

Equipowerful surface distributions (DRAFT)

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Abstract

An alternative approach to the Cauchy stress (together with higher order stresses, usually called hyper-stresses) is proposed, relying on the notion of power expended on test velocity fields, instead of the notion of resultant force. The stress tensor arises as the appropriate descriptor of equipowerful classes, on test velocity fields of grade 1, of surface distributions expending no power on test velocity fields of grade 0. Two different distributions characterized by the same 1-stress (serialized name for the Cauchy stress) do not belong to the same equipowerful class of higher grade in general. In order to characterize higher equipowerful classes we need one more hyper-stress for each increment of the grade of the theory. It can be shown that a 2-stress corresponds to edge forces and surface double forces, besides the ordinary surface distribution. Similarly a 3-stress corresponds to vertex forces, edge double forces and surface triple forces, besides the previous ones.

Contents

1	Introduction	1
2	Affine model	2
3	Two-dimensional affine bodies	4
4	Second gradient model	6
5	Properties of the moment \mathbb{L}	7
6	Velocity patterns	9

1 Introduction

We propose an alternative approach to Cauchy stress, and to higher order stresses (usually called *hyper-stresses*), relying on the notion of power expended on test velocities instead of resultant force.

We ask ourselves the following question: what is the entity characterizing surface force distributions which are *equipowerful* on any spatial velocity field belonging to a polynomial space of an assigned degree $m > 0$? The Cauchy stress tensor arises as the appropriate entity characterizing equipowerful classes on test velocity fields of degree 1, made up of surface force distributions expending no power on velocity fields of degree 0. From our point of view the *mean stress*, introduced by Signorini, precedes logically the notion of pointwise stress (it seems satisfactory the former being an integral quantity).

Indeed, any two distributions belonging to the same powerful class characterized by a value of the 1-stress (serialized name for the Cauchy stress) are not equipowerful on test velocity fields of degree greater than 1, in general. If we want to tell them apart we need to introduce more refined equipowerful classes, increasing the grade of the theory. These new equipowerful classes will be characterized by a new hyper-stress. We can show that a 2-stress describes a class characterized by edge forces and surface double forces, besides the ordinary surface distribution. Similarly a 3-stress describes a class characterized by vertex forces, edge double forces and surface triple forces, besides the previous ones.

First we show how to build on a body in a regular shape \mathcal{R} a force distribution belonging to an equipowerful class. To this end velocity fields on increasing degree are used. Corresponding power expressions are described through “moment tensors” and “mean stress tensor” of increasing order.

Then we show a naive method for building force distributions adapted to bodies in the shape of parallelepipeds. This method is by no means restrictive because, whatever the grade m of the theory is, it is sufficient to work on n -intervals (i.e., with parallelepiped cuts of a body - a n -dimensional manifold - embedded in the ambient space, of dimension $N \geq n$). This is allowed by recent results in geometric measure theory which, starting from entities defined on n -intervals, show how it is possible to cope with the more general case of sets of finite perimeter ([6]).

The notion of equipower relies on the notion of virtual power introduced in [3]. The force distributions characterizing to equipowerful classes of grade 2 correspond those described in [1].

2 Affine model

Let us consider a body in the shape \mathcal{R} embedded in a 3-dimensional Euclidean space \mathcal{E} . Let the test velocity fields be the given by

$$v(x) = v_o + G(x - x_o), \quad (1)$$

with

$$v_o \in \mathcal{V}, \quad G : \mathcal{U} \rightarrow \mathcal{V}. \quad (2)$$

Let the *working* be

$$\mathcal{W}_{\mathcal{R}}(v) := \mathcal{W}_{\mathcal{R}}^{out}(v) + \mathcal{W}_{\mathcal{R}}^{in}(v) \quad (3)$$

with the *outer* and the *inner* working given by

$$\mathcal{W}_{\mathcal{R}}^{out}(v) := (b \cdot v_o + L \cdot G) \text{vol}(\mathcal{R}), \quad (4)$$

$$\mathcal{W}_{\mathcal{R}}^{in}(v) := -(z \cdot v_o + T \cdot G) \text{vol}(\mathcal{R}). \quad (5)$$

Let a rigid test velocity be

$$v(x) = v_o + W(x - x_o), \quad (6)$$

whith W antisymmetric. Assuming the inner working be zero for any rigid test velocity (*principle of material frame indifference*) leads to the constitutive requirements

$$z = o, \quad \text{skw } T = O. \quad (7)$$

Assuming the total working be zero for any test velocity (*principle of null working*) leads to the following balance equations

$$\begin{aligned} b &= o, \\ \text{skw } L &= O, \\ \text{sym } L &= T. \end{aligned} \tag{8}$$

We want to find out examples of equipowerful force distributions. Those distributions are characterized, through (4), by same value of b and L . Any representation satisfying the balance equations will also illustrate the meaning of the stress tensor T .¹

The first and more interesting example is given by the traction

$$t := Ln, \tag{9}$$

where n is the outward unit normal vector field on $\partial\mathcal{R}$, assumed to be piecewise regular. The outer working is

$$\mathcal{W}_{\mathcal{R}}^{\text{out}}(v) = \int_{\partial\mathcal{R}} t \cdot v = \int_{\partial\mathcal{R}} t \cdot v_o + \int_{\partial\mathcal{R}} t \cdot Gr = \int_{\partial\mathcal{R}} Ln \cdot v_o + \int_{\partial\mathcal{R}} Ln \cdot Gr, \tag{10}$$

where $r := (x - x_o)$. The first of the last two integrals is zero because

$$\int_{\partial\mathcal{R}} Ln = L \int_{\partial\mathcal{R}} n = o. \tag{11}$$

The second integral will transform, by the divergence theorem, like

$$\begin{aligned} \mathcal{W}_{\mathcal{R}}^{\text{out}}(v) &= \int_{\partial\mathcal{R}} Ln \cdot Gr = \int_{\partial\mathcal{R}} r \otimes Ln \cdot G = L \int_{\partial\mathcal{R}} r \otimes n \cdot G \\ &= L \int_{\mathcal{R}} \text{grad } r \cdot G = L \cdot G \text{ vol}(\mathcal{R}). \end{aligned} \tag{12}$$

As an alternative we can consider the force defined by the field on \mathcal{R}

$$p(x) := Ar_c, \tag{13}$$

where A is an endomorphism of \mathcal{V} , $r_c = (x - x_c)$ and x_c is the center of the shape \mathcal{R} . The outer working in the finer model is

$$\begin{aligned} \mathcal{W}_{\mathcal{R}}^{\text{out}}(v) &= \int_{\mathcal{R}} p \cdot v = \int_{\mathcal{R}} p \cdot v_o + \int_{\mathcal{R}} p \cdot Gr \\ &= \int_{\mathcal{R}} Ar_c \cdot v_o + \int_{\mathcal{R}} r \otimes Ar_c \cdot G. \end{aligned} \tag{14}$$

Since it turns out

$$\int_{\mathcal{R}} Ar_c = 0, \tag{15}$$

then the last expression can be transformed in the following way

$$\begin{aligned} \int_{\mathcal{R}} r \otimes Ar_c \cdot G &= A \int_{\mathcal{R}} r \otimes r_c \cdot G \\ &= A \int_{\mathcal{R}} r_c \otimes r_c \cdot G = AJ \cdot G, \end{aligned} \tag{16}$$

¹All of that can even be applied to a rigid body if we only look at it as a constrained affine body, the stress T taking the meaning of reactive stress.

where J is the Euler tensor. We can get $\mathcal{W}_{\mathcal{R}}^{out}(v) = L \cdot G \text{vol}(\mathcal{R})$ by setting $A := LJ^{-1} \text{vol}(\mathcal{R})$. The expression for p becomes

$$p(x) = L \left(\frac{J}{\text{vol}(\mathcal{R})} \right)^{-1} r_c. \quad (17)$$

Let us consider now any L (or equivalently, through (8), any T) and let us try to build up a force distribution on bodies in the shape \mathcal{R}_ε , whose working is given by $L \cdot G \text{vol}(\mathcal{R}_\varepsilon)$. For any sequence $\mathcal{R}_\varepsilon \subset \mathcal{B}_\varepsilon$, being \mathcal{B}_ε a sequence of spheres of decreasing radius ε , it turns out that for any ε both distributions (9) and (17) belong to the same equipowerful class. But it happens that $\lim_{\varepsilon \rightarrow 0} \|p(x)\| = \infty$, while the expression (9) holds constant.

Another remark is the following. If we replace the surface force distribution (9) with

$$t := Ln + \tilde{t}, \quad (18)$$

for the properties of Ln already used, we get to the conclusion

$$\int_{\partial\mathcal{R}} \tilde{t} \cdot v_o = 0, \quad \int_{\partial\mathcal{R}} \tilde{t} \cdot Gr = 0. \quad (19)$$

Hence \tilde{t} is a distribution expending no working on velocity fields of degree 0 and 1.

We can obtain a more interesting case by considering the surface force distribution (18) together with a constant volume distribution

$$p(x) := p_o, \quad (20)$$

and assuming $x_o = x_c$, the center of the shape. This time we get to the conclusion

$$p_o \cdot v_o + \frac{1}{\text{vol}(\mathcal{R})} \int_{\partial\mathcal{R}} \tilde{t} \cdot v_o = 0, \quad \int_{\partial\mathcal{R}} \tilde{t} \cdot Gr = 0. \quad (21)$$

At last it is worth noting that, for an antisymmetric L , the distribution (18) belong to the same equipowerful class defined, through (8), by the 1-stress T .

3 Two-dimensional affine bodies

Let us consider a shell with thickness distension, in the shape of a surface \mathcal{F} , oriented and with piecewise regular boundary. Let the test velocity fields be

$$v(x) = v_o + Gr, \quad (22)$$

$$g(x) = G n(x), \quad (23)$$

where n is the unit normal field and g describes both thickness distension velocity and spin. Let the outer working be

$$\mathcal{W}_{\mathcal{F}}^{out}(v, g) = \int_{\mathcal{F}} p \cdot v + \int_{\partial\mathcal{F}} t \cdot v + \int_{\mathcal{F}} c \cdot g + \int_{\partial\mathcal{F}} q \cdot g. \quad (24)$$

We are looking for a system of forces such that, at corresponding velocities (22),

$$\mathcal{W}_{\mathcal{F}}^{out}(v, g) = (b \cdot v_o + L \cdot G) \text{area}(\mathcal{F}), \quad (25)$$

with $b = o$. One such a system of forces is defined by

$$p = (k_1 + k_2)Ln, \quad t = Lm, \quad c = Ln, \quad q = o, \quad (26)$$

where k_1 and k_2 are the principal curvatures and m denotes the outward unit normal field of tangent vectors on the boundary. By substituting (22) and (26) into (24) we get

$$\begin{aligned} \mathcal{W}_{\mathcal{F}}^{out}(v, g) &= \int_{\partial\mathcal{F}} Lm \cdot v_o + \int_{\mathcal{F}} (k_1 + k_2)Ln \cdot v_o + \int_{\partial\mathcal{F}} Lm \cdot Gr \\ &+ \int_{\mathcal{F}} (k_1 + k_2)Ln \cdot Gr + \int_{\mathcal{F}} Ln \cdot Gn. \end{aligned} \quad (27)$$

We can define at each $x \in \mathcal{F}$ the orthogonal projection onto the tangent space $P : \mathcal{V} \rightarrow \mathcal{T}$ together with its complement $V := I - P$. This allows the following decomposition

$$L \cdot G = (LV^{\top}) \cdot (GV) + (LP^{\top}) \cdot (GP). \quad (28)$$

By the divergence theorem for a tangent vector field the first term in (27) can be transformed as follows

$$\int_{\partial\mathcal{F}} Lm \cdot v_o = \int_{\partial\mathcal{F}} m \cdot L^{\top}v_o = \int_{\mathcal{F}} \operatorname{div}_s(PL^{\top}v_o) = - \int_{\mathcal{F}} (k_1 + k_2)Ln \cdot v_o. \quad (29)$$

This result comes from the definition

$$\operatorname{div}_s u := \operatorname{tr}(P \operatorname{grad}_s u) \quad (30)$$

where u is a tangent vector field. Using the expressions $V = n \otimes n$ and $P = I - n \otimes n$, for any tangent vector a we have

$$\begin{aligned} P \operatorname{grad}_s (PL^{\top}v_o)a &= -P((\operatorname{grad}_s n)a \otimes n + n \otimes (\operatorname{grad}_s n)a)L^{\top}v_o \\ &= -(n \otimes (\operatorname{grad}_s n)a)L^{\top}v_o = -(Ln \cdot v_o)(\operatorname{grad}_s n)a. \end{aligned} \quad (31)$$

Hence

$$\begin{aligned} \operatorname{div}_s (PL^{\top}v_o) &= \operatorname{tr}(P \operatorname{grad}_s (PL^{\top}v_o)) \\ &= -(Ln \cdot v_o) \operatorname{tr}(\operatorname{grad}_s n) = -(Ln \cdot v_o)(k_1 + k_2). \end{aligned} \quad (32)$$

For the same reasons the third term in (27) will transform as follows

$$\begin{aligned} \int_{\partial\mathcal{F}} Lm \cdot Gr &= \int_{\partial\mathcal{F}} m \cdot L^{\top}Gr = \int_{\mathcal{F}} \operatorname{div}_s(PL^{\top}Gr) \\ &= \int_{\mathcal{F}} \operatorname{tr}(P \operatorname{grad}_s(PL^{\top}Gr)) \\ &= \int_{\mathcal{F}} \operatorname{tr}(PL^{\top}G - (Ln \cdot Gr) \operatorname{grad}_s n) \\ &= \int_{\mathcal{F}} (LP^{\top}) \cdot (GP) - \int_{\mathcal{F}} (k_1 + k_2)Ln \cdot Gr \end{aligned} \quad (33)$$

The last term in (27), according to the definition of V , can be put into the form

$$\int_{\mathcal{F}} Ln \cdot Gn = \int_{\mathcal{F}} (LV^{\top}) \cdot (GV). \quad (34)$$

Substituting (29), (33), (34) into (27) we get, by (28),

$$\mathcal{W}_{\mathcal{F}}^{out}(v, g) = \int_{\mathcal{F}} L \cdot G = L \cdot G \text{ area}(\mathcal{F}). \quad (35)$$

Thus condition (25) holds with $b = o$, for any value of L .

As an alternative let us consider a system of forces defined by

$$p(x) = Ar_c, \quad t = o, \quad c = Bn, \quad q = o, \quad (36)$$

where A is an endomorphism of \mathcal{U} and x_c is the center of the shape \mathcal{F} , not necessarily on \mathcal{F} . By substituting (22), (23) and (36) into (24) we get

$$\begin{aligned} \mathcal{W}_{\mathcal{F}}^{out}(v, g) &= \int_{\mathcal{F}} Ar_c \cdot v_o + \int_{\mathcal{F}} Ar_c \cdot Gr + \int_{\mathcal{F}} Bn \cdot Gn \\ &= A \int_{\mathcal{F}} r_c \cdot v_o + A \int_{\mathcal{F}} r \otimes r_c \cdot G + B \int_{\mathcal{F}} n \otimes n \cdot G \\ &= \left(A \int_{\mathcal{F}} r_c \otimes r_c + B \int_{\mathcal{F}} n \otimes n \right) \cdot G. \end{aligned} \quad (37)$$

Condition (25) can be fulfilled by setting $B := Ah^2$, $A := LJ^{-1} \text{ area}(\mathcal{F})$, with

$$J := \int_{\mathcal{F}} \left(r_c \otimes r_c + n \otimes nh^2 \right), \quad (38)$$

and h denoting the thickness. The expressions for p and c turn out to be

$$p(x) = L \left(\frac{J}{\text{area}(\mathcal{F})} \right)^{-1} r_c, \quad c(x) = L \left(\frac{J}{\text{area}(\mathcal{F})} \right)^{-1} h^2 n(x). \quad (39)$$

4 Second gradient model

Let us consider a body in the shape \mathcal{R} embedded in a 3-dimensional Eucliden space \mathcal{E} . Let the test velocity fields be the given by

$$v(x) = v_o + Gr + \frac{1}{2} \mathbb{G}rr. \quad (40)$$

where

$$v_o \in \mathcal{V}, \quad G : \mathcal{U} \rightarrow \mathcal{V}, \quad \mathbb{G} : \mathcal{U} \rightarrow \text{Lin}(\mathcal{U}, \mathcal{V}), \quad (41)$$

with the following symmetry assumption²

$$\mathbb{G}uv = \mathbb{G}vu, \quad \forall u \in \mathcal{U}, \forall v \in \mathcal{U}. \quad (42)$$

Let the outer working and the inner working be

$$\mathcal{W}_{\mathcal{R}}^{out}(v) = (b \cdot v_o + L \cdot G + \mathbb{L} \cdot \mathbb{G}) \text{ vol}(\mathcal{R}), \quad (43)$$

$$\mathcal{W}_{\mathcal{R}}^{in}(v) = -(z \cdot v_o + T \cdot G + \mathbb{T} \cdot \mathbb{G}) \text{ vol}(\mathcal{R}). \quad (44)$$

²Strictly speaking this would be a *simmetry* assumption only if \mathbb{G} were interpreted as a tensor $\mathbb{G} : \mathcal{U} \times \mathcal{U} \rightarrow \mathcal{V}$. This assumption is motivated by the fact that both the first and the second gradient of the field (40) would be independent of the *skew* part of \mathbb{G} .

Both the moment \mathbb{L} and the stress \mathbb{T} so defined inherit the symmetry property (42) from \mathbb{G} . Let a rigid test velocity be

$$v(x) = v_o + Wr \quad (45)$$

where W is antisymmetric. The principle of material frame indifference leads to the constitutive requirements

$$z = 0, \quad \text{skw } T = O. \quad (46)$$

The principle of null working leads to the following balance equations

$$\begin{aligned} b &= o, \\ \text{skw } L &= O, \\ \text{sym } L &= T, \\ \mathbb{L} &= \mathbb{T}. \end{aligned} \quad (47)$$

We want to find out examples of equipowerful force distributions, for a given value of \mathbb{L} and with $b = o$, $L = O$. This will also be useful to illustrate the meaning of the stress \mathbb{T} .

5 Properties of the moment \mathbb{L}

First observe that

$$\int_{\partial\mathcal{R}} r \otimes \mathbb{L}n = \mathbb{L} \int_{\partial\mathcal{R}} r \otimes n = \mathbb{L} \int_{\mathcal{R}} \text{grad } r = \mathbb{L} \text{vol}(\mathcal{R}). \quad (48)$$

The working corresponding to the second gradient \mathbb{G} can then be put in the form

$$\mathbb{L} \cdot \mathbb{G} \text{vol}(\mathcal{R}) = \int_{\partial\mathcal{R}} r \otimes \mathbb{L}n \cdot \mathbb{G} = \int_{\partial\mathcal{R}} \mathbb{L}n \cdot \mathbb{G}r. \quad (49)$$

At this stage we don't know whether the integral above can be interpreted as the outer working of a force distribution whatsoever or not. Assuming the boundary $\partial\mathcal{R}$ to be piecewise regular, let us consider a partition

$$\partial\mathcal{R} = \bigcup_i^N \mathcal{F}_i \quad (50)$$

into regular faces with piecewise regular boundary and non overlapping interiors. From (49) we get

$$\mathbb{L} \cdot \mathbb{G} \text{vol}(\mathcal{R}) = \sum_i^N \int_{\mathcal{F}_i} \mathbb{L}n \cdot \mathbb{G}r. \quad (51)$$

For any \mathcal{F}_i we can define at each $x \in \mathcal{F}_i$ the orthogonal projection onto the tangent space $P : \mathcal{V} \rightarrow \mathcal{T}$ together with its complement $V := I - P$. This allows the following decomposition

$$\mathbb{L}n \cdot \mathbb{G}r = (\mathbb{L}n)V^T \cdot (\mathbb{G}r)V + (\mathbb{L}n)P^T \cdot (\mathbb{G}r)P. \quad (52)$$

Now we may guess that the second term in the above expression is nothing but the working of forces $(\mathbb{L}n)m$ along the boundary $\partial\mathcal{F}_i$. So let us examine the expression

$$\int_{\partial\mathcal{F}_i} \mathbb{L}nm \cdot \mathbb{G}rr \quad (53)$$

and try to relate it to (52). By applying the divergence theorem we get

$$\begin{aligned} \int_{\partial\mathcal{F}_i} \mathbb{L}nm \cdot \mathbb{G}rr &= \int_{\partial\mathcal{F}_i} m \cdot (\mathbb{L}n)^\top \mathbb{G}rr \\ &= \int_{\mathcal{F}_i} \operatorname{div}_s (P(\mathbb{L}n)^\top \mathbb{G}rr) \\ &= \int_{\mathcal{F}_i} \operatorname{tr} (P \operatorname{grad}_s (P(\mathbb{L}n)^\top \mathbb{G}rr)). \end{aligned} \quad (54)$$

By using the symmetry property of \mathbb{G} we can derive the following expression for the gradient applied to any tangent vector a

$$\begin{aligned} P \operatorname{grad}_s (P(\mathbb{L}n)^\top \mathbb{G}rr) a &= 2P(\mathbb{L}n)^\top \mathbb{G}ra + P(\mathbb{L}(\operatorname{grad}_s n)a)^\top \mathbb{G}rr - (n \otimes (\operatorname{grad}_s n)a)(\mathbb{L}n)^\top \mathbb{G}rr \\ &= 2P(\mathbb{L}n)^\top \mathbb{G}ra + P(\mathbb{L}(\operatorname{grad}_s n)a)^\top \mathbb{G}rr - (\mathbb{L}nn \cdot \mathbb{G}rr)(\operatorname{grad}_s n)a, \end{aligned} \quad (55)$$

from which we finally obtain

$$\begin{aligned} \operatorname{div}_s (P(\mathbb{L}n)^\top \mathbb{G}rr) &= 2(\mathbb{L}n)P^\top \cdot \mathbb{G}rP \\ &\quad + (k_1 \mathbb{L}a_1 a_1 + k_2 \mathbb{L}a_2 a_2 - (k_1 + k_2) \mathbb{L}nn) \cdot \mathbb{G}rr. \end{aligned} \quad (56)$$

Here a_1 and a_2 are eigenvectors of the Weingarten tensor $(\operatorname{grad}_s n)$, k_1 and k_2 are the corresponding principal curvatures. Hence

$$\begin{aligned} \int_{\partial\mathcal{F}_i} \mathbb{L}nm \cdot \mathbb{G}rr &= 2 \int_{\mathcal{F}_i} (\mathbb{L}n)P^\top \cdot \mathbb{G}rP \\ &\quad + \int_{\mathcal{F}_i} (k_1 \mathbb{L}a_1 a_1 + k_2 \mathbb{L}a_2 a_2 - (k_1 + k_2) \mathbb{L}nn) \cdot \mathbb{G}rr \end{aligned} \quad (57)$$

from which we get

$$\begin{aligned} \int_{\mathcal{F}_i} (\mathbb{L}n)P^\top \cdot \mathbb{G}rP &= \frac{1}{2} \int_{\partial\mathcal{F}_i} \mathbb{L}nm \cdot \mathbb{G}rr \\ &\quad + \frac{1}{2} \int_{\mathcal{F}_i} (-k_1 \mathbb{L}a_1 a_1 - k_2 \mathbb{L}a_2 a_2 + (k_1 + k_2) \mathbb{L}nn) \cdot \mathbb{G}rr. \end{aligned} \quad (58)$$

Substituting (58) into (51) through (52) we finally obtain

$$\begin{aligned} \mathbb{L} \cdot \mathbb{G} \operatorname{vol}(\mathcal{R}) &= \sum_i^N \left(\int_{\mathcal{F}_i} \mathbb{L}nn \cdot \mathbb{G}rn + \frac{1}{2} \int_{\partial\mathcal{F}_i} \mathbb{L}nm \cdot \mathbb{G}rr \right. \\ &\quad \left. + \frac{1}{2} \int_{\mathcal{F}_i} (-k_1 \mathbb{L}a_1 a_1 - k_2 \mathbb{L}a_2 a_2 + (k_1 + k_2) \mathbb{L}nn) \cdot \mathbb{G}rr \right). \end{aligned} \quad (59)$$

6 Velocity patterns

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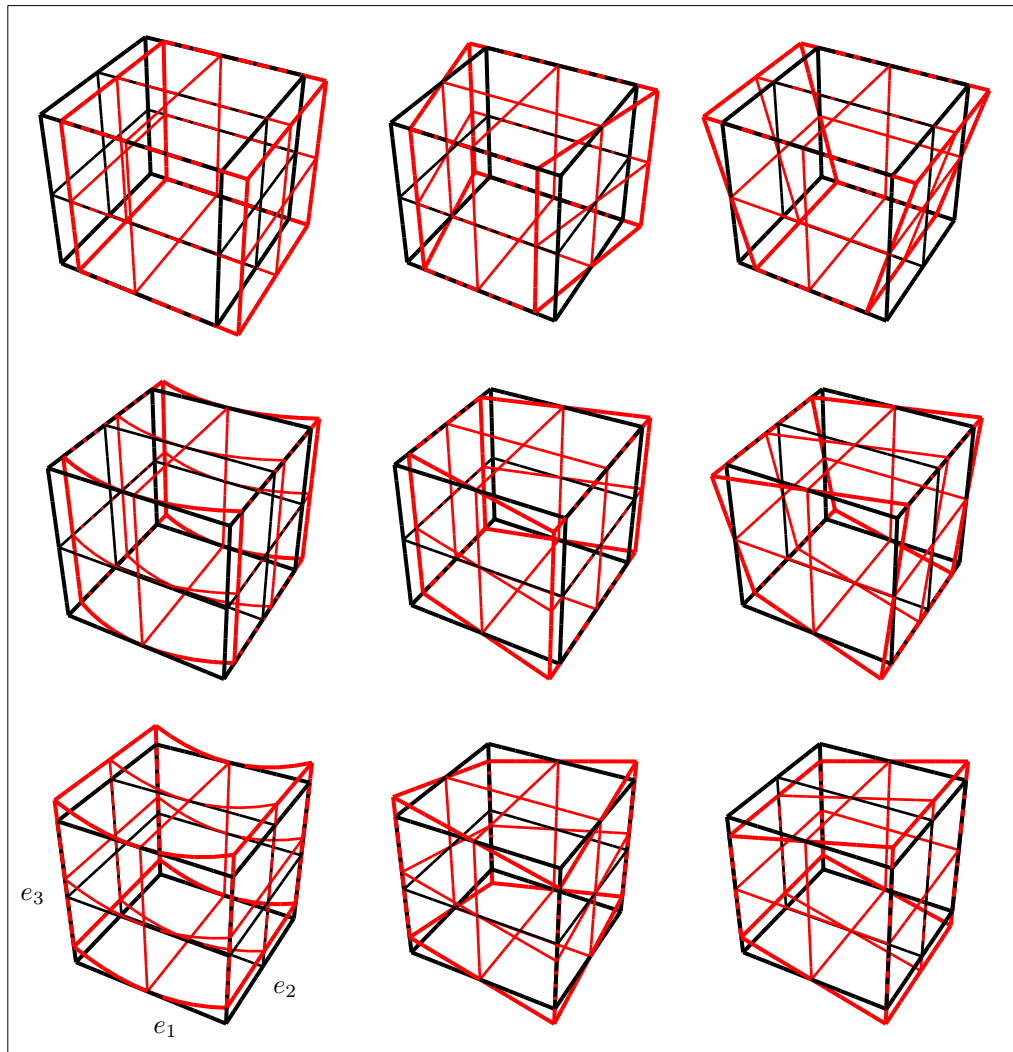


Figure 1: Velocity patterns corresponding to \mathbb{G}_{e_1} .

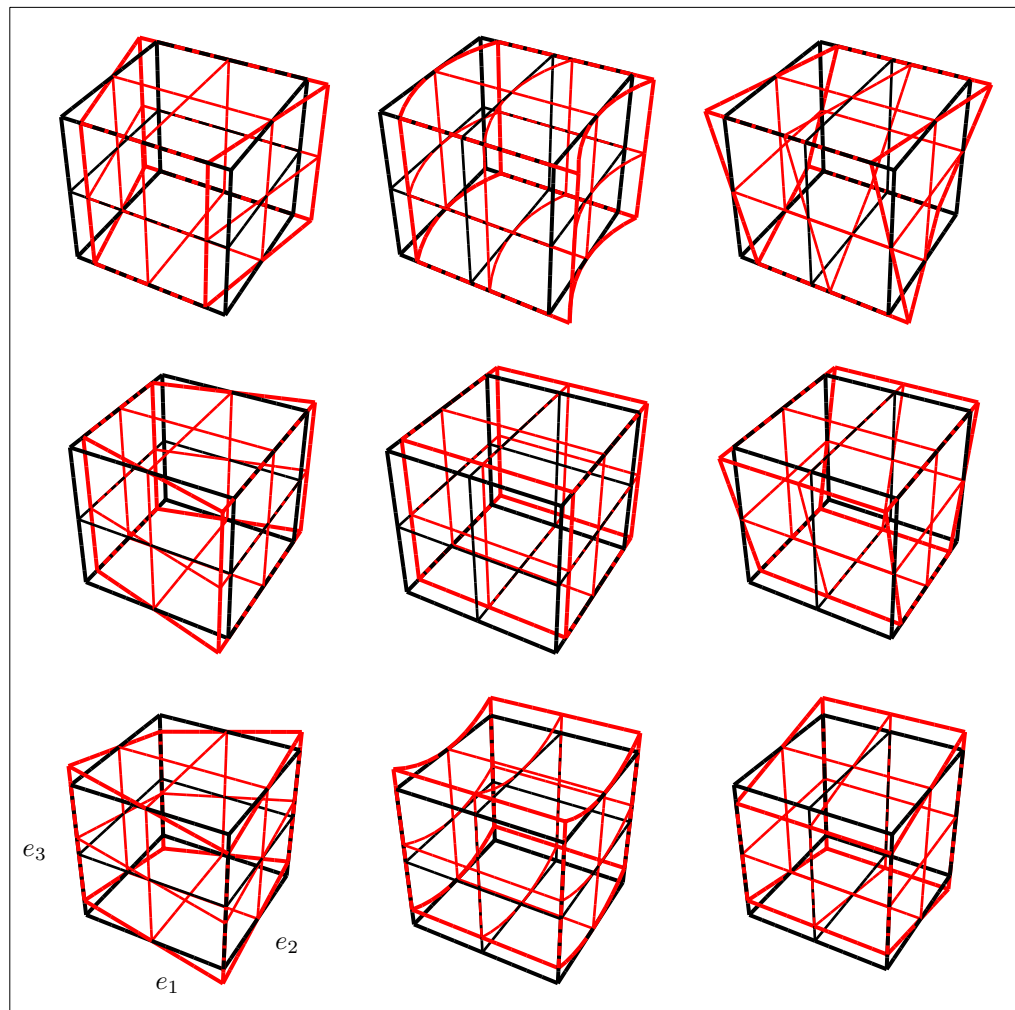
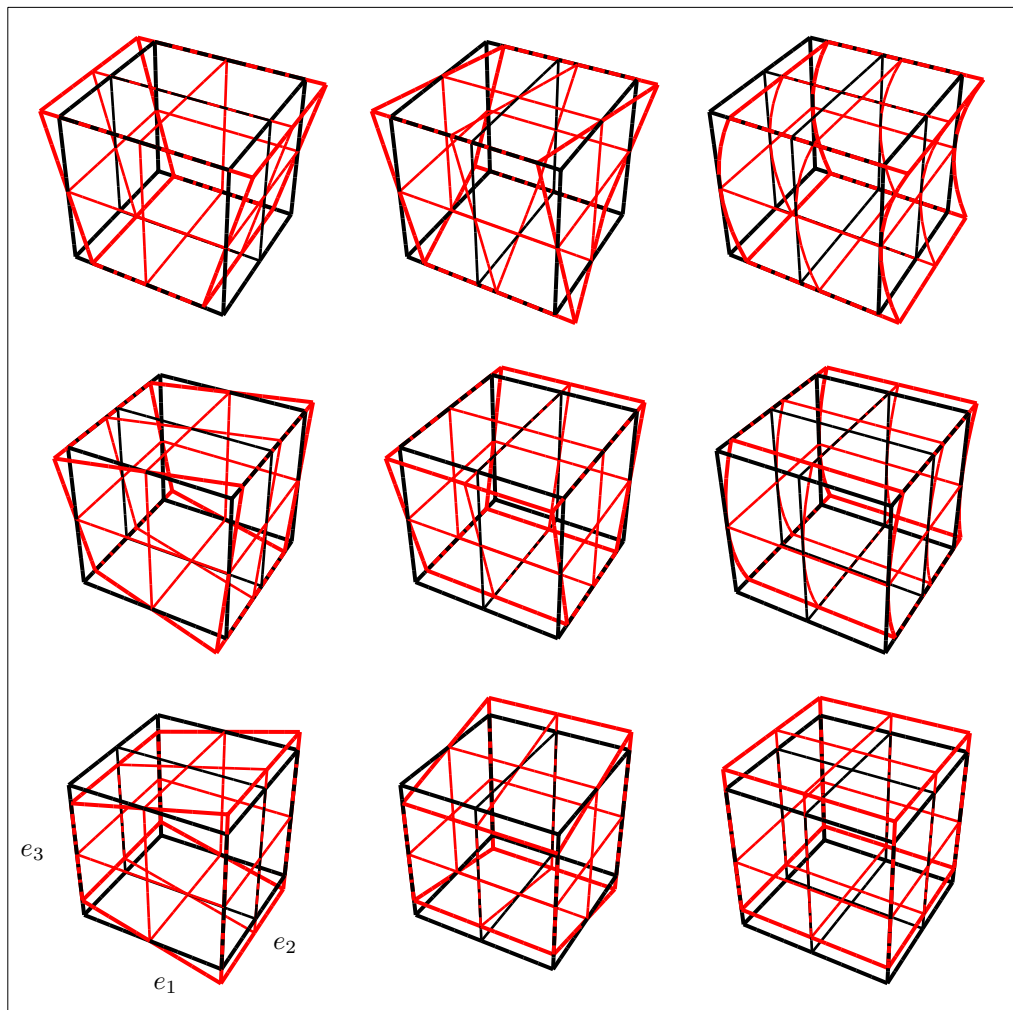


Figure 2: Velocity patterns corresponding to \mathbb{G}_{e_2} .

Figure 3: Velocity patterns corresponding to \mathbb{G}_{e_3} .