Homeostatic states and relaxed configurations of a spherical thick shell

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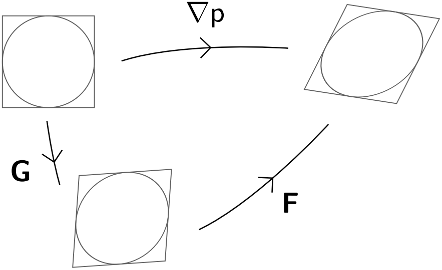
Kröner-Lee scheme



$$\mathbf{F} := \nabla \mathrm{p}\, \mathbf{G}^{-1}$$

Spherically symmetric shapes

Transplacement

$$\mathsf{p}(\mathbf{x},\tau) \;=\; \mathbf{x}_\mathsf{o} + \rho\,(\xi,\tau)\,\mathbf{e}_r(\vartheta,\varphi)\,,$$

Transplacement gradient and remodeling

$$\nabla\mathsf{p}|_\mathbf{x} = \rho'(\xi)\,\mathbf{P}_r + \xi^{-1}\rho\,(\xi)\,\mathbf{P}_h$$

$$\mathbf{G}(\mathbf{x}) = \gamma_r(\xi)\,\mathbf{P}_r + \gamma_h(\xi)\,\mathbf{P}_h$$

Effective stretch

$$\mathbf{F}(\mathbf{x}) := \nabla\mathsf{p}|_\mathbf{x}\mathbf{G}(\mathbf{x})^{-1} = \lambda_r(\xi)\,\mathbf{P}_r + \lambda_h(\xi)\,\mathbf{P}_h$$

General setting

Effective stretches

$$\lambda_h(\xi) := \frac{\rho(\xi)}{\xi\,\gamma_h(\xi)}, \quad \lambda_r(\xi) := \frac{\rho'(\xi)}{\gamma_r(\xi)}$$

Jacobian determinant

$$J_p(\xi) := (\rho(\xi)/\xi)^2\,\rho'(\xi)$$

Relaxed Jacobian

$$J(\xi) := \gamma_h^2(\xi)\gamma_r(\xi)$$

General setting

Balance

$$2\left(\mathrm{S}_r(\xi) - \mathrm{S}_h(\xi)\right) + \xi\,\mathrm{S}_r'(\xi) = 0$$

$$\mathrm{S}_r(\xi_\mp) = -\,\pi_\mp$$

$\pi_\mp$ outer *reference* pressure.

General setting

Cauchy stress

$$\mathrm{T}_h(\xi) = \frac{\rho(\xi)}{\xi}\mathrm{S}_h(\xi)/J_p(\xi)\,, \quad \mathrm{T}_r(\xi) = \rho'(\xi)\,\mathrm{S}_r(\xi)/J_p(\xi)$$

Balance equations

$$\frac{\rho(\xi)}{\xi}\Big(2\left(\mathrm{T}_r(\xi) - \mathrm{T}_h(\xi)\right)\rho'(\xi) + \rho(\xi)\,\mathrm{T}_r'(\xi)\Big) = 0$$

$$\mathrm{T}_r(\xi_\mp) = -\,\widehat{p}_\mp$$

$\widehat{p}_\mp$ outer *actual* pressure

$$\pi_\mp = \left(\rho\left(\xi_\mp\right)/\xi_\mp\right)^2 \widehat{p}_\mp$$

Strain energy function

Elastically incompressible material

$$\lambda_h^2(\xi)\lambda_r(\xi) = 1 \tag{1}$$

Fung strain energy function

$$\phi(\lambda_h) = \frac{c}{H}\left(-1 + e^{\frac{1}{2}\Gamma\left(\lambda_h^2 - 1\right)^2}\right) \tag{2}$$

Because of the incompressibility constraint the response function turns out to be defined only for the deviatoric part of the stress:

$$\mathrm{T}_h(\xi) - \mathrm{T}_r(\xi) = \frac{\lambda_h(\xi)}{2}\,\phi'(\lambda_h(\xi)) \tag{3}$$

Initial homeostatic state

For a given shape and fixed values of the outer pressure, find the stress and the transformation stretch. It is assumed that the hoop stress field is uniform across the thickness.

Results: It is shown that there exists only one solution (although no formal proof is given). The main point is that the transplacement is the identity map. This also makes Piola stress and Cauchy stress indistinguishable.

Initial homeostatic state

The reference shape is the same as the given shape:

$$\rho(\xi) = \xi \quad \Rightarrow \quad \lambda_h \gamma_h = 1,\ \lambda_r \gamma_r = 1$$

We assume that a homeostatic state is characterized by a uniform hoop stress $\mathrm{S}_h^\diamond$. The balance equation gives the following solution for the radial stress

$$\mathrm{S}_r(\xi) = \mathrm{S}_h^\diamond + \frac{C}{\xi^2}$$

By enforcing the two boundary conditions

$$\mathrm{S}_r(\xi_\mp) = -\,\pi_\mp$$

we get both C and $\mathrm{S}_h^\diamond$

Initial homeostatic state

Piola stress field

$$\mathrm{S}_r(\xi) = \frac{\pi_+ \left(\xi^2 - \xi_-^2\right) \xi_+^2 + \pi_- \left(\xi_+^2 - \xi^2\right) \xi_-^2}{\xi^2 \left(\xi_-^2 - \xi_+^2\right)}$$

$$\mathrm{S}_h^\diamond = \frac{\pi_+ \, \xi_+^2 - \pi_- \, \xi_-^2}{\left(\xi_-^2 - \xi_+^2\right)}$$

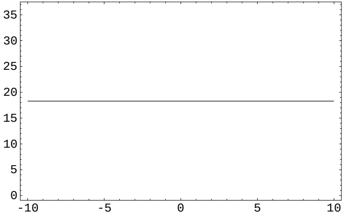
Initial homeostatic state

Numerical values

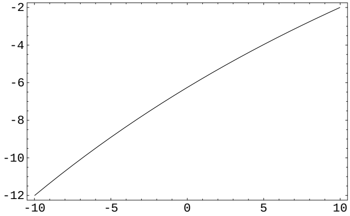
$$\begin{aligned}
\xi_m &= 2000/20\ \mu\text{m} \\
L &= H/1.18^2\ \mu\text{m} \\
c &= 0.8769 \times 10^{-6} \times 10^{-1}\,\text{N}\,\mu\text{m}^{-2} = 876.9 \times 10^{-1}\ \text{kPa} \\
\Gamma &= 12.99 \\
H &= 27.8\ \mu\text{m} \\
\pi_+ &= 2 \times 10^{-9}\,\text{N}\,\mu\text{m}^{-2} = 2\ \text{kPa} \\
\pi_- &= 12 \times 10^{-9}\,\text{N}\,\mu\text{m}^{-2} = 12\ \text{kPa}
\end{aligned}$$

Stress

$\mathrm{S}_h^\diamond$

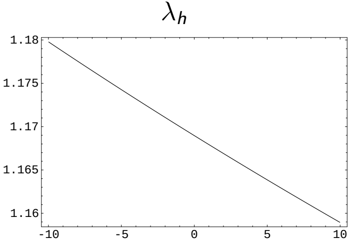


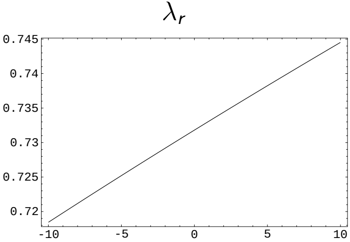
S_r



Initial homeostatic state

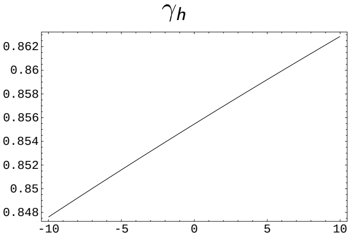
Effective stretch





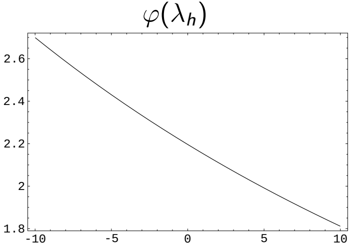
Initial homeostatic state

Transformation stretch



Initial homeostatic state

Strain energy density



Unloaded shape

After setting the outer pressure to zero, let the body relax while keeping the transformation stretch unchanged.

Results: The configuration the body reaches is not stress free.

Unloaded shape

Unloading transplacement

Since both γ_h and γ_r are left unchanged the transplacement is isochoric. From

$$\lambda_h(\xi) = \frac{\rho(\xi)}{\xi\,\gamma_h(\xi)}\,, \quad \lambda_r(\xi) = \frac{\rho'(\xi)}{\gamma_r(\xi)}\,, \quad \lambda_h^2(\xi)\lambda_r(\xi) = 1$$

we get

$$\rho(\xi) = \left(\xi^3 + (\rho_m^3 - \xi_m^3)\right)^{1/3} \tag{4}$$

where $\rho_m := \rho(\xi_m)$ is an integration constant.

Unloaded shape

From material response and balance equation

$$\mathrm{T}_h(\xi) - \mathrm{T}_r(\xi) = \frac{\lambda_h(\xi)}{2}\,\phi'(\lambda_h(\xi))$$
$$2\left(\mathrm{T}_r(\xi) - \mathrm{T}_h(\xi)\right)\rho'(\xi) + \rho(\xi)\,\mathrm{T}_r'(\xi) = 0$$

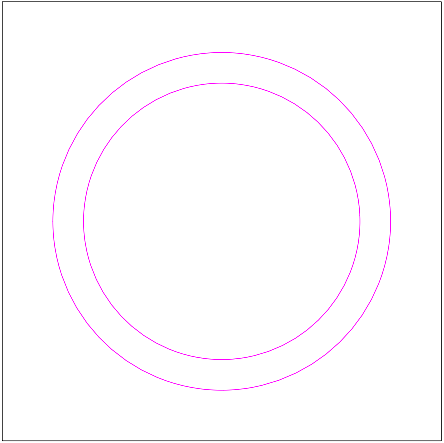
we get the boundary value problem

$$-\lambda_h(\xi)\,\phi'(\lambda_h(\xi))\,\rho'(\xi) + \rho(\xi)\,\mathrm{T}_r'(\xi) = 0$$

$$\mathrm{T}_r(\xi_\mp) = 0$$

which we can solve (possibly numerically) for both T_r and ρ_m, by using also $\lambda_h(\xi) := \rho(\xi)/\xi\,\gamma_h(\xi)$. Finally we compute T_h.

Reference shape



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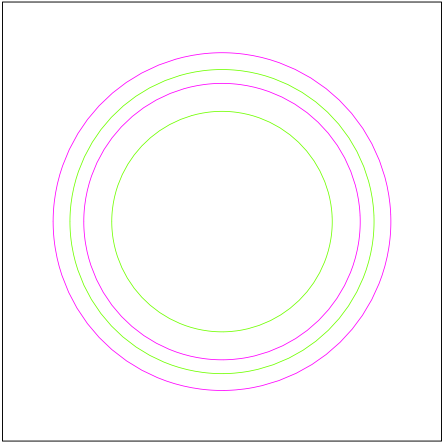
Unloaded shape



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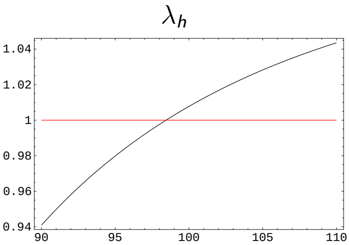
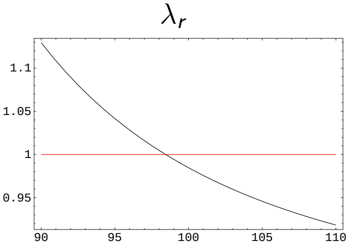
Unloaded shape

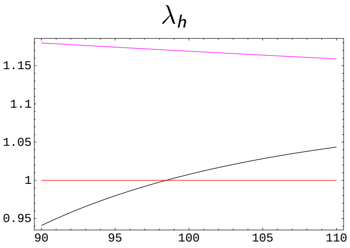
Residual effective stretch

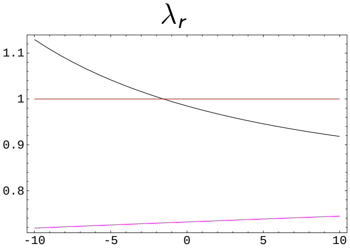




Unloaded shape

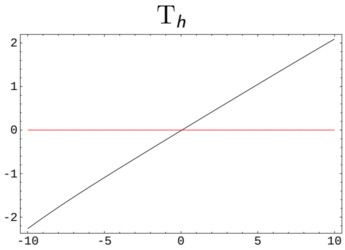
Residual effective stretch

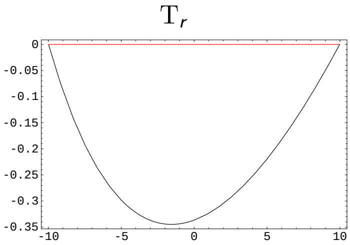




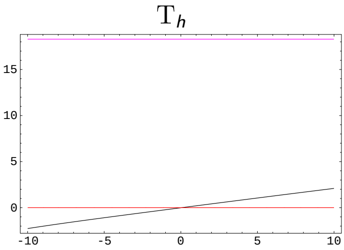
Unloaded shape

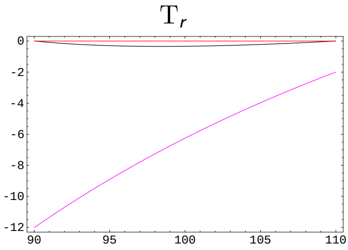
Residual stress





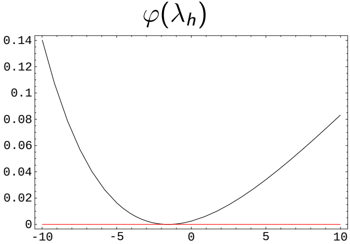
Residual stress





Unloaded shape

Residual strain energy density



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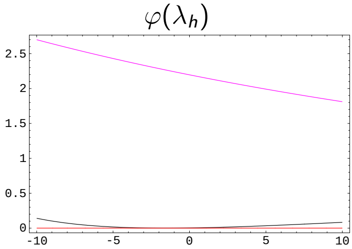
Residual strain energy density



Relaxed configuration

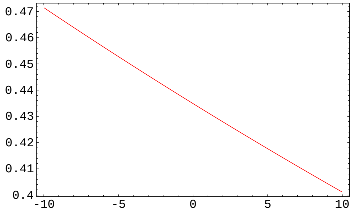
Does a relaxed configuration exist?

$$\nabla \mathsf{p}|_{\mathbf{x}} = \rho'(\xi)\,\mathbf{P}_r + \xi^{-1}\rho\,(\xi)\,\mathbf{P}_h$$

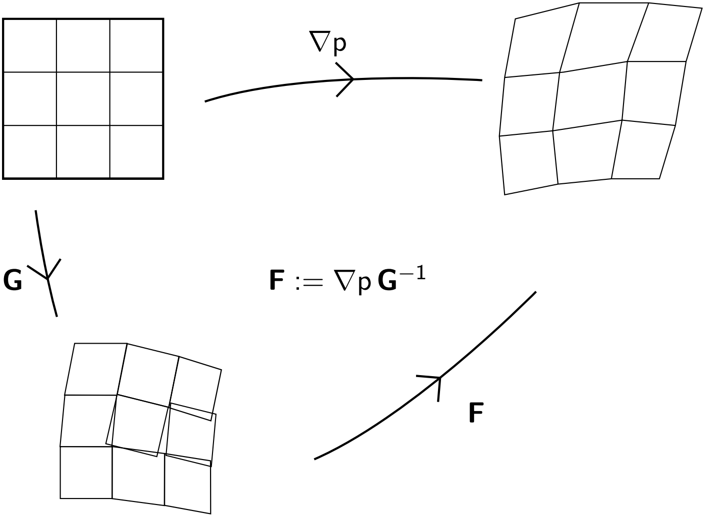
$$\lambda_r(\xi) = \frac{\rho'(\xi)}{\gamma_r(\xi)}\,, \quad \lambda_h(\xi) = \frac{\rho(\xi)}{\xi\,\gamma_h(\xi)}$$

$$\lambda_r(\xi) = 1 \,\&\, \lambda_h(\xi) = 1 \quad \Rightarrow \quad \gamma_r(\xi) - \big(\gamma_h(\xi) + \xi\gamma_h'(\xi)\big) = 0$$

$$\gamma_r(\xi) - \big(\gamma_h(\xi) + \xi\gamma_h'(\xi)\big)$$

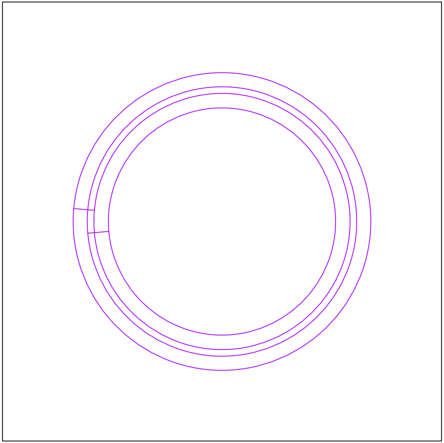


Kröner-Lee scheme



$$\mathbf{F} := \nabla p \, \mathbf{G}^{-1}$$

Unloaded two layer shape



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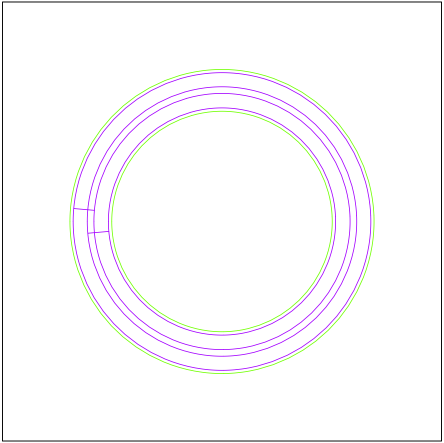
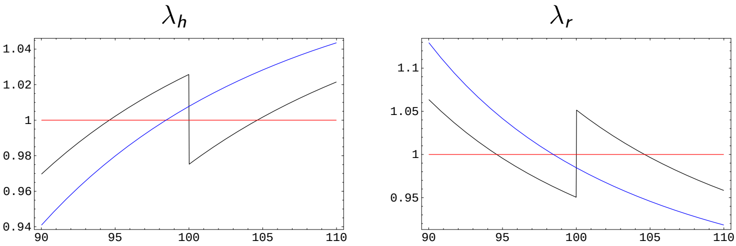


Unloaded two layer shape

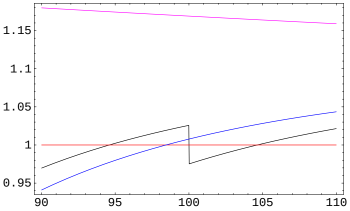
Residual effective stretch



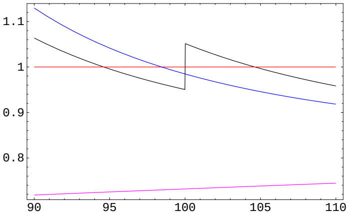
Unloaded two layer shape

Residual effective stretch

λ_h

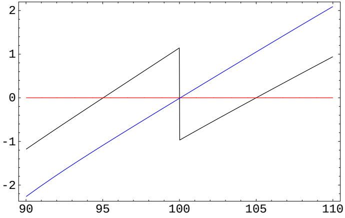


λ_r

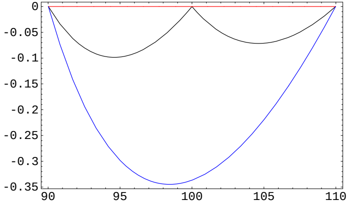


Residual stress

T_h



T_r



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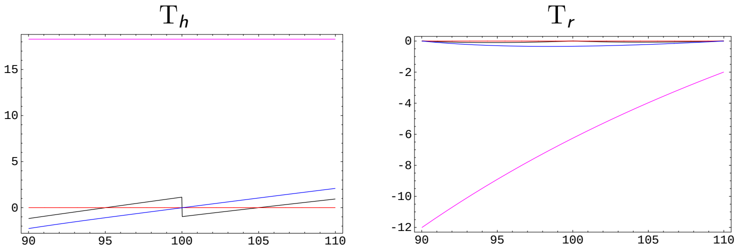
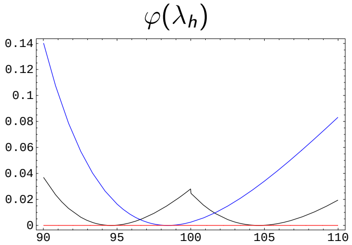
Residual stress



Unloaded two layer shape

Residual strain energy density



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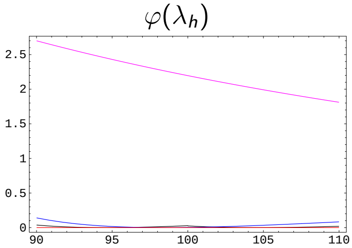
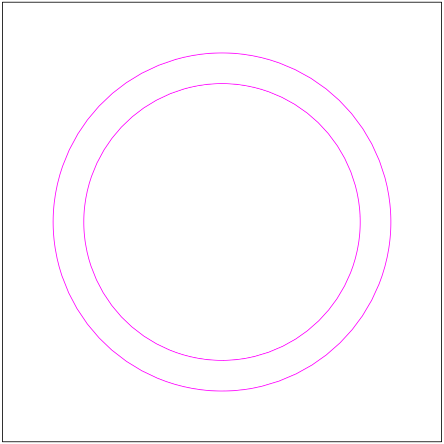
Residual strain energy density



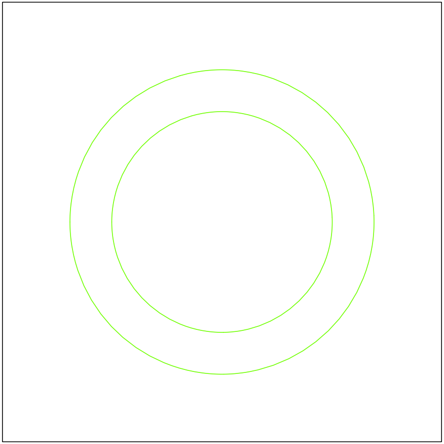
Slice

Reference shape



Slice

Unloaded shape



Slice

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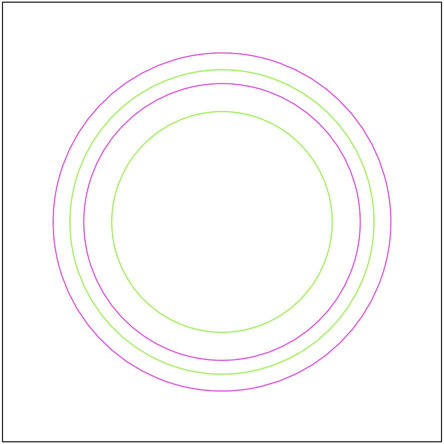

Slice

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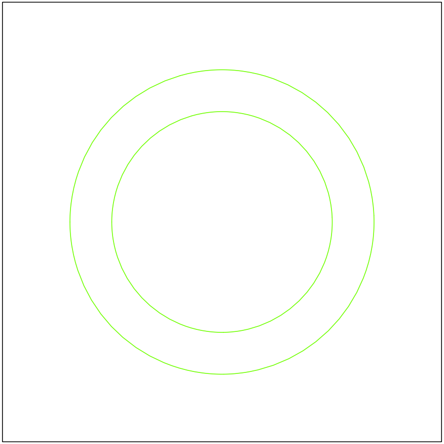

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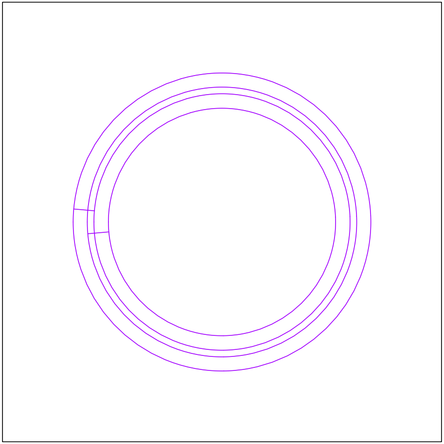

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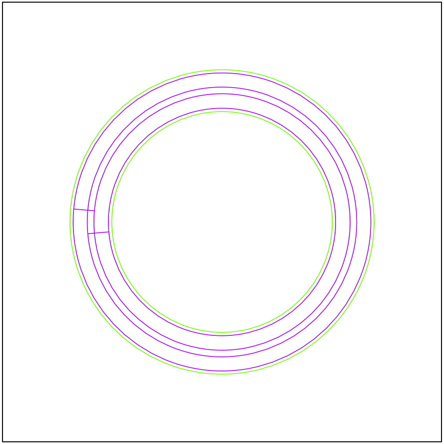

Slice

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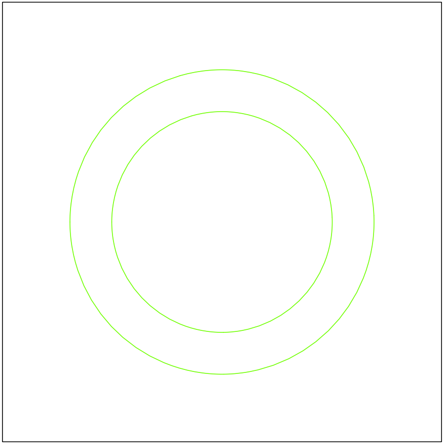
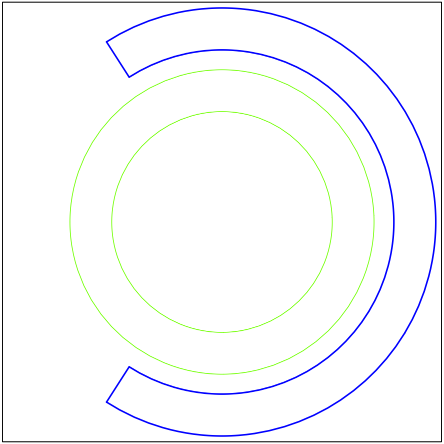
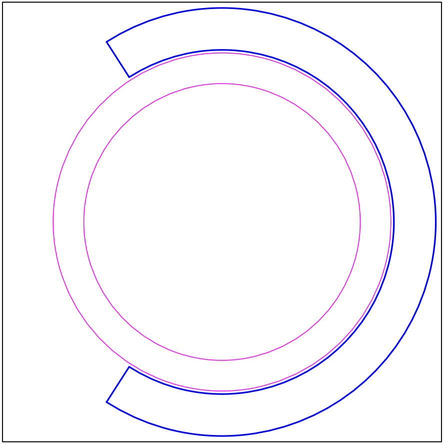


Open slice



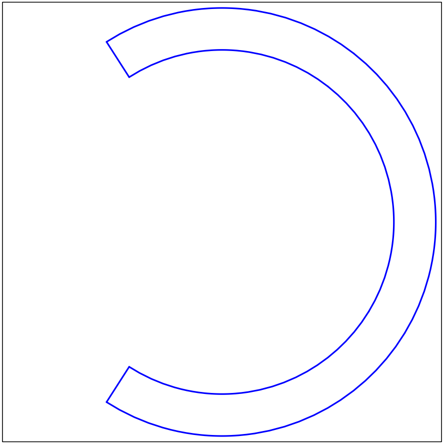
Slice

Open slice



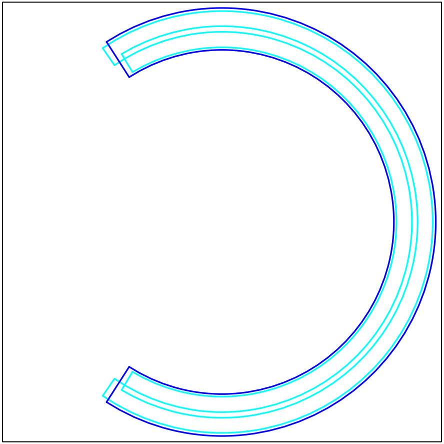
Slice

Open slice



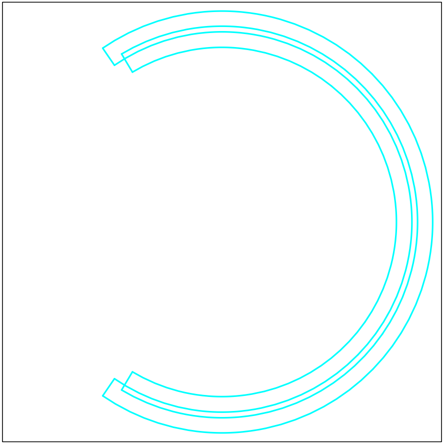
Slice

Open slice split into 2 layers



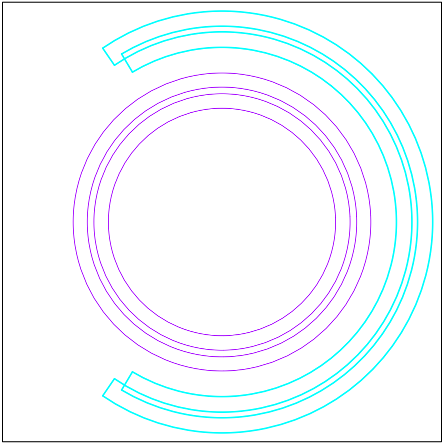
Slice

Open slice split into 2 layers



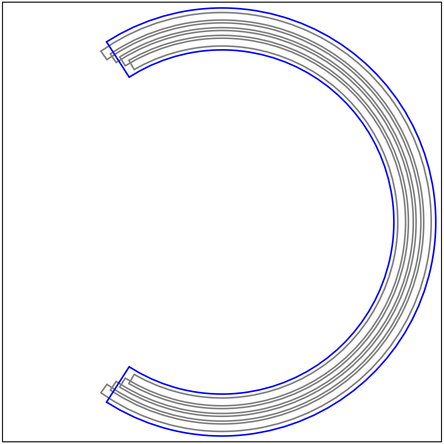
Slice

Open slice split into 2 layers



Slice

Open slice split into 4 layers



Slice

Open slice split into 6 layers