{\mathcal{D}} \mathsf{S} \cdot (\nabla \tilde{\mathrm{v}} \nabla \mathsf{b}) = \int_{\mathcal{D}} (\mathbb{S} \cdot \mathrm{D}\tilde{\mathrm{v}}) \mathbb{J} \tag{2.8}$$

[1] $\mathcal{D}$ is the image of $\mathcal{B}$ through a chart χ, while $\mathsf{b} := \chi^{-1}$. For these definitions and more take a look at Appendix A.

where $\mathbb{J} := \det \mathbb{P}$ and

$$\mathrm{Dv} := \nabla \mathrm{v}|_b \, \mathbb{P}|_b^{-1} = (\nabla \mathrm{v}|_b \, \nabla \mathsf{b}|_x) \left(\nabla \mathsf{b}|_x^{-1} \, \mathbb{P}|_b^{-1}\right) \tag{2.9}$$

We define the warp

$$F|_b := \nabla p|_b \, \mathbb{P}|_b^{-1} \tag{2.10}$$

The Cauchy stress T is defined by the following relation

$$\int_{\mathcal{D}} (\mathbb{S} \cdot \mathrm{D}\tilde{\mathrm{v}}) \mathbb{J} = \int_{\mathsf{p}(\mathcal{D})} (\mathbb{S} \cdot \mathrm{D}\tilde{\mathrm{v}}) \frac{\mathbb{J}}{\det \nabla \mathsf{p}} = \int_{\mathsf{p}(\mathcal{D})} T \cdot \nabla \tilde{\mathrm{v}} \, \nabla p^{-1} \tag{2.11}$$

Note that

$$T \cdot \nabla \tilde{\mathrm{v}} \, \nabla p^{-1} = T \cdot \nabla \tilde{\mathrm{v}} \, \nabla \mathsf{b} \, \nabla \mathsf{p}^{-1} \tag{2.12}$$

and

$$(\mathbb{S} \cdot \mathrm{D}\tilde{\mathrm{v}}) \frac{\mathbb{J}}{\det \nabla \mathsf{p}} = (\mathbb{S} \cdot \nabla \tilde{\mathrm{v}} \, \mathbb{P}^{-1}) \frac{1}{\det F} = (\mathbb{S} \cdot \nabla \tilde{\mathrm{v}} \, \nabla \mathsf{b} \, (\mathbb{P} \nabla \mathsf{b})^{-1}) \frac{1}{\det F} \tag{2.13}$$

Hence

$$T \, \nabla \mathsf{p}^{\,-\mathsf{T}} = \frac{1}{\det F} \mathbb{S} \, (\mathbb{P} \nabla \mathsf{b})^{\,-\mathsf{T}} \tag{2.14}$$

from which we arrive at the classical formula

$$T = \frac{1}{\det F} \, \mathbb{S} \, F^{\mathsf{T}} \tag{2.15}$$

The stress $\mathbb{S}$ is a Piola-like stress tensor, like the stress S in (2.8). The two stress tensors are related by

$$\mathsf{S} \cdot (\nabla \tilde{\mathrm{v}} \nabla \mathsf{b}) = (\mathbb{S} \cdot \mathrm{D}\tilde{\mathrm{v}}) \mathbb{J} \tag{2.16}$$

Substituting (2.9) we get

$$\mathsf{S} \cdot (\nabla \tilde{\mathrm{v}} \nabla \mathsf{b}) = (\mathbb{S} \cdot \mathrm{D}\tilde{\mathrm{v}}) \mathbb{J} = \mathbb{S} \cdot \nabla \tilde{\mathrm{v}} \, \nabla \mathsf{b} (\mathbb{P} \, \nabla \mathsf{b})^{-1} \mathbb{J} \tag{2.17}$$

Hence

$$\mathbb{S} = \frac{1}{\mathbb{J}} \, \mathsf{S} \, (\mathbb{P} \nabla \mathsf{b})^{\mathsf{T}} \tag{2.18}$$

Chapter 3

Spherical shapes of a thick spherical shell (3D model)

3.1 Placements and gradients

Let us consider a family of placements defined by[1]

$$p(b) = \mathsf{p}(x) = \mathsf{p}_\kappa(\xi, \theta, \phi) := x_o + \rho(\xi)\, \mathsf{a}_r(\kappa(\xi, \theta, \phi)), \tag{3.1}$$

with $x = \kappa(\xi, \theta, \phi)$ and $b = \mathsf{b}(x)$. The gradient of p turns out to be such that

$$\nabla p|_b\, \gamma_\theta(b) = \nabla \mathsf{p}|_x\, \mathsf{a}_\theta(x) = \frac{\rho(\xi)}{\xi}\, \mathsf{a}_\theta(x), \tag{3.2}$$

$$\nabla p|_b\, \gamma_\phi(b) = \nabla \mathsf{p}|_x\, \mathsf{a}_\phi(x) = \frac{\rho(\xi)}{\xi}\, \mathsf{a}_\phi(x), \tag{3.3}$$

$$\nabla p|_b\, \gamma_r(b) = \nabla \mathsf{p}|_x\, \mathsf{a}_r(x) = \rho'(\xi)\, \mathsf{a}_r(x). \tag{3.4}$$

By using the orthogonal projectors in $\mathsf{V}\mathcal{E}$

$$N(x) := \mathsf{a}_r(x) \otimes \mathsf{a}_r(x), \tag{3.5}$$

$$P(x) := I - N(x), \tag{3.6}$$

the gradient of p can be given the expression

$$\nabla p|_b \nabla \mathsf{b}|_x = \nabla \mathsf{p}|_x = \frac{\rho(\xi)}{\xi} P(x) + \rho'(\xi) N(x). \tag{3.7}$$

The relaxed stance, resembling ∇p, will be given as

$$\mathbb{P}|_b\, \nabla \mathsf{b}|_x = \alpha_h(\xi)\, P(x) + \alpha_r(\xi)\, N(x) \tag{3.8}$$

From the inverse

$$\nabla \mathsf{b}|_x^{-1}\, \mathbb{P}|_b^{-1} = \frac{1}{\alpha_h(\xi)} P(x) + \frac{1}{\alpha_r(\xi)} N(x) \tag{3.9}$$

[1] For definitions of b, p, p_κ see appendix A.

we can obtain the warp expression

$$
\begin{aligned}
F|_b &= (\nabla p|_b \, \nabla \mathsf{b}|_x) \left(\nabla \mathsf{b}|_x^{-1} \, \mathbb{P}|_b^{-1}\right) \\
&= \left(\frac{\rho(\xi)}{\xi} P(x) + \rho'(\xi) N(x)\right) \left(\frac{1}{\alpha_h(\xi)} P(x) + \frac{1}{\alpha_r(\xi)} N(x)\right) \\
&= \frac{\rho(\xi)}{\xi \alpha_h(\xi)} P(x) + \frac{\rho'(\xi)}{\alpha_r(\xi)} N(x) \\
&= \lambda_h(\xi) \, P(x) + \lambda_r(\xi) \, N(x) \qquad (3.10)
\end{aligned}
$$

with

$$
\lambda_h(\xi) := \frac{\rho(\xi)}{\xi \alpha_h(\xi)}, \quad \lambda_r(\xi) := \frac{\rho'(\xi)}{\alpha_r(\xi)}. \qquad (3.11)
$$

Hence

$$
(F|_b)^{\mathsf{T}} F|_b = \lambda_h(\xi)^2 \, P(x) + \lambda_r(\xi)^2 \, N(x). \qquad (3.12)
$$

Let us consider in a complete motion the expression of the relaxed stance

$$
\mathbb{P}(\tau)|_b \, \nabla \mathsf{b}|_x = \alpha_h(\xi, \tau) \, P(x) + \alpha_r(\xi, \tau) \, N(x) \qquad (3.13)
$$

By differentiating with respect to time

$$
\dot{\mathbb{P}}(\tau)|_b \, \nabla \mathsf{b}|_x = \dot{\alpha}_h(\xi, \tau) P(x) + \dot{\alpha}_r(\xi, \tau) N(x) \qquad (3.14)
$$

we get the expression for the corresponding velocity field (2.4)

$$
\mathbb{V}|_b = \dot{\mathbb{P}}|_b \mathbb{P}|_b^{-1} = (\dot{\mathbb{P}}|_b \nabla \mathsf{b}|_x)(\mathbb{P}|_b \nabla \mathsf{b}|_x)^{-1} = \beta_h(\xi, \tau) \, P(x) + \beta_r(\xi, \tau) \, N(x) \qquad (3.15)
$$

with

$$
\beta_h(\xi, \tau) := \frac{\dot{\alpha}_h(\xi, \tau)}{\alpha_h(\xi, \tau)}, \quad \beta_r(\xi, \tau) := \frac{\dot{\alpha}_r(\xi, \tau)}{\alpha_r(\xi, \tau)}. \qquad (3.16)
$$

By differentiating (3.7) with respect to time and using (2.3) we get

$$
\nabla \mathrm{v}|_b \nabla \mathsf{b}|_x = \nabla \dot{p}|_b \nabla \mathsf{b}|_x = \nabla \dot{\mathsf{p}}|_x = \frac{\dot{\rho}(\xi)}{\xi} P(x) + \dot{\rho}'(\xi) N(x). \qquad (3.17)
$$

The gradient (2.9) has the following expression

$$
\begin{aligned}
\mathrm{Dv} &= (\nabla \mathrm{v}|_b \, \nabla \mathsf{b}|_x) \left(\nabla \mathsf{b}|_x^{-1} \, \mathbb{P}|_b^{-1}\right) \qquad (3.18) \\
&= \left(\frac{\dot{\rho}(\xi)}{\xi} P(x) + \dot{\rho}'(\xi) N(x)\right) \left(\frac{1}{\alpha_h(\xi)} P(x) + \frac{1}{\alpha_r(\xi)} N(x)\right) \\
&= \frac{\dot{\rho}(\xi)}{\xi \alpha_h(\xi)} P(x) + \frac{\dot{\rho}'(\xi)}{\alpha_r(\xi)} N(x) \\
&= \frac{v(\xi)}{\xi \alpha_h(\xi)} P(x) + \frac{v'(\xi)}{\alpha_r(\xi)} N(x) \\
&= (\dot{\lambda}_h + \lambda_h \beta_h) P(x) + (\dot{\lambda}_r + \lambda_r \beta_r) N(x) \qquad (3.19) \\
&= (g_h + \lambda_h \beta_h) P(x) + (g_r + \lambda_r \beta_r) N(x) \qquad (3.20)
\end{aligned}
$$

with

$$v(\xi,\tau) := \dot\rho(\xi,\tau) \tag{3.21}$$

and

$$g_h(\xi,\tau) := \dot\lambda_h(\xi,\tau), \quad g_r(\xi,\tau) := \dot\lambda_r(\xi,\tau). \tag{3.22}$$

Hence[2]

$$\begin{aligned}\int_{\mathcal{D}} \Big(\mathbb{C}\cdot\tilde{\mathbb{V}} + (\mathbb{S}\cdot \mathrm{D}\tilde{\mathrm{v}})\mathbb{J}\Big) &= \int_{\mathcal{D}} \Big(2\mathbb{s}_h(\tilde g_h + \lambda_h\tilde\beta_h)\mathbb{J} + \mathbb{s}_r(\tilde g_r + \lambda_r\tilde\beta_r)\mathbb{J} + 2\mathbb{c}_h\,\tilde\beta_h + \mathbb{c}_r\,\tilde\beta_r\Big)\\ &= \int_{\mathcal{D}} \Big(2\mathbb{s}_h\mathbb{J}\tilde g_h + \mathbb{s}_r\mathbb{J}\tilde g_r + 2(\mathbb{c}_h + \lambda_h\mathbb{s}_h\mathbb{J})\tilde\beta_h + (\mathbb{c}_r + \lambda_r\mathbb{s}_r\mathbb{J})\tilde\beta_r\Big)\\ &= \int_{\mathcal{D}} \left(2\lambda_h\mathbb{s}_h\mathbb{J}\,\frac{\tilde v}{\rho} + \lambda_r\mathbb{s}_r\mathbb{J}\,\frac{\tilde v'}{\rho'} + 2\mathbb{c}_h\,\tilde\beta_h + \mathbb{c}_r\,\tilde\beta_r\right)\end{aligned} \tag{3.23}$$

where $\mathbb{J} = \alpha_h^2\alpha_r$.

3.2 Energetic response

We look at the power of the hyperelastic part of the stress, which we denote by $\check{\mathbb{C}}$ and $\check{\mathbb{S}}$, as the time derivative of a potential along any trajectory

$$\int_{\mathcal{D}} \check\varpi(F,\mathbb{P}) = \int_{\mathcal{D}} (\check{\mathbb{C}}\cdot\mathbb{V} + (\check{\mathbb{S}}\cdot \mathrm{Dv})\mathbb{J}) = \int_{\mathcal{D}} \dot\psi(F,\mathbb{P}) \tag{3.24}$$

From (3.23) we get

$$\begin{aligned}\dot\psi(\lambda_h,\lambda_r;\alpha_h,\alpha_r) &= 2\check{\mathbb{s}}_h\mathbb{J}g_h + \check{\mathbb{s}}_r\mathbb{J}g_r + 2(\check{\mathbb{c}}_h + \lambda_h\check{\mathbb{s}}_h\mathbb{J})\beta_h + (\check{\mathbb{c}}_r + \lambda_r\check{\mathbb{s}}_r\mathbb{J})\beta_r\\ &= 2\check{\mathbb{s}}_h\mathbb{J}\dot\lambda_h + \check{\mathbb{s}}_r\mathbb{J}\dot\lambda_r + 2(\check{\mathbb{c}}_h + \lambda_h\check{\mathbb{s}}_h\mathbb{J})\frac{\dot\alpha_h}{\alpha_h} + (\check{\mathbb{c}}_r + \lambda_r\check{\mathbb{s}}_r\mathbb{J})\frac{\dot\alpha_r}{\alpha_r}\end{aligned} \tag{3.25}$$

In order for a potential ψ to exist, the following relations should hold

$$\check{\mathbb{s}}_h = \frac{1}{2}\frac{\partial\psi}{\partial\lambda_h}\mathbb{J}^{-1} \tag{3.26}$$

$$\check{\mathbb{s}}_r = \frac{\partial\psi}{\partial\lambda_r}\mathbb{J}^{-1} \tag{3.27}$$

$$\check{\mathbb{c}}_h = \frac{\alpha_h}{2}\frac{\partial\psi}{\partial\alpha_h} - \lambda_h\check{\mathbb{s}}_h\mathbb{J} \tag{3.28}$$

$$\check{\mathbb{c}}_r = \alpha_r\frac{\partial\psi}{\partial\alpha_r} - \lambda_r\check{\mathbb{s}}_r\mathbb{J} \tag{3.29}$$

Note how the last two conditions relate the stress $\check{\mathbb{C}}$ to the stress $\check{\mathbb{S}}$. Now let us assume that[3]

$$\psi(\lambda_h,\lambda_r;\alpha_h,\alpha_r) = \varphi(\lambda_h,\lambda_r)\mathbb{J} = \varphi(\lambda_h,\lambda_r)\alpha_h^2\alpha_r \tag{3.30}$$

[2]For safe transformations of the expression of the inner working we used a Mathematica Notebook [math/stress-dec/decomp-13.nb]. One may look at [math/stress-dec/decomp-13.nb.pdf].

[3]This assumption deserves at least some motivation.

The previous expressions get the simpler form

$$\check{\mathbb{s}}_h = \frac{1}{2}\frac{\partial\varphi}{\partial\lambda_h} \tag{3.31}$$

$$\check{\mathbb{s}}_r = \frac{\partial\varphi}{\partial\lambda_r} \tag{3.32}$$

$$\check{\mathbb{c}}_h = \mathbb{J}(\varphi - \lambda_h\check{\mathbb{s}}_h) \tag{3.33}$$

$$\check{\mathbb{c}}_r = \mathbb{J}(\varphi - \lambda_r\check{\mathbb{s}}_r) \tag{3.34}$$

3.3 Dissipation inequality

$$\varpi(F,\mathbb{P}) = \dot{\psi}(F,\mathbb{P}) + \varpi^+ \tag{3.35}$$

where

$$\varpi(F,\mathbb{P}) := \mathbb{C}\cdot\mathbb{V} + (\mathbb{S}\cdot \mathrm{Dv})\mathbb{J} \tag{3.36}$$

$$\mathbb{s}_h = \frac{1}{2}\frac{\partial\varphi}{\partial\lambda_h} + \mathbb{s}_h^+ \tag{3.37}$$

$$\mathbb{s}_r = \frac{\partial\varphi}{\partial\lambda_r} + \mathbb{s}_r^+ \tag{3.38}$$

$$\mathbb{c}_h = \mathbb{J}(\varphi - \lambda_h\mathbb{s}_h) + \mathbb{c}_h^+ \tag{3.39}$$

$$\mathbb{c}_r = \mathbb{J}(\varphi - \lambda_r\mathbb{s}_r) + \mathbb{c}_r^+ \tag{3.40}$$

3.4 Incompressibility and reactive stress

Biological materials are usually incompressible. From the incompressibility condition

$$\det F = \lambda_h^2\lambda_r = 1 \tag{3.41}$$

we get the following relations for any isochoric motion

$$\lambda_r = \frac{1}{\lambda_h^2}, \quad \frac{\dot{\lambda}_r}{\lambda_r} = -2\frac{\dot{\lambda}_h}{\lambda_h}, \quad \frac{g_r}{\lambda_r} = -2\frac{g_h}{\lambda_h} \tag{3.42}$$

Because of the incompressibility the stress has a reactive part, which we denote by $\hat{\mathbb{C}}$ and $\hat{\mathbb{S}}$, whose working vanishes for any isochoric test velocity. From (3.23)

$$\begin{aligned}\int_{\mathcal{D}}\Big(\hat{\mathbb{C}}\cdot\tilde{\mathbb{V}} + (\hat{\mathbb{S}}\cdot \mathrm{D}\tilde{\mathrm{v}})\mathbb{J}\Big) &= \int_{\mathcal{D}}\Big(2\hat{\mathbb{s}}_h\mathbb{J}\tilde{g}_h + \hat{\mathbb{s}}_r\mathbb{J}\tilde{g}_r + 2(\hat{\mathbb{c}}_h + \lambda_h\hat{\mathbb{s}}_h\mathbb{J})\tilde{\beta}_h + (\hat{\mathbb{c}}_r + \lambda_r\hat{\mathbb{s}}_r\mathbb{J})\tilde{\beta}_r\Big) \\ &= \int_{\mathcal{D}}\left(2\mathbb{J}(\lambda_h\hat{\mathbb{s}}_h - \lambda_r\hat{\mathbb{s}}_r)\frac{\tilde{g}_h}{\lambda_h} + 2(\hat{\mathbb{c}}_h + \lambda_h\hat{\mathbb{s}}_h\mathbb{J})\tilde{\beta}_h + (\hat{\mathbb{c}}_r + \lambda_r\hat{\mathbb{s}}_r\mathbb{J})\tilde{\beta}_r\right) = 0\end{aligned} \tag{3.43}$$

Hence the reactive stress is characterized by

$$\lambda_h\hat{\mathbb{s}}_h = \lambda_r\hat{\mathbb{s}}_r \tag{3.44}$$

$$\hat{\mathbb{c}}_h = -\lambda_h\hat{\mathbb{s}}_h\mathbb{J} \tag{3.45}$$

$$\hat{\mathbb{c}}_r = -\lambda_r\hat{\mathbb{s}}_r\mathbb{J} = \hat{\mathbb{c}}_h \tag{3.46}$$

Note how the last two conditions relate the stress $\hat{\mathbb{C}}$ to the stress $\hat{\mathbb{S}}$. This allows us to give the reactive stress the general form

$$\hat{\mathfrak{s}}_h = \hat{\pi}/\lambda_h \tag{3.47}$$
$$\hat{\mathfrak{s}}_r = \hat{\pi}/\lambda_r \tag{3.48}$$
$$\hat{\mathfrak{c}}_h = -\hat{\pi}\mathbb{J} \tag{3.49}$$
$$\hat{\mathfrak{c}}_r = -\hat{\pi}\mathbb{J} \tag{3.50}$$

where π is a *pressure*.

3.5 Energetic response for an incompressible material

The power of the hyperelastic part of the stress is the time derivative of a potential along a trajectory in any isochoric motion. Let us consider the restriction of the energy to isochoric motions

$$\psi_{\mathrm{I}}(\lambda_h;\alpha_h,\alpha_r) := \psi(\lambda_h,\frac{1}{\lambda_h^2};\alpha_h,\alpha_r) \tag{3.51}$$

From (3.25) we get

$$\begin{aligned}\dot{\psi}_{\mathrm{I}}(\lambda_h;\alpha_h,\alpha_r) &= 2\check{\mathfrak{s}}_h\mathbb{J}g_h + \check{\mathfrak{s}}_r\mathbb{J}g_r + 2(\check{\mathfrak{c}}_h + \lambda_h\check{\mathfrak{s}}_h\mathbb{J})\beta_h + (\check{\mathfrak{c}}_r + \lambda_r\check{\mathfrak{s}}_r\mathbb{J})\beta_r \\ &= 2\check{\mathfrak{s}}_h\mathbb{J}\dot{\lambda}_h + \check{\mathfrak{s}}_r\mathbb{J}\dot{\lambda}_r + 2(\check{\mathfrak{c}}_h + \lambda_h\check{\mathfrak{s}}_h\mathbb{J})\frac{\dot{\alpha}_h}{\alpha_h} + (\check{\mathfrak{c}}_r + \lambda_r\check{\mathfrak{s}}_r\mathbb{J})\frac{\dot{\alpha}_r}{\alpha_r} \\ &= 2\check{\mathfrak{s}}_h\mathbb{J}\dot{\lambda}_h - 2\check{\mathfrak{s}}_r\mathbb{J}\dot{\lambda}_h\frac{\lambda_r}{\lambda_h} + 2(\check{\mathfrak{c}}_h + \lambda_h\check{\mathfrak{s}}_h\mathbb{J})\frac{\dot{\alpha}_h}{\alpha_h} + (\check{\mathfrak{c}}_r + \lambda_r\check{\mathfrak{s}}_r\mathbb{J})\frac{\dot{\alpha}_r}{\alpha_r} \\ &= 2\mathbb{J}(\lambda_h\check{\mathfrak{s}}_h - \lambda_r\check{\mathfrak{s}}_r)\frac{\dot{\lambda}_h}{\lambda_h} + 2(\check{\mathfrak{c}}_h + \lambda_h\check{\mathfrak{s}}_h\mathbb{J})\frac{\dot{\alpha}_h}{\alpha_h} + (\check{\mathfrak{c}}_r + \lambda_r\check{\mathfrak{s}}_r\mathbb{J})\frac{\dot{\alpha}_r}{\alpha_r}\end{aligned} \tag{3.52}$$

For the hyperelastic stress let us consider the decomposition

$$\begin{aligned}\lambda_h\check{\mathfrak{s}}_h &= \lambda_h\check{\mathfrak{s}}_h^o + \check{\pi} \\ \lambda_r\check{\mathfrak{s}}_r &= \lambda_r\check{\mathfrak{s}}_r^o + \check{\pi}\end{aligned} \tag{3.53}$$

where

$$\check{\pi} := \frac{2\lambda_h\check{\mathfrak{s}}_h + \lambda_r\check{\mathfrak{s}}_r}{3} \tag{3.54}$$

is the *spherical part*, while the *deviatoric part* have the property

$$2\lambda_h\check{\mathfrak{s}}_h^o + \lambda_r\check{\mathfrak{s}}_r^o = 0 \tag{3.55}$$

It is convenient to define also

$$\begin{aligned}\check{\mathfrak{c}}_h^o &:= \check{\mathfrak{c}}_h + \check{\pi}\mathbb{J} \\ \check{\mathfrak{c}}_r^o &:= \check{\mathfrak{c}}_r + \check{\pi}\mathbb{J}\end{aligned} \tag{3.56}$$

By substituting decompositions (3.53) and (3.56) into (3.52) we get

$$\dot{\psi}_{\mathrm{I}}(\lambda_h;\alpha_h,\alpha_r) = 6\mathbb{J}\,\lambda_h\check{\mathfrak{s}}_h^o\frac{\dot{\lambda}_h}{\lambda_h} + 2(\check{\mathfrak{c}}_h^o + \lambda_h\check{\mathfrak{s}}_h^o\mathbb{J})\frac{\dot{\alpha}_h}{\alpha_h} + (\check{\mathfrak{c}}_r^o - 2\lambda_h\check{\mathfrak{s}}_h^o\mathbb{J})\frac{\dot{\alpha}_r}{\alpha_r} \tag{3.57}$$

In order for the potential $\psi_{\rm I}$ to exist the following relations should hold

$$\check{\mathbb{s}}_h^o = \frac{1}{6}\frac{\partial\psi_{\rm I}}{\partial\lambda_h}\mathbb{J}^{-1} \tag{3.58}$$

$$\check{\mathbb{s}}_r^o = -\frac{\lambda_h^3}{3}\frac{\partial\psi_{\rm I}}{\partial\lambda_h}\mathbb{J}^{-1} \tag{3.59}$$

$$\check{\mathbb{c}}_h^o = \frac{\alpha_h}{2}\frac{\partial\psi_{\rm I}}{\partial\alpha_h} - \lambda_h\check{\mathbb{s}}_h^o\mathbb{J} \tag{3.60}$$

$$\check{\mathbb{c}}_r^o = \alpha_r\frac{\partial\psi_{\rm I}}{\partial\alpha_r} + 2\lambda_h\check{\mathbb{s}}_h^o\mathbb{J} \tag{3.61}$$

Note how the last two conditions relate the stress $\check{\mathbb{C}}$ to the stress $\check{\mathbb{S}}$. Note also that the spherical part $\check{\pi}$ of the stress is not determined by $\psi_{\rm I}$. Now let us consider the restriction

$$\varphi_{\rm I}(\lambda_h) := \varphi(\lambda_h, \frac{1}{\lambda_h^2}) \tag{3.62}$$

and assume that

$$\psi_{\rm I}(\lambda_h;\alpha_h,\alpha_r) = \varphi_{\rm I}(\lambda_h)\mathbb{J} = \varphi_{\rm I}(\lambda_h)\alpha_h^2\alpha_r \tag{3.63}$$

The previous expressions become

$$\check{\mathbb{s}}_h^o = \frac{1}{6}\frac{\partial\varphi_{\rm I}}{\partial\lambda_h} \tag{3.64}$$

$$\check{\mathbb{s}}_r^o = -\frac{\lambda_h^3}{3}\frac{\partial\varphi_{\rm I}}{\partial\lambda_h} \tag{3.65}$$

$$\check{\mathbb{c}}_h^o = \mathbb{J}(\varphi_{\rm I} - \lambda_h\check{\mathbb{s}}_h^o) \tag{3.66}$$

$$\check{\mathbb{c}}_r^o = \mathbb{J}(\varphi_{\rm I} + 2\lambda_h\check{\mathbb{s}}_h^o) \tag{3.67}$$

3.6 Spherical and deviatoric stress (a summary)

As a consequence of the characterization of energy and stress for an incompressible material it is useful to derive a new version of expression (3.23) in terms of spherical and deviatoric parts of both stress and velocity gradient.

Whatever the stress be, hyperelastic or reactive, we can consider the decomposition

$$\begin{aligned}\lambda_h\mathbb{s}_h &= \lambda_h\mathbb{s}_h^o + \pi\\ \lambda_r\mathbb{s}_r &= \lambda_r\mathbb{s}_r^o + \pi\end{aligned} \tag{3.68}$$

where

$$\pi := \frac{2\lambda_h\mathbb{s}_h + \lambda_r\mathbb{s}_r}{3} \tag{3.69}$$

is the *spherical part*, while the *deviatoric part* have the property

$$2\lambda_h\mathbb{s}_h^o + \lambda_r\mathbb{s}_r^o = 0 \tag{3.70}$$

It is conveniet to define also the following decomposition for the remodelling stress

$$\begin{aligned}\mathbb{c}_h &= \mathbb{c}_h^o - \pi\mathbb{J}\\ \mathbb{c}_r &= \mathbb{c}_r^o - \pi\mathbb{J}\end{aligned} \tag{3.71}$$

The same kind of decomposition can be conceived for the velocity gradients as well. Let us define

$$d := \frac{1}{3}\left(2\frac{g_h}{\lambda_h} + \frac{g_r}{\lambda_r}\right) \tag{3.72}$$

$$e := \frac{2\beta_h + \beta_r}{3} \tag{3.73}$$

and consider the decompositions

$$\begin{aligned} g_h &= g_h^o + \lambda_h\, d \\ g_r &= g_r^o + \lambda_r\, d \end{aligned} \tag{3.74}$$

$$\begin{aligned} \beta_h &= \beta_h^o + e \\ \beta_r &= \beta_r^o + e \end{aligned} \tag{3.75}$$

By substituting (3.68), (3.70), (3.71), (3.74) and (3.75) into (3.23) we get[4].

$$\begin{aligned} &\int_{\mathcal{D}} \left(\mathbb{C}\cdot\tilde{\mathbb{V}} + (\mathbb{S}\cdot \mathrm{D}\tilde{\mathbb{v}})\mathbb{J}\right) \\ &= \int_{\mathcal{D}} \left(2\mathbb{s}_h^o\mathbb{J}\tilde{g}_h^o + \mathbb{s}_r^o\mathbb{J}\tilde{g}_r^o + 2(\mathbb{c}_h^o + \lambda_h\mathbb{s}_h^o\mathbb{J})\tilde{\beta}_h^o + (\mathbb{c}_r^o + \lambda_r\mathbb{s}_r^o\mathbb{J})\tilde{\beta}_r^o + 3\pi\mathbb{J}\,\tilde{d} + (2\mathbb{c}_h^o + \mathbb{c}_r^o)\,\tilde{e}\right) \\ &= \int_{\mathcal{D}} \left(6\mathbb{s}_h^o\mathbb{J}\tilde{g}_h^o + 2(\mathbb{c}_h^o - \mathbb{c}_r^o + 3\lambda_h\mathbb{s}_h^o\mathbb{J})\tilde{\beta}_h^o + 3\pi\mathbb{J}\,\tilde{d} + (2\mathbb{c}_h^o + \mathbb{c}_r^o)\,\tilde{e}\right) \\ &= \int_{\mathcal{D}} \left(6\mathbb{s}_h^o\mathbb{J}(\tilde{g}_h^o + \lambda_h\tilde{\beta}_h^o) + 2\mathbb{c}_h^o(\tilde{e} + \tilde{\beta}_h^o) + \mathbb{c}_r^o(\tilde{e} - 2\tilde{\beta}_h^o) + 3\pi\mathbb{J}\,\tilde{d}\right) \\ &= \int_{\mathcal{D}} \left(2(-p + \lambda_h\mathbb{s}_h^o)\mathbb{J}\,\frac{\tilde{v}}{\rho} - (p + 2\lambda_h\mathbb{s}_h^o)\mathbb{J}\,\frac{\tilde{v}'}{\rho'} + 2(p\mathbb{J} + \mathbb{c}_h^o)\,\tilde{\beta}_h + (p\mathbb{J} + \mathbb{c}_r^o)\,\tilde{\beta}_r\right) \end{aligned} \tag{3.76}$$

From the characterization of the stress, summarized at the end of sect. 3.5,

$$\mathbb{s}_h^o = \check{\mathbb{s}}_h^o = \frac{1}{6}\frac{\partial\varphi_{\mathrm{I}}}{\partial\lambda_h} \tag{3.77}$$

$$\mathbb{s}_r^o = \check{\mathbb{s}}_r^o = -\frac{\lambda_h^3}{3}\frac{\partial\varphi_{\mathrm{I}}}{\partial\lambda_h} \tag{3.78}$$

$$\mathbb{c}_h^o = \check{\mathbb{c}}_h^o = \mathbb{J}(\varphi_{\mathrm{I}} - \lambda_h\check{\mathbb{s}}_h^o) \tag{3.79}$$

$$\mathbb{c}_r^o = \check{\mathbb{c}}_r^o = \mathbb{J}(\varphi_{\mathrm{I}} + 2\lambda_h\check{\mathbb{s}}_h^o) \tag{3.80}$$

Finally, the complete expressions for both reactive and hyperelastic stress are

$$\mathbb{s}_h = \hat{\mathbb{s}}_h + \check{\mathbb{s}}_h = \frac{\hat{\pi}}{\lambda_h} + \check{\mathbb{s}}_h^o + \frac{\check{\pi}}{\lambda_h} = \frac{\pi}{\lambda_h} + \frac{1}{6}\frac{\partial\varphi_{\mathrm{I}}}{\partial\lambda_h} \tag{3.81}$$

$$\mathbb{s}_r = \hat{\mathbb{s}}_r + \check{\mathbb{s}}_r = \frac{\hat{\pi}}{\lambda_r} + \check{\mathbb{s}}_r^o + \frac{\check{\pi}}{\lambda_r} = \frac{\pi}{\lambda_r} - \frac{\lambda_h^3}{3}\frac{\partial\varphi_{\mathrm{I}}}{\partial\lambda_h} \tag{3.82}$$

$$\mathbb{c}_h = \hat{\mathbb{c}}_h + \check{\mathbb{c}}_h = -\hat{\pi}\mathbb{J} + \check{\mathbb{c}}_h^o - \check{\pi}\mathbb{J} = -\pi\mathbb{J} + \mathbb{J}(\varphi_{\mathrm{I}} - \lambda_h\check{\mathbb{s}}_h^o) = \mathbb{J}\left(\varphi_{\mathrm{I}} - \frac{\lambda_h}{6}\frac{\partial\varphi_{\mathrm{I}}}{\partial\lambda_h} - \pi\right) \tag{3.83}$$

$$\mathbb{c}_r = \hat{\mathbb{c}}_r + \check{\mathbb{c}}_r = -\hat{\pi}\mathbb{J} + \check{\mathbb{c}}_r^o - \check{\pi}\mathbb{J} = -\pi\mathbb{J} + \mathbb{J}(\varphi_{\mathrm{I}} + 2\lambda_h\check{\mathbb{s}}_h^o) = \mathbb{J}\left(\varphi_{\mathrm{I}} + \frac{\lambda_h}{3}\frac{\partial\varphi_{\mathrm{I}}}{\partial\lambda_h} - \pi\right) \tag{3.84}$$

where the total spherical stress $\pi := \hat{\pi} + \check{\pi}$ is made of two undistinguishable parts.

[4] See [math/stress-dec/decomp-13.nb.pdf]

3.7 Cauchy stress

From (2.15), remembering that $\det F = 1$, we can easily compute the Cauchy stress components

$$t_h = \mathfrak{s}_h \lambda_h \qquad t_r = \mathfrak{s}_r \lambda_r \tag{3.85}$$

3.8 Strain energy function for an incompressible material

Let us assume that the material is incompressible ($\det F = 1$) and consider the Fung strain energy, as in ([3], p. 395, (8.4)),

$$\varphi_{\mathrm{F}} = c(e^q - 1) \tag{3.86}$$

with

$$q = c_1\delta_1^2 + c_2\delta_2^2 + 2c_3\delta_1\delta_2 \tag{3.87}$$

where, denoting by λ_1 and λ_2 the principal stretches,

$$\delta_1 := \frac{1}{2}\left(\lambda_1^2 - 1\right), \quad \delta_2 := \frac{1}{2}\left(\lambda_2^2 - 1\right) \tag{3.88}$$

are the eigenvalues of the Green-Lagrange tensor

$$D := \frac{1}{2}\left(F^{\mathsf{T}}F - I\right) \tag{3.89}$$

Fung strain energy is defined per undeformed (*relaxed* in this context) surface area and it is deviced for a direct thin shell model (as remarked in [3]). For an isotropic shell its expression turns into

$$q = 2(c_1 + c_3)\delta_h^2 = 2\Gamma\,\delta_h^2 = 2\Gamma\,\frac{1}{4}(\lambda_h^2 - 1)^2 = \frac{\Gamma}{2}(\lambda_h^2 - 1)^2 \tag{3.90}$$

where $\Gamma = c_1 + c_3$. According to [7] and [6] the material parameter identification for aneurysmal tissue was performed on experimental data (in [5]) by using the balance equation

$$c\Gamma e^q(\lambda_h^2 - 1) = \frac{p\mathrm{R}}{2} \tag{3.91}$$

where the right hand side expression is the Laplace formula for the stress in a spherical membrane with radius R and inner pressure p. The best-fit values were $c = 0.88$ N/m and $\Gamma = (c_1 + c_3) = 12.99$. The thickness of the sample was $\mathrm{H} = 27.8 \times 10^{-6}$ m.

To recover the response function for the Cauchy stress implicitly used on the left side of (3.91), we first divide both terms by the actual thickness h

$$\frac{c}{\mathrm{h}}\Gamma e^q(\lambda_h^2 - 1) = \frac{p\mathrm{R}}{2\mathrm{h}} \tag{3.92}$$

so getting a balance equation for the mean Cauchy stress. Then observe that the derivative of (3.86) with respect to λ is

$$\frac{\partial \varphi_{\mathrm{F}}}{\partial \lambda_h} = 2c\Gamma e^q \lambda_h(\lambda_h^2 - 1) \tag{3.93}$$

Hence

$$\breve{\sigma}(\lambda) = \frac{c}{\mathrm{h}}\Gamma e^q(\lambda_h^2 - 1) = \frac{1}{2\lambda_h \mathrm{h}}\frac{\partial \varphi_{\mathrm{F}}}{\partial \lambda_h} \tag{3.94}$$

By using explicitly the stretches λ_h and λ_r and the incompressibility condition we get at last

$$\breve{\sigma}(\lambda_h) = \frac{1}{2\lambda_h \mathrm{h}}\frac{\partial \varphi_{\mathrm{F}}}{\partial \lambda} = \frac{1}{2\lambda_h\lambda_r \mathrm{H}}\frac{\partial \varphi_{\mathrm{F}}}{\partial \lambda_h} = \frac{\lambda_h}{2\mathrm{H}}\frac{\partial \varphi_{\mathrm{F}}}{\partial \lambda_h} \tag{3.95}$$

whose graph[5] is shown in Fig. 3.7.

It is interesting to note that by replacing (3.95) into (3.92) we get the balance equation

$$\breve{\sigma}(\lambda_h) = \frac{p\mathrm{R}}{2\mathrm{h}} \tag{3.96}$$

for a thin shell with thickness different from that of the sample used in the experiments. This fact allows us to show how the stress depends on the undeformed (*relaxed* in this context) radius to thickness ratio, in order to guess the effect of growth on the stress. To this end, denoting by $\bar{\mathrm{R}}$ and $\bar{\mathrm{h}}$ the relaxed values of radius and thickness, and by k their ratio, we get

$$\frac{p\mathrm{R}}{2\mathrm{h}} = \frac{p\lambda_h\bar{\mathrm{R}}}{2\lambda_r\bar{\mathrm{h}}} = \frac{p\lambda_h^3}{2}k \tag{3.97}$$

The balance equation (3.96) becomes

$$\frac{\lambda_h}{2\mathrm{H}}\frac{\partial \varphi_{\mathrm{F}}}{\partial \lambda_h} = \frac{p\lambda_h^3}{2}k \tag{3.98}$$

Fig. 3.8 shows how the pressure p is related, for a fixed value of k, to the stretch λ_h through (3.96), while Fig. 3.9 shows how, for a fixed value of p, the stress decreases for decreasing values of k, so making it convenient for the thickness to grow.

Now come to the question about how the Fung strain energy can be related to the strain energy in (3.62). From (3.85) the Cauchy stress is given through (3.64) by

$$t_h = \mathfrak{s}_h\lambda_h = \frac{\lambda_h}{6}\frac{\partial \varphi_{\mathrm{I}}}{\partial \lambda_h} + \pi \tag{3.99}$$

$$t_r = \mathfrak{s}_r\lambda_r = -\frac{\lambda_h}{3}\frac{\partial \varphi_{\mathrm{I}}}{\partial \lambda_h} + \pi \tag{3.100}$$

In order to reproduce the conditions of the experimental setup we may assume the radial stress t_r was linear in the shell thickness. As a consequence its mean value would be

$$t_r = -\frac{p}{2} \tag{3.101}$$

Substituting this value into the expression above and solving for π we get

$$t_h = \frac{\lambda_h}{2}\frac{\partial \varphi_{\mathrm{I}}}{\partial \lambda_h} - \frac{p}{2} \tag{3.102}$$

The balance equation

$$t_h = \frac{p\mathrm{R}}{2\mathrm{h}} \tag{3.103}$$

[5] From [math/Fung-Humphrey/Humphrey-09.nb] or [math/Fung-Humphrey/Humphrey-09.nb.pdf]

will transform, through (3.97) and (3.102), into

$$\frac{\lambda_h}{2}\frac{\partial\varphi_{\mathrm{I}}}{\partial\lambda_h} = \frac{p}{2}(\lambda_h^3 k - 1) \tag{3.104}$$

Comparing this expression with (3.98) we get

$$\frac{\partial\varphi_{\mathrm{I}}}{\partial\lambda_h} = \frac{\partial\varphi_{\mathrm{F}}}{\partial\lambda_h}\frac{1}{\mathrm{H}}\frac{\lambda_h^3 k - 1}{\lambda_h^3 k} \tag{3.105}$$

As the ratio k in the experiments was of order 10^2 while λ_h was near to 1, we may set

$$\varphi_{\mathrm{I}}(\lambda_h) := \frac{1}{\mathrm{H}}\varphi_{\mathrm{F}}(\lambda_h) \tag{3.106}$$

3.9 Modified Fung strain energy

All the information collected through experiments is confined to values $\lambda > 1$, as they were performed on a membrane in tension. An important issue is how to device extensions of the Fung energy to values $\lambda < 1$. Here are some proposals of ideal extensions, waiting for experimental data.

The energy should be expected to rise to infinity for $\lambda_h \to 0$, as it does for $\lambda_h \to \infty$. To this end the strain energy function can be modified by multiplying the original expression by $(2\lambda_h + \lambda_r)/3$

$$\varphi_{\mathrm{F_m}}(\lambda_h) = \frac{1}{3}c\big(e^q - 1\big)\left(2\lambda_h + \frac{1}{\lambda_h^2}\right) \tag{3.107}$$

A comparison[6] is showed in Fig. 3.1, Fig. 3.2 and Fig. 3.4, Fig. 3.5. A different way of modifying the Fung strain energy is to set

$$q := 2\,\Gamma\delta_h^2 \qquad \text{if } \lambda_h \geq 1 \tag{3.108}$$

$$q := c_1(\delta_h^2 + \delta_r^2) + 2c_3\delta_h\delta_r \qquad \text{if } \lambda_h < 1 \tag{3.109}$$

The underlying idea is to exchange λ_r and λ_h when $\lambda_r > 1$. After substituting Γ for (c_1+c_3), as in the first case, we assume $c_3 = \Gamma/3$ in order to enforce continuity of the energy function up to the second derivative, thus obtaining

$$q := \frac{\Gamma(\lambda_h^2 - 1)^2(1 + 2\lambda_h^2 - \lambda_h^6 + \lambda_h^8)}{6\lambda_h^8} \qquad \text{if } \lambda_h < 1 \tag{3.110}$$

Look at Fig. 3.3 and Fig. 3.6 for a comparison.[7] As an alternative we may set

$$q := c_1\delta_r^2 + 2c_3\delta_h\delta_r \qquad \text{if } \lambda_h < 1 \tag{3.111}$$

assuming here $c_3 = \Gamma/4$ to enforce continuity up to the second derivative, thus obtaining

$$q := \frac{\Gamma}{16}\left(\frac{1}{\lambda_h^4} - 1\right)\left(2\lambda_h^2 + \frac{3}{\lambda_h^4} - 5\right) \qquad \text{if } \lambda_h < 1 \tag{3.112}$$

Graphs of energy and stress can hardly be distinguished from the previous ones.[8]

[6]From [math/Fung-tv/Fung-tv-14.nb], or [math/Fung-tv/Fung-tv-14.nb.pdf]
[7]From [math/Fung-adc/Fung-adc-06-v1.nb], or [math/Fung-adc/Fung-adc-06-v1.nb.pdf].
[8]Look at [math/Fung-adc/Fung-adc-06-v2.nb], or [math/Fung-adc/Fung-adc-06-v2.nb.pdf].

3.10 Pressure from inside or from outside

A pressure π which is a traction orthogonal to the body shape boundary $p(\partial\mathcal{B})$ can be defined by

$$\int_{\partial\mathcal{B}} \mathrm{t}\cdot\tilde{\mathrm{v}} = \int_{p(\partial\mathcal{B})} \pi\, n\cdot\tilde{\mathrm{v}} \tag{3.113}$$

As $p(\mathsf{b}(x)) = \mathsf{p}(x)$, by using the area transformation formula we get

$$\int_{p(\partial\mathcal{B})} \pi\, n\cdot\tilde{\mathrm{v}} = \int_{\mathsf{p}(\partial\mathcal{D})} \pi\, n\cdot\tilde{\mathrm{v}} = \int_{\partial\mathcal{D}} \pi\, n\cdot\tilde{\mathrm{v}}\ \det\nabla p\, \|(\nabla p^{-\mathsf{T}} m)\| \tag{3.114}$$

where the unit external normal vector n on $\mathsf{p}(\partial\mathcal{D})$ and the unit external normal vector m on $\partial\mathcal{D}$ are related by

$$n = \frac{\nabla p^{-\mathsf{T}} m}{\|(\nabla p^{-\mathsf{T}} m)\|} \tag{3.115}$$

By substituting this expression we obtain

$$\int_{p(\partial\mathcal{B})} \pi\, n\cdot\tilde{\mathrm{v}} = \int_{\partial\mathcal{D}} \pi\,(\nabla p^{-\mathsf{T}} m)\cdot\tilde{\mathrm{v}}\ \det\nabla p \tag{3.116}$$

In the case at hand it turns out that

$$(\nabla p^{-\mathsf{T}} m)\cdot\tilde{\mathrm{v}}\ \det\nabla p = \tilde{v}\,(\lambda_h\alpha_h)^2 \tag{3.117}$$

3.11 Local balance equations

If there is no bulk brute force distribution, the balance equations, as they arise from (2.5) and (3.43), are[9]

$$\xi\alpha_h\big(-2\alpha_r\mathbb{s}_h + 2\alpha_h\mathbb{s}_r + \xi(2\mathbb{s}_r\alpha_h' + \alpha_h\mathbb{s}_r')\big) = 0 \tag{3.118}$$

$$2\xi^2\frac{\mathbb{b}_h - \mathbb{c}_h}{\alpha_h} = 0 \tag{3.119}$$

$$\xi^2\frac{\mathbb{b}_r - \mathbb{c}_r}{\alpha_r} = 0 \tag{3.120}$$

where $\mathbb{b}_h$ and $\mathbb{b}_r$ are the components of the external bulk remodelling couple. Arranging terms in a different way and dropping factor ξ the three balance equations turn into

$$-2\mathbb{J}\frac{\mathbb{s}_h}{\alpha_h} + 2\mathbb{J}\frac{\mathbb{s}_r}{\alpha_r} + \xi\mathbb{J}\left(2\frac{\alpha_h'}{\alpha_h}\frac{\mathbb{s}_r}{\alpha_r} + \frac{\mathbb{s}_r'}{\alpha_r}\right) = 0 \tag{3.121}$$

$$\mathbb{b}_h - \mathbb{c}_h = 0 \tag{3.122}$$

$$\mathbb{b}_r - \mathbb{c}_r = 0 \tag{3.123}$$

Adding dissipative terms to expressions (3.83) and (3.84), as in Sec. 3.3, the inner remodelling couple is constitutively given by

$$\mathbb{c}_h = \mathbb{J}(\varphi - \lambda_h\mathbb{s}_h) + \mathbb{c}_h^+ \tag{3.124}$$

$$\mathbb{c}_r = \mathbb{J}(\varphi - \lambda_r\mathbb{s}_r) + \mathbb{c}_r^+ \tag{3.125}$$

[9] See [math/stress-dec/decomp-13.nb] or [math/stress-dec/decomp-13.nb.pdf].

For describing a growing spherical aneurysm the following constitutive prescriptions have been deviced

$$\mathbb{c}_h^+ := \mathfrak{v}_h \frac{\dot{\alpha}_h}{\alpha_h} \tag{3.126}$$

$$\mathbb{c}_r^+ := \mathfrak{v}_r \frac{\dot{\alpha}_r}{\alpha_r} \tag{3.127}$$

$$\mathbb{b}_h := \mathfrak{g}_h (\alpha_h^\star - \bar{\alpha}_h) \tag{3.128}$$

$$\mathbb{b}_r := -\mathfrak{g}_r (\mathbb{t}_h^\star - \lambda_h \mathbb{s}_h) \tag{3.129}$$

The first two expressions describe just a resistance to growth. The last two expressions describe a control action on growth with two concurrent goals: the first one is to reach a fixed value $\alpha_h^\star$ of the average stretch $\bar{\alpha}_h$, the second one is to reach a fixed value $\mathbb{t}_h^\star$ of the Cauchy stress $\lambda_h \mathbb{s}_h$.

Substituting all of these expressions into the balance equations we get

$$-2\mathbb{J}\frac{\mathbb{s}_h}{\alpha_h} + 2\mathbb{J}\frac{\mathbb{s}_r}{\alpha_r} + \xi\mathbb{J}\left(2\frac{\alpha_h'}{\alpha_h}\frac{\mathbb{s}_r}{\alpha_r} + \frac{\mathbb{s}_r'}{\alpha_r}\right) = 0 \tag{3.130}$$

$$\mathfrak{v}_h \frac{\dot{\alpha}_h}{\alpha_h} = -\mathbb{J}(\varphi - \lambda_h \mathbb{s}_h) + \mathfrak{g}_h (\alpha_h^\star - \bar{\alpha}_h) \tag{3.131}$$

$$\mathfrak{v}_r \frac{\dot{\alpha}_r}{\alpha_r} = -\mathbb{J}(\varphi - \lambda_r \mathbb{s}_r) - \mathfrak{g}_r (\mathbb{t}_h^\star - \lambda_h \mathbb{s}_h) \tag{3.132}$$

If the stress $\mathbb{S}$ is replaced by stress S through (2.18), the balance equation (3.130) will turn into

$$-2\mathsf{s}_h + 2\mathsf{s}_r + \xi \mathsf{s}_r' = 0 \tag{3.133}$$

The boundary equations, at the outer side and at the inner side, are respectively

$$\rho_{out}^2\, \mathfrak{t}_{out} - \xi_{out}^2\, \mathsf{s}_r = 0 \tag{3.134}$$

$$\rho_{in}^2\, \mathfrak{t}_{in} + \xi_{in}^2\, \mathsf{s}_r = 0 \tag{3.135}$$

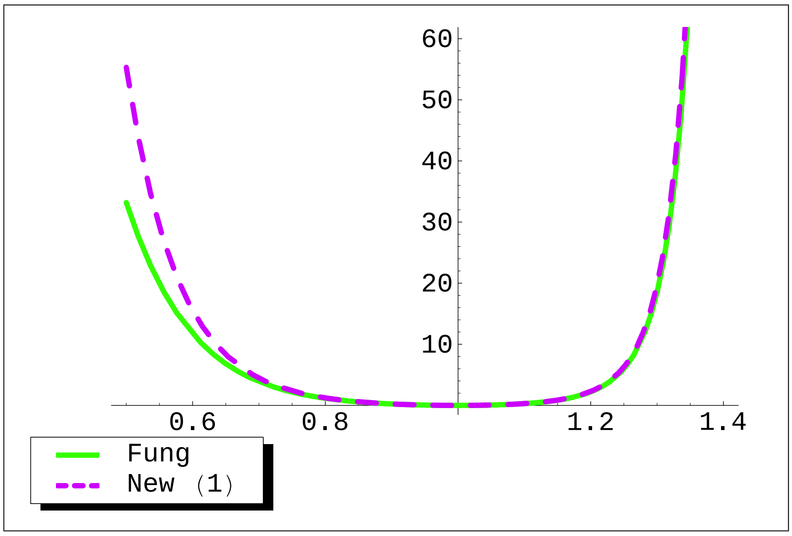

Figure 3.1: $\varphi(\lambda_h)$

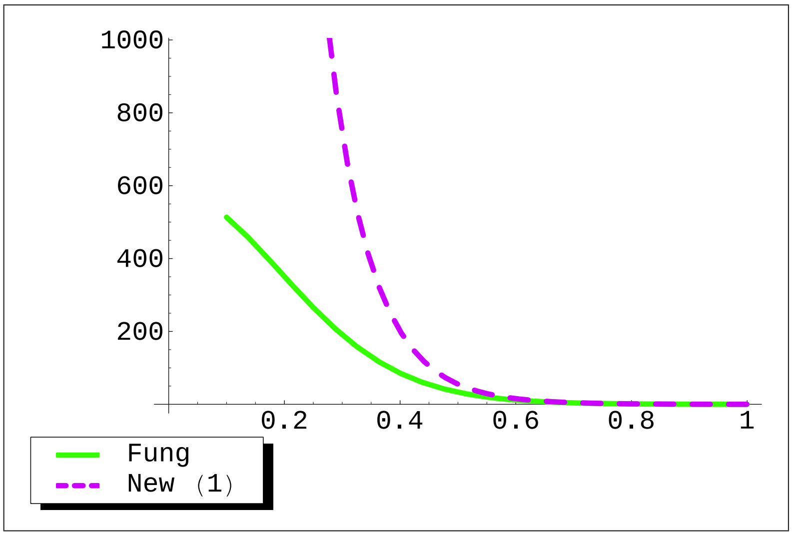

Figure 3.2: $\varphi(\lambda_h)$

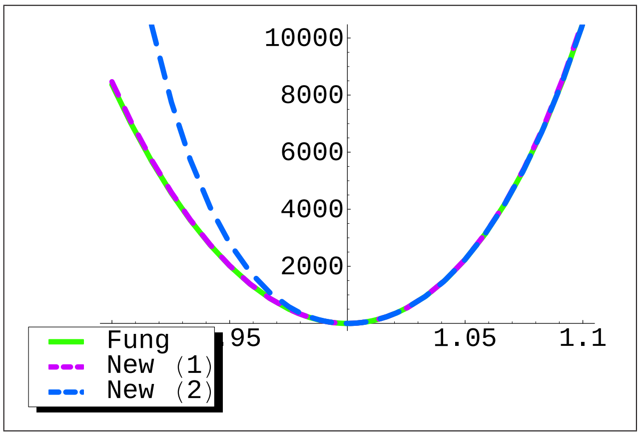

Figure 3.3: $\varphi(\lambda_h)$

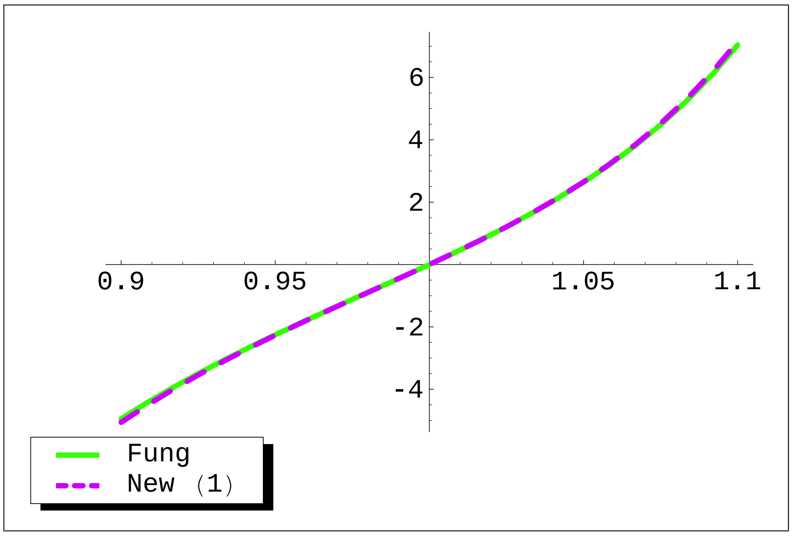

Figure 3.4: $\partial\varphi(\lambda_h)/\partial\lambda_h$

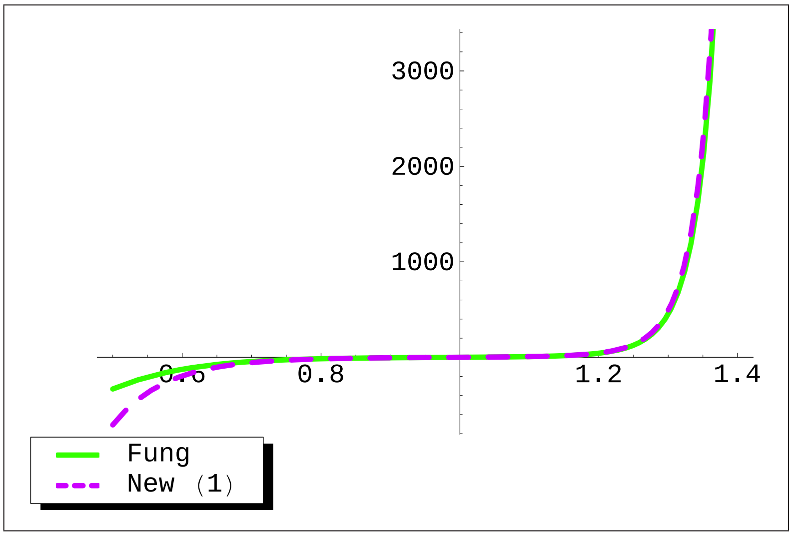

Figure 3.5: $\partial\varphi(\lambda_h)/\partial\lambda_h$

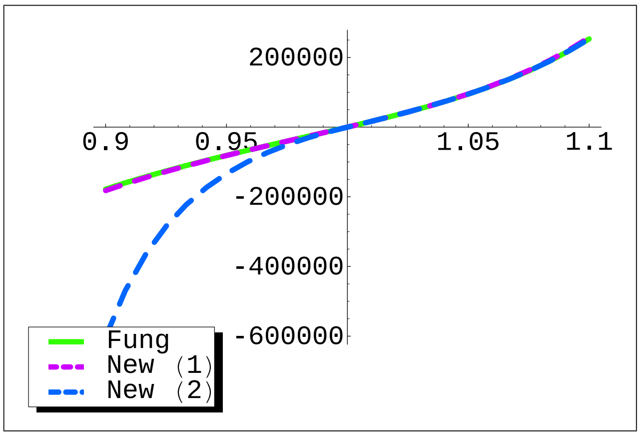

Figure 3.6: $\partial\varphi(\lambda_h)/\partial\lambda_h$

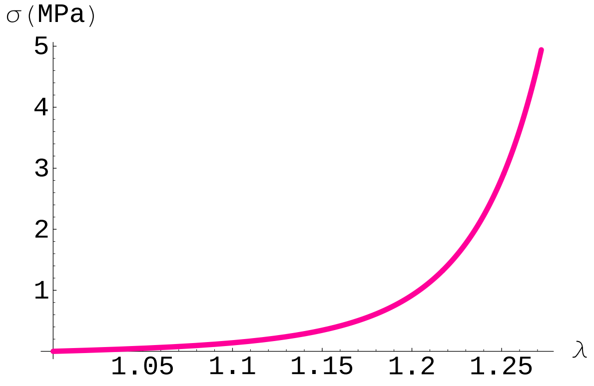

Figure 3.7: Cauchy stress response given by Fung strain energy

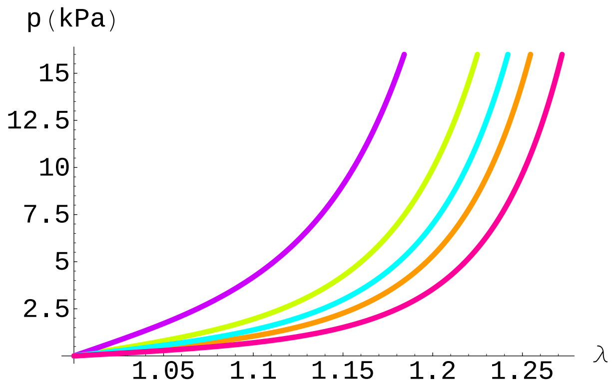

Figure 3.8: Inner pressure versus hoop stretch ($k = 50, 107, 152, 200, 300$, from left to right)

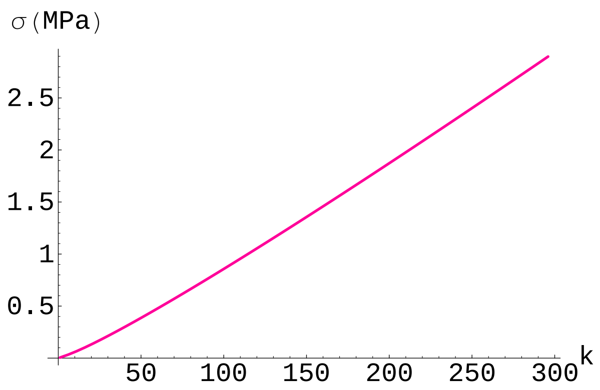

Figure 3.9: Cauchy stress versus relaxed radius to thickness ratio (inner pressure $p = 10\,\mathrm{kPa}$)

Chapter 4

Affine placements of a thin spherical shell (3D model)

4.1 Placement

All the matter in this section is intented to be used to build up the shell model as a two-dimensional continuum.

Thinking of a shell as a thin body we can consider the following affine approximation for a generic placement [see appendix A]

$$p(b) = \mathsf{p}(x) = \mathsf{p}(\kappa(r_o, \theta, \phi)) + (\xi - r_o)\nabla\mathsf{p}|_{\kappa(r_o,\theta,\phi)}\mathsf{a}_r(\kappa(r_o, \theta, \phi)), \tag{4.1}$$

with $x = \kappa(\xi, \theta, \phi)$ and $b = \mathsf{b}(x)$. By defining

$$\mathsf{l}(\kappa(r_o, \theta, \phi)) := \nabla\mathsf{p}|_{\kappa(r_o,\theta,\phi)}\mathsf{a}_r(\kappa(r_o, \theta, \phi)) \tag{4.2}$$

and setting $\zeta := (\xi - r_o)$, we can rewrite the placement expression as

$$p(b) = \mathsf{p}(\kappa(\xi, \theta, \phi)) = \mathsf{p}(\kappa(r_o, \theta, \phi)) + \zeta\, \mathsf{l}(\kappa(r_o, \theta, \phi)). \tag{4.3}$$

Computing the gradient from this expression we get

$$\nabla\mathsf{p}|_x\mathsf{a}_r(x) = \mathsf{l}(\kappa(r_o, \theta, \phi)), \tag{4.4}$$

$$\nabla\mathsf{p}|_x\mathsf{a}_\theta(x) = \nabla\mathsf{p}|_{\kappa(r_o,\theta,\phi)}\mathsf{a}_\theta(\kappa(\xi, \theta, \phi)) + \zeta\, \nabla\mathsf{l}|_{\kappa(r_o,\theta,\phi)}\mathsf{a}_\theta(\kappa(\xi, \theta, \phi)), \tag{4.5}$$

$$\nabla\mathsf{p}|_x\mathsf{a}_\phi(x) = \nabla\mathsf{p}|_{\kappa(r_o,\theta,\phi)}\mathsf{a}_\phi(\kappa(\xi, \theta, \phi)) + \zeta\, \nabla\mathsf{l}|_{\kappa(r_o,\theta,\phi)}\mathsf{a}_\phi(\kappa(\xi, \theta, \phi)). \tag{4.6}$$

By introducing $\bar{p}$ and l such that

$$\bar{p}(b) = \mathsf{p}(\kappa(r_o, \theta, \phi)), \tag{4.7}$$
$$l(b) = \mathsf{l}(\kappa(r_o, \theta, \phi)) \tag{4.8}$$

expression (4.3) can be rewritten as

$$p(b) = \bar{p}(b) + \zeta\, l(b). \tag{4.9}$$

Pulling back to $\mathcal{B}$ the gradient expressions (4.4), (4.5),(4.6), and dropping subscript b, we get

$$\nabla p\,\gamma_3 = l, \tag{4.10}$$
$$\nabla p\,\gamma_1 = (\nabla\bar{p} + \zeta\,\nabla l)\,\gamma_1, \tag{4.11}$$
$$\nabla p\,\gamma_2 = (\nabla\bar{p} + \zeta\,\nabla l)\,\gamma_2. \tag{4.12}$$

Let us define now P_o and L_o such that[1]

$$P_o\,\gamma_1 = \nabla\bar{p}\,\gamma_1, \tag{4.13}$$
$$P_o\,\gamma_2 = \nabla\bar{p}\,\gamma_2, \tag{4.14}$$
$$P_o\,\gamma_3 = l, \tag{4.15}$$

$$L_o\,\gamma_1 = \nabla l\,\gamma_1, \tag{4.16}$$
$$L_o\,\gamma_2 = \nabla l\,\gamma_2, \tag{4.17}$$
$$L_o\,\gamma_3 = 0. \tag{4.18}$$

The above definitions[2] can be summarized by the following shorthand

$$P_o := (\nabla\bar{p} \mid l), \tag{4.19}$$
$$L_o := (\nabla l \mid 0). \tag{4.20}$$

The expression for ∇p can now be given the form

$$\nabla p = P_o + \zeta\,L_o \tag{4.21}$$

By defining[3]

$$\mathbb{P}_o := (\bar{\mathbb{P}} \mid \mathbb{l}) \tag{4.22}$$
$$\mathbb{L}_o := (\bar{\mathbb{L}} \mid 0) \tag{4.23}$$

we can give the relaxed stance $\mathbb{P}$ the same form as ∇p

$$\mathbb{P} = \mathbb{P}_o + \zeta\,\mathbb{L}_o \tag{4.24}$$

The warp

$$F|_b := \nabla p|_b\,\mathbb{P}|_b^{-1} \tag{4.25}$$

becomes, dropping again subscript b,

$$F = (P_o + \zeta\,L_o)(\mathbb{P}_o + \zeta\,\mathbb{L}_o)^{-1} \tag{4.26}$$

Neglecting terms of order $o(\zeta)$ we get the following expression

$$F = F_o + \zeta\,B_o \tag{4.27}$$

with

$$F_o := P_o\mathbb{P}_o^{-1} \tag{4.28}$$
$$B_o := (L_o - F_o\mathbb{L}_o)\mathbb{P}_o^{-1} \tag{4.29}$$

Here a clear definition of $\mathbb{P}^{-1}$ and $\mathbb{P}_o^{-1}$ should be given!

[1] Here subscript o stands for *defined at* $\xi = r_o$ (the "middle surface") while an overbar denotes the restriction of p to the middle surface.

[2] Note that these definitions do not depend on time since vectors γ_i do not.

[3] Here overbars denote linear transformations from the two-dimensional vector space tangent to the middle surface into $\mathsf{V}\mathcal{E}$.

4.2 Velocity fields

Differentiating with respect to time from the definitions above we get

$$\dot{p} = \dot{\bar{p}} + \zeta\,\dot{l} \tag{4.30}$$

$$\nabla\dot{p} = \dot{P}_o + \zeta\,\dot{L}_o \tag{4.31}$$

with

$$\dot{P}_o = (\nabla\dot{\bar{p}} \mid \dot{l}) \tag{4.32}$$

$$\dot{L}_o = (\nabla\dot{l} \mid 0) \tag{4.33}$$

We can also define

$$\bar{\mathrm{v}} := \dot{\bar{p}} \tag{4.34}$$

$$\mathrm{w} := \dot{l} \tag{4.35}$$

and

$$\mathrm{v} := \dot{p} = \bar{\mathrm{v}} + \zeta\,\mathrm{w} \tag{4.36}$$

It easy to check the consistency of the definition above

$$\nabla\mathrm{v} = (\nabla\dot{\bar{p}} \mid \dot{l}) + \zeta\,(\nabla\dot{l} \mid 0) = (\nabla\bar{\mathrm{v}} \mid \mathrm{w}) + \zeta\,(\nabla\mathrm{w} \mid 0) = \dot{P}_o + \zeta\,\dot{L}_o = \nabla\dot{p} \tag{4.37}$$

As the derivative of the remodelling stance is

$$\dot{\mathbb{P}} = \dot{\mathbb{P}}_o + \zeta\,\dot{\mathbb{L}}_o \tag{4.38}$$

with

$$\dot{\mathbb{P}}_o = (\dot{\bar{\mathbb{P}}} \mid \dot{\mathbb{l}}) \tag{4.39}$$

$$\dot{\mathbb{L}}_o = (\dot{\bar{\mathbb{L}}} \mid 0) \tag{4.40}$$

the remodelling velocity, neglecting terms of order $o(\zeta)$, turns out to be

$$\begin{aligned} \mathbb{V} &:= \dot{\mathbb{P}}\mathbb{P}^{-1} \\ &= (\dot{\mathbb{P}}_o + \zeta\,\dot{\mathbb{L}}_o)({\mathbb{P}_o}^{-1} - \zeta\,{\mathbb{P}_o}^{-1}\mathbb{L}_o{\mathbb{P}_o}^{-1}) \\ &= \dot{\mathbb{P}}_o{\mathbb{P}_o}^{-1} + \zeta\,(\dot{\mathbb{L}}_o - \dot{\mathbb{P}}_o{\mathbb{P}_o}^{-1}\mathbb{L}_o){\mathbb{P}_o}^{-1} \end{aligned} \tag{4.41}$$

Hence

$$\mathbb{V} = \mathbb{V}_o + \zeta\,\mathbb{W}_o \tag{4.42}$$

with

$$\mathbb{V}_o := \dot{\mathbb{P}}_o{\mathbb{P}_o}^{-1} \tag{4.43}$$

$$\mathbb{W}_o := (\dot{\mathbb{L}}_o - \mathbb{V}_o\mathbb{L}_o){\mathbb{P}_o}^{-1} \tag{4.44}$$

Chapter 5

Growing shells (2D model)

Warning: this section has only been roughly shaped (rephrasing sect. 4) and has some inconsinstencies! It is still not clear whether $\mathcal{B}$ should be a simple 2D-manifold or a fiber bundle. The main point is how to make $\mathbb{P}$ invertible. Another point is whether the shear strain *is hidden somewhere or not.*

5.1 Placements

We define a shell as a smooth manifold $\mathcal{B}$ (fiber boundle with one-dimensional fiber) with boundary $\partial\mathcal{B}$ and call **complete placement** any smooth embedding

$$(\bar{p}, l, \bar{\mathbb{P}}, \mathbb{l}, \bar{\mathbb{L}}) : \mathcal{B} \to \mathcal{E} \times \mathsf{V}\mathcal{E} \times (\mathsf{V}\mathcal{E} \otimes \mathsf{V}\mathcal{E}) \times \mathsf{V}\mathcal{E} \times (\mathsf{V}\mathcal{E} \otimes \mathsf{V}\mathcal{E}) \tag{5.1}$$

such that for any **body point** $(b, \nu) \in \mathcal{B}$, $\bar{p}(b)$ is a place in the three-dimensional Euclidean space $\mathcal{E}$, $l(b)$ is a vector in the translation space $\mathsf{V}\mathcal{E}$ at $\bar{p}(b) \in \mathcal{E}$, the image of $\bar{p}$ is a two-dimensional manifold $\mathcal{S}$.

A **complete motion** is a family of complete placements smootly parametrized by the *time line* $\mathbb{R}$. We call base velocity

$$\bar{\mathrm{v}}|_b := \dot{\bar{p}}|_b \tag{5.2}$$

$$\mathrm{w}|_b := \dot{l}|_b \tag{5.3}$$

Extended gradients

$$P := (\nabla\bar{p} \mid l), \tag{5.4}$$

$$L := (\nabla l \mid 0), \tag{5.5}$$

which stand for P and L such that, at any point (b, ν),

$$P|_b\,\gamma_1(b) = \nabla\bar{p}|_b\,\gamma_1(b), \tag{5.6}$$

$$P|_b\,\gamma_2(b) = \nabla\bar{p}|_b\,\gamma_2(b), \tag{5.7}$$

$$P|_b\,\nu = l(b), \tag{5.8}$$

$$L|_b\,\gamma_1(b) = \nabla l|_b\,\gamma_1(b), \tag{5.9}$$

$$L|_b\,\gamma_2(b) = \nabla l|_b\,\gamma_2(b), \tag{5.10}$$

$$L|_b\,\nu = 0. \tag{5.11}$$

Extended remodeling couples

$$\mathbb{P} := (\bar{\mathbb{P}} \mid \mathbb{0}), \tag{5.12}$$
$$\mathbb{L} := (\bar{\mathbb{L}} \mid 0). \tag{5.13}$$

Remodelling velocity

$$\mathbb{V}|_b := \dot{\mathbb{P}}|_b \mathbb{P}|_b^{-1} \tag{5.14}$$
$$\mathbb{W}|_b := \dot{\mathbb{L}}|_b \mathbb{P}|_b^{-1} \tag{5.15}$$
$$\tag{5.16}$$

Warp

$$F|_b := P|_b \, \mathbb{P}|_b^{-1}, \tag{5.17}$$
$$B|_b := (L|_b - F|_b \mathbb{L}|_b) \, \mathbb{P}|_b^{-1}. \tag{5.18}$$

5.2 Balance

Denoting by a tilde any test velocity field, belonging to the corresponding space of realizable velocities, we assume the total working be zero

$$\int_{\mathcal{B}} \Big(\mathrm{b} \cdot \tilde{\mathrm{v}} + \mathbb{B} \cdot \tilde{\mathbb{V}} + \mathbb{T} \cdot \tilde{\mathbb{W}} \Big) + \int_{\partial \mathcal{B}} \mathrm{t} \cdot \tilde{\mathrm{v}} + \int_{\mathcal{B}} - \Big(\mathrm{s} \cdot \tilde{\mathrm{v}} + \mathbb{C} \cdot \tilde{\mathbb{V}} + \mathrm{S} \cdot \nabla \tilde{\mathrm{v}} + \mathrm{M} \cdot \nabla \tilde{\mathrm{w}} \Big) = 0. \tag{5.19}$$

Chapter 6

Axially symmetric spherical shell (2D model)

Chapter 7

Multilayered shells

$$(p_1, l_1) : \mathcal{B} \to \mathcal{E} \times \mathsf{V}\mathcal{E} \tag{7.1}$$
$$(p_2, l_2) : \mathcal{B} \to \mathcal{E} \times \mathsf{V}\mathcal{E} \tag{7.2}$$
$$(p_3, l_3) : \mathcal{B} \to \mathcal{E} \times \mathsf{V}\mathcal{E} \tag{7.3}$$

Appendix A

Backstage

We describe here all the geometric machinery which makes it possible to perform computations on scalar expressions by means of coordinate systems, parametrizations, vector bases. All this matter, usually left backstage, deserves nevertheless a clear definition.

A.1 Body chart

Denoting by $\mathcal{E}$ the standard Euclidean space, let us assume that a chart (or an atlas))

$$\chi : \mathcal{B} \to \mathcal{E} \tag{A.1}$$

is given together with a parametrization (or a family of local parametrizations)

$$\kappa : \mathcal{K} \to \mathcal{D}, \qquad \mathcal{K} \subset \mathbb{R}^n, \tag{A.2}$$

where

$$\mathcal{D} := \chi(\mathcal{B}) \subset \mathcal{E} \tag{A.3}$$

will be called the *dummy shape*, or simply *dummy*. These maps induce two different parametrizations of $\mathcal{B}$

$$\mathsf{b} := \chi^{-1} \; : \mathcal{D} \to \mathcal{B}, \tag{A.4}$$

$$\mathsf{b}_\kappa := \chi^{-1} \circ \kappa \; : \mathcal{K} \to \mathcal{B}. \tag{A.5}$$

They are both useful, even though the second one, giving rise to representations in $\mathbb{R}^n$, will be placed behind the first one giving rise to representations in $\mathcal{E}$.

Through (A.4) and (A.5) any placement (2.1) can be given one of the following representations

$$\mathsf{p} := p \circ \mathsf{b} \; : \mathcal{D} \to \mathcal{E}, \tag{A.6}$$

$$\mathsf{p}_\kappa := p \circ \mathsf{b}_\kappa \; : \mathcal{K} \to \mathcal{E}. \tag{A.7}$$

The vectors tangent to curves on $\mathcal{B}$ through $b = \mathsf{b}(x)$ are

$$\gamma_i(b) = \nabla \mathsf{b}|_x \, \mathsf{a}_i(x) \tag{A.8}$$

The gradient maps

$$\nabla p : \mathsf{T}\mathcal{B} \to \mathsf{V}\mathcal{E}, \tag{A.9}$$

$$\nabla \mathsf{p} : \mathsf{T}\mathcal{D} \to \mathsf{V}\mathcal{E}, \tag{A.10}$$

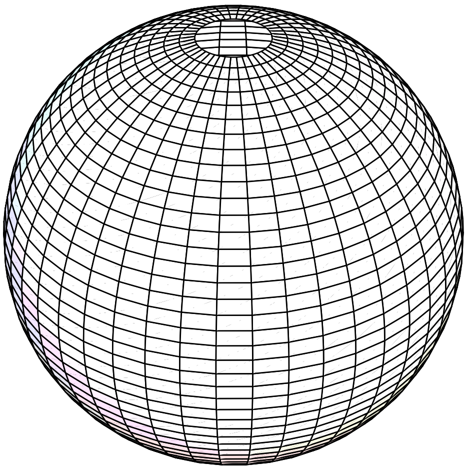

Figure A.1: The dummy shape $\mathcal{D} := \chi(\mathcal{B}) \subset \mathcal{E}$

are related by

$$\nabla \mathsf{b} : \mathsf{T}\mathcal{D} \to \mathsf{T}\mathcal{B}, \tag{A.11}$$

through the tangent vector transformations

$$\nabla p|_b \gamma_i(b) = \nabla p|_b \nabla \mathsf{b}|_x \, \mathsf{a}_i(x) = \nabla \mathsf{p}|_x \, \mathsf{a}_i(x) \tag{A.12}$$

at $x = \chi(b)$. Hence

$$\nabla p|_b \nabla \mathsf{b}|_x = \nabla \mathsf{p}|_x \tag{A.13}$$

Here $\mathsf{V}\mathcal{E}$ is the translation vector space of $\mathcal{E}$, while

$$\mathsf{T}\mathcal{E} = \mathcal{E} \times \mathsf{V}\mathcal{E} \tag{A.14}$$

is the tanget bundle of $\mathcal{E}$. Note that the fiber at the place $x \in \mathcal{E}$ is

$$\mathsf{T}_x\mathcal{E} = \mathsf{V}\mathcal{E} \tag{A.15}$$

A.2 Parametrization of a sphere

Let us assume that $\mathcal{D}$ is a ball without an axis. A parametrization (A.2) suitable for our purposes is defined, denoting by $\{\mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3\}$ a basis in $\mathsf{V}\mathcal{E}$, through the expression

$$\chi(b) = \kappa(\xi, \theta, \phi) = x_o + \xi(\cos\phi\,\cos\theta\,\mathsf{e}_3 + \cos\phi\,\sin\theta\,\mathsf{e}_1 + \sin\phi\,\mathsf{e}_2), \tag{A.16}$$

with parameters ranging in the following intervals

$$\begin{aligned} r_0 - \epsilon \leq & \xi \leq r_0 + \epsilon, \\ -\pi < & \theta \leq \pi, \\ -\frac{\pi}{2} < & \phi < \frac{\pi}{2}. \end{aligned} \tag{A.17}$$

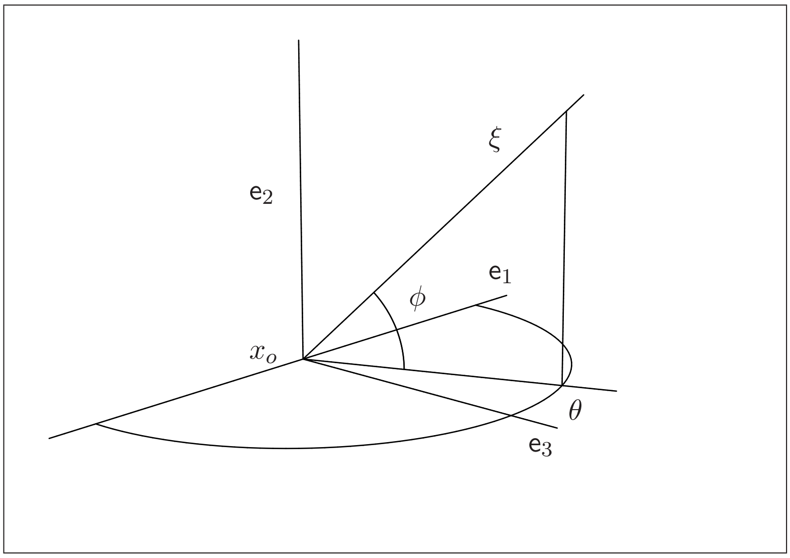

Figure A.2: Spherical coordinates in $\mathcal{E}$

The points left out are those belonging to the diameter in the e_2 direction. The vectors tangent to coordinate curves on $\mathcal{D}$ through $x = \kappa(\xi, \theta, \phi)$ turn out to be

$$\begin{aligned} \mathsf{a}_3(x) \equiv \mathsf{a}_r(x) &= \cos\phi\, \cos\theta\, \mathsf{e}_3 + \cos\phi\, \sin\theta\, \mathsf{e}_1 + \sin\phi\, \mathsf{e}_2, \\ \mathsf{a}_1(x) \equiv \mathsf{a}_\theta(x) &= \xi(-\cos\phi\, \sin\theta\, \mathsf{e}_3 + \cos\phi \cos\theta\, \mathsf{e}_1), \\ \mathsf{a}_2(x) \equiv \mathsf{a}_\phi(x) &= \xi(-\sin\phi\, \cos\theta\, \mathsf{e}_3 - \sin\phi\, \sin\theta\, \mathsf{e}_1 + \cos\phi\, \mathsf{e}_2), \end{aligned} \tag{A.18}$$

with norms

$$\begin{aligned} \|\mathsf{a}_r(x)\| &= 1, \\ \|\mathsf{a}_\theta(x)\| &= \xi\, \cos\phi, \\ \|\mathsf{a}_\phi(x)\| &= \xi, \end{aligned} \tag{A.19}$$

and

$$\mathrm{vol}\,(\mathsf{a}_r(x), \mathsf{a}_\theta(x), \mathsf{a}_\phi(x)) = \xi^2\, \cos\phi. \tag{A.20}$$

It will also be useful to derive the expression for the gradient of a_r

$$\nabla \mathsf{a}_r|_x \mathsf{a}_\theta(x) = \frac{1}{\xi}\, \mathsf{a}_\theta(x), \tag{A.21}$$

$$\nabla \mathsf{a}_r|_x \mathsf{a}_\phi(x) = \frac{1}{\xi}\, \mathsf{a}_\phi(x). \tag{A.22}$$

Note that parametrization (A.16) can be expressed as

$$\begin{aligned} \chi(b) = \kappa(\xi, \theta, \phi) &= \kappa(r_o, \theta, \phi) + (\xi - r_o)\mathsf{a}_r(\kappa(r_o, \theta, \phi)) \\ &= \bar{\kappa}(\theta, \phi) + (\xi - r_o)\mathsf{a}_r(\bar{\kappa}(\theta, \phi)) \end{aligned} \tag{A.23}$$

with

$$\bar{\kappa}(\theta, \phi) := \kappa(r_o, \theta, \phi) \tag{A.24}$$

$$\bar{\mathsf{b}}(\bar{\kappa}(\theta, \phi)) := \chi^{-1}(\bar{\kappa}(\theta, \phi)) \tag{A.25}$$

A.3 Integration

Integration on $\mathcal{B}$ is defined through the chart χ. If α is a scalar field on $\mathcal{B}$ we assume

$$\int_{\mathcal{B}} \alpha(b) = \int_{\mathcal{D}} \alpha(\mathsf{b}(x)) \tag{A.26}$$

In turn, through the parametrization κ and its Jacobian (A.20), the second integral can be expressed as

$$\begin{aligned}\int_{\mathcal{D}} \alpha(\mathsf{b}(x)) &= \int_{\mathcal{K}} \alpha(\mathsf{b}(\kappa(\xi,\theta,\phi)))\, \xi^2 \cos\phi \\ &= \int_{\xi_1}^{\xi_2} \left(\int_{\theta_1}^{\theta_2} \left(\int_{\phi_1}^{\phi_2} \alpha(\mathsf{b}_\kappa(\xi,\theta,\phi))\, \xi^2 \cos\phi \, d\phi \right) d\theta \right) d\xi \end{aligned} \tag{A.27}$$

When α is independent of ϕ and θ (i.e. α is a spherically symmetric field) and the parameter intervals are those defined in (A.17), the integral becomes

$$\int_{\mathcal{D}} \alpha = 4\pi \int_{r_0-\epsilon}^{r_0+\epsilon} \alpha_\kappa(\xi)\, \xi^2 \, d\xi \tag{A.28}$$

A.4 Parametrization of a spherical surface

Let us assume that $\bar{\mathcal{D}}$ is a spherical surface without poles. The previous parametrization can be adapted to this case by holding ξ fixed in the following way

$$\bar{\chi}(b) = \bar{\kappa}(\theta,\phi) = \kappa(r_o,\theta,\phi) = x_o + r_o(\cos\phi\,\cos\theta\,\mathsf{e}_3 + \cos\phi\,\sin\theta\,\mathsf{e}_1 + \sin\phi\,\mathsf{e}_2), \tag{A.29}$$

with parameters ranging in the following intervals

$$-\pi < \theta \le \pi, \tag{A.30}$$

$$-\frac{\pi}{2} < \phi < \frac{\pi}{2}. \tag{A.31}$$

The points left out are the two poles in the e_2 direction. The vectors tangent to coordinate curves on $\bar{\mathcal{D}}$ through $x = \bar{\kappa}(\theta,\phi)$ turn out to be

$$\mathsf{a}_1(x) \equiv \mathsf{a}_\theta(x) = r_o(-\cos\phi\,\sin\theta\,\mathsf{e}_3 + \cos\phi\cos\theta\,\mathsf{e}_1), \tag{A.32}$$

$$\mathsf{a}_2(x) \equiv \mathsf{a}_\phi(x) = r_o(-\sin\phi\,\cos\theta\,\mathsf{e}_3 - \sin\phi\,\sin\theta\,\mathsf{e}_1 + \cos\phi\,\mathsf{e}_2), \tag{A.33}$$

with norms

$$\|\mathsf{a}_\theta(x)\| = r_o\,\cos\phi, \tag{A.34}$$

$$\|\mathsf{a}_\phi(x)\| = r_o, \tag{A.35}$$

and

$$\text{area}\,(\mathsf{a}_\theta(x), \mathsf{a}_\phi(x)) = r_o^2\,\cos\phi. \tag{A.36}$$

In order to get a basis in $\mathsf{V}\mathcal{E}$ we can add a third unit vector defined by

$$\mathsf{a}_r(x) \equiv \mathsf{a}_3(x) := \frac{\mathsf{a}_1(x) \times \mathsf{a}_2(x)}{\|\mathsf{a}_1(x)\|\,\|\mathsf{a}_2(x)\|}. \tag{A.37}$$

The gradient of a_r is

$$\nabla \mathsf{a}_r|_x \mathsf{a}_\theta(x) = \frac{1}{r_o}\, \mathsf{a}_\theta(x), \tag{A.38}$$

$$\nabla \mathsf{a}_r|_x \mathsf{a}_\phi(x) = \frac{1}{r_o}\, \mathsf{a}_\phi(x). \tag{A.39}$$

Appendix B

Finite element implementation

We look at a *finite element method* as a *direct method* for finding solutions of problems stated by a working balance principle. Test functions are endowed in the abstract model definition, instead of arising from a variational principle or by a reformulation in weak form of a problem initially stated in terms of partial differential equations.[1]

[1]To this respect even the choice of *interpolating functions* could be interpreted as pertaining to the modeling and not simply a matter of approximation. From this point of view on should face the problem of defining changes of observer and enforce invariance of the inner working.

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