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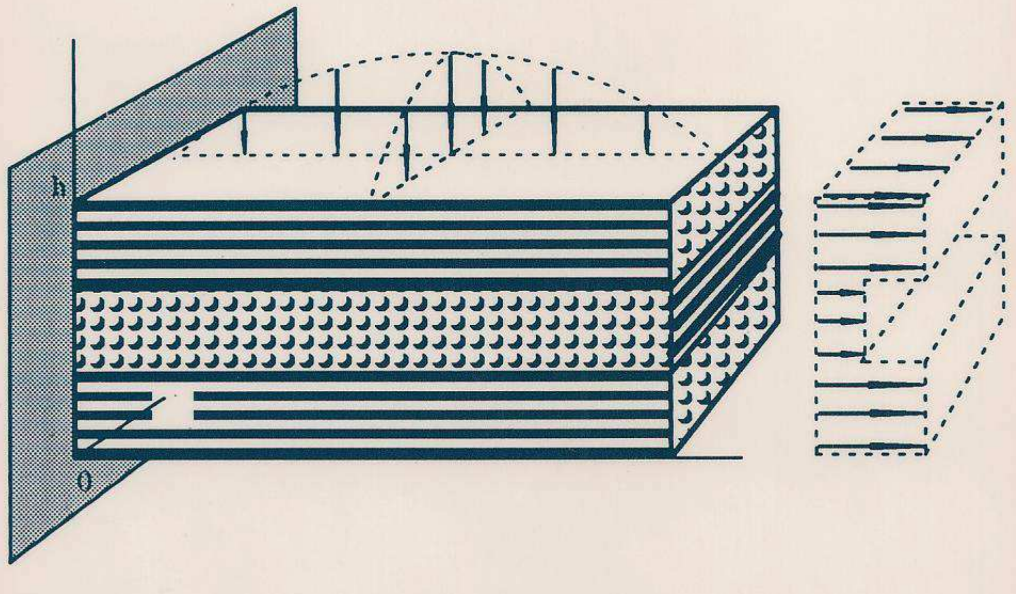

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NON-STANDARD MODELS FOR THIN-WALLED BEAMS WITH A VIEW TO APPLICATIONS

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ABSTRACT

A *direct* theory of a one-dimensional structured continuum is introduced in order to study the postbuckling behaviour of thin-walled beams. A simply supported beam bent by end couples is analyzed showing that, in the case of non symmetric cross sections, lateral buckling gives rise to imperfection sensitivity. Then an axially loaded beam is studied taking also into account the interaction between torsional and flexural buckling. The results obtained prove that in this case imperfection sensitivity, though slighter than in the previous case, arises also for symmetric cross sections.

1. INTRODUCTION

In a previous work (Tatone and Rizzi, 1991) the writers have shown how some relevant aspects of the mechanical behavior of thin-walled beams can be described by means of a one-dimensional continuum. This continuum is endowed with a 'minimal' local structure introduced *ab initio*, that is without any resort to other continuum theories. It can easily be seen that the classical (linear) theories of Wagner (1929), Kappus (1937) and Vlasov (1961), can be recovered by introducing suitable assumptions to the proposed theory and, in particular, by imposing an internal constraint.

The aim of this work is to analyze, making use of the cited theory, two cases of bifurcation of initially straight beams that can be considered exemplary in many aspects.

The first case regards a simply supported beam free to warp and loaded at the end sections by conservative bending couples. The buckling analysis performed will show that, while for the buckling load fair assumptions lead to the classical result of Timoshenko (1910), beams with non symmetric sections can be *imperfection sensitive*, as a consequence of the coupling between torsion, warping and bending.

The second case consists of a cantilever beam, with warping restrained at the clamped end section, compressed by a force applied at the free end, showing both flexural and torsional buckling. By exploiting the exact kinematics of the model, the post buckling analysis has been carried out within the framework of the asymptotic bifurcation theory. The results show that *also* beams with symmetric cross sections can show slight imperfection sensitivity, as a consequence of the interaction of the buckling modes.

It must be pointed out that all the results have been obtained in closed form, despite the complexity of the problems involved. This proves that the proposed model is a simple and effective tool for the investigation of the most relevant aspects of the behavior of thin-walled beams. Such a property, in the opinion of the writers, can be exploited for numerical implementations.

2. ONE-DIMENSIONAL CONTINUUM MODEL

The body we call 'beam' can be thought of as a continuous array of 'sections' along a smooth curve (the beam axis). We will consider only motions of the body in which the beam axis remains a smooth curve while each section can undergo a rigid motion plus a warping. We will give an exact description of the motion, apart from the warping that will be taken into account by means of a single scalar parameter.

To this end we regard a beam as a differentiable manifold $\mathcal{B}$ whose shapes are in $\mathcal{E} \times \mathcal{F}$, where $\mathcal{E}$ is a three-dimensional Euclidean space (its translation space will be denoted by $\mathcal{V}$), $\mathcal{F}$ is the four-dimensional product manifold $SO(\mathcal{V}) \times \mathcal{R}$, $\mathcal{R}$ being the field of real numbers.

We call *reference placement* a function

$$\kappa : \mathcal{B} \to \mathcal{E} \times \mathcal{F}, \tag{1}$$

such that $\kappa : \mathcal{B} \to \kappa(\mathcal{B})$ is a diffeomorphism and $\mathcal{C}_\kappa := p(\kappa(\mathcal{B}))$, denoting by p the natural projection of $\kappa(\mathcal{B})$ into $\mathcal{E}$, is a smooth curve (the beam axis). Then we call $\mathbf{X}(s)$ a parametrization of $\mathcal{C}_\kappa$, $s \in [0, \ell]$ being a curve-length parameter.

With respect to the reference shape $\kappa(\mathcal{B})$, any other shape assumed by the beam in a given motion at a given time t, can be described by a function $\mathbf{x}$ which maps $\mathcal{C}_\kappa$ to $\mathcal{C}_t$ (the beam axis in the new shape) and, for each point of $\mathcal{C}_\kappa$, by a proper rotation $\mathbf{R}$ and a scalar quantity α. By using the parametrization of $\mathcal{C}_\kappa$ we will regard $\mathbf{x}, \mathbf{R}$ and α as functions on $[0, \ell] \times \mathcal{I}$ ($\mathcal{I}$ being the time inteval in which the motion is defined) and will assume them to be sufficiently smooth. In our interpretation, $\mathbf{R}(s,t)$ and $\alpha(s,t)$ describe the rotation and the *warping*, with respect to the reference shape, of the beam cross section corresponding to the point $\mathbf{x}(s,t)$ of the beam axis $\mathcal{C}_t$.

The velocity is defined by

$$\mathbf{v} := \dot{\mathbf{x}}, \quad \mathbf{W} := \dot{\mathbf{R}}\mathbf{R}^T, \quad a := \dot{\alpha}, \tag{2}$$

where a dot denotes differentiation with respect to time, while a suitable definition of strain measures, suggested by a similar definition (Capriz, 1981), is given as follows

$$\mathbf{e} := \mathbf{R}^T\mathbf{x}' - \mathbf{X}', \quad \mathbf{E} := \mathbf{R}^T\mathbf{R}', \quad \beta := \alpha', \tag{3}$$

where a prime denotes differentiation with respect to s.

Corresponding to each point $\mathbf{x}(s,t)$, the contact actions are characterized by a vector $\mathbf{t}$, (the contact force) a skew-symmetric tensor $\mathbf{T}$ (the contact couple) and a scalar quantity ϖ, which in this context will be given the meaning of *bimoment*, while the body actions are characterized by a vector $\mathbf{b}$, a skew-symmetric tensor $\mathbf{B}$ and a scalar quantity η. Such a characterization of the actions results from the following expression of the mechanical power

$$\mathcal{W} := \int_0^\ell (\mathbf{b} \cdot \mathbf{v} + \mathbf{B} \cdot \mathbf{W} + \eta a) dS + [\mathbf{t} \cdot \mathbf{v} + \mathbf{T} \cdot \mathbf{W} + \varpi a]_0^\ell, \tag{4}$$

where use has been made of the following definition for the inner product of skew-symmetric tensors $\mathbf{B} \cdot \mathbf{W} = \frac{1}{2}\mathrm{tr}(\mathbf{B}\mathbf{W}^T)$.

The conditions for the power to be objective are expressed by the following balance equations

$$\mathbf{t}' + \mathbf{b} = \mathbf{0},$$
$$\mathbf{T}' + \mathbf{B} - \mathbf{x}' \wedge \mathbf{t} = \mathbf{0}. \tag{5}$$

Substituting these relations into the expression (5) we obtain the stress power formula

$$\mathcal{W} = \int_0^\ell (\mathbf{s} \cdot \dot{\mathbf{e}} + \mathbf{S} \cdot \dot{\mathbf{E}} + \pi a + \varpi a')ds, \tag{6}$$

where $\mathbf{s} := \mathbf{R}^T\mathbf{t}$, $\mathbf{S} := \mathbf{R}^T\mathbf{T}\mathbf{R}$, $\dot{\mathbf{e}} = \mathbf{v}' - \mathbf{W}\mathbf{x}'$, $\dot{\mathbf{E}} = \mathbf{W}'$, and

$$\pi = \eta + \varpi'. \tag{7}$$

It is worth noting that this expression is a special case of (4.17) in (Green and Laws, 1966).

By choosing an orthonormal basis $(\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3)$ of $\mathcal{V}$, let us consider a beam shape whose axis is straight and the sections are orthogonal to it. If we assume $\mathbf{D}_1 = \mathbf{X}'$, we can introduce the components of $\mathbf{s}, \mathbf{S}, \mathbf{E}$, which will be used in the sequel

$$\begin{aligned}
\mathbf{s} &= Q_1\mathbf{D}_1 + Q_2\mathbf{D}_2 + Q_3\mathbf{D}_3,\\
\mathbf{S} &= M_1\mathbf{D}_2 \wedge \mathbf{D}_3 + M_2\mathbf{D}_3 \wedge \mathbf{D}_1 + M_3\mathbf{D}_1 \wedge \mathbf{D}_2,\\
\mathbf{E} &= \mu_1\mathbf{D}_2 \wedge \mathbf{D}_3 + \mu_2\mathbf{D}_3 \wedge \mathbf{D}_1 + \mu_3\mathbf{D}_1 \wedge \mathbf{D}_2.
\end{aligned} \tag{8}$$

It also turns out to be useful to define the vector $\mathbf{u} := \mathbf{x} - \mathbf{X}$ and its components v_i with respect to the basis

$$\mathbf{d}_i := \mathbf{R}\mathbf{D}_i, \qquad i = 1, 2, 3. \tag{9}$$

Assuming that the material is elastic and homogeneous, it can be proved that the most general constitutive relations for the contact actions are of the form

$$\mathcal{S} = \hat{\mathcal{S}}(\mathbf{e}, \mathbf{E}, \alpha, \beta), \tag{10}$$

where $\mathcal{S}$ stands for one of the $\mathbf{s}, \mathbf{S}, \pi, \varpi$.

If we want to cast into our model the property that, as shown by experimental observations, the warping is an effect of the other strain measures, it seems reasonable to assume an internal constraint of the form

$$\alpha = \hat{\alpha}(\mathbf{e}, \mathbf{E}), \quad \hat{\alpha}(\mathbf{o}, \mathbf{O}) = 0, \tag{11}$$

that, for any objective scalar function $\hat{\alpha}$, turns out to be objective, any change of frame leaving both $\mathbf{e}$ and $\mathbf{E}$ unchanged. By limiting ourselves to consider a linear constraint between warping and torsion, we put (11) in the form

$$\alpha = k\mu_1, \quad (k \in \mathcal{R}) \qquad \Rightarrow \qquad \beta = k\mu_1'. \tag{12}$$

Such a choice, despite its simplicity, proves to be effective for the cases we are interested in studying. Incidentally, it must be said that such a constraint is the one incorporated in the Vlasov's model. Further, we will assume the material to be shear–indeformable and require

$$\mathbf{e} = \epsilon\mathbf{X}'. \tag{13}$$

By the axiom of determinism for constrained materials—which states that the stress is determined by the motion only to within an arbitrary additive part which does no work in any motion compatible with the constraint—the indetermined part of the stress corresponding to the above constraints, which will be denoted by a subscript v, must satisfy the following scalar relations

$$Q_{1v} = 0,\ M_{1v} = -k\pi_v,\ M_{2v} = 0,\ M_{3v} = 0,\ \varpi_v = 0, \tag{14}$$

while Q_{2v}, Q_{3v}, as well as π_v, can take any value.

The constitutive functions, consistently with all the material assumptions made till now, will be assumed to be

$$\begin{aligned}
&Q_1 = a\epsilon + \frac{1}{2}p\mu_1^2, \quad Q_2 = Q_{2v}, \quad Q_3 = Q_{3v},\\
&M_1 = (c + q_2\mu_1 + q_3\mu_3 + p\epsilon + r\beta)\mu_1 + M_{1v},\\
&M_2 = b_2\mu_2 + \frac{1}{2}q_2\mu_1^2, \quad M_3 = b_3\mu_3 + \frac{1}{2}q_3\mu_1^2,\\
&\varpi = h\beta + \frac{1}{2}r\mu_1^2, \quad \pi = \pi_v.
\end{aligned} \tag{15}$$

If we also assume $\eta = 0$, from (7) and $(15)_8$ follows that $\pi_v = \varpi'$. Then, by virtue of (12), the relations $(15)_4, (15)_7$ and $(15)_8$ finally become

$$\begin{aligned}
M_1 &= (c + q_2\mu_1 + q_3\mu_3 + p\epsilon)\mu_1 - hk^2\mu_1'',\\
\varpi &= hk\mu_1' + \frac{1}{2}r\mu_1^2,\\
\pi &= hk\mu_1'' + r\mu_1\mu_1'.
\end{aligned} \tag{16}$$

By transforming the strain-displacement relations (3) and the balance equation (5) into their scalar form, one obtains, by means of (15) and (16), a set of nonlinear field equations that, supplied with the boundary conditions relative to each case examined, gives rise to a nonlinear boundary value problem.

3. BIFURCATION ANALYSIS

The nonlinear boundary value problems we are going to face, are of the form

$$F(u, \lambda) = 0. \tag{17}$$

A solution (u, λ) consists of a set of functions describing a shape, and a scalar parameter λ affecting the applied actions, such that the balance equations (5), together with the appropriate boundary conditions, are satisfied.

The graph of the solutions (u, λ) can be thought of as being made of regular branches. Usually a branch $(u^f(\tau), \lambda^f(\tau))$ crossing the point $(0, 0)$, where $u = 0$ denotes the reference shape, is easily computed and one is interested in finding a bifurcation point on it. Denoting by $(u^b(t), \lambda^b(t))$ an intersecting branch, assume the parametrization of the two branches be such that $u^b(0) = u^f(\tau_c)$, $\lambda^b(0) = \lambda^f(\tau_c) = \lambda_c$, and define the new set of functions

$$v := u^b - u^f, \tag{18}$$

made up with the differences of the corresponding functions in u^b and u^f.

As v inherits the parametrization from the intersecting branches, by assuming there exists a diffeomorphism relating the parameters t and τ, we can give the following series expansions near $t = 0$

$$\begin{aligned}
v_t &= \tilde{v}_c t + \frac{1}{2}\tilde{\tilde{v}}_c t^2 + o(t^2),\\
\lambda_t &= \lambda_c + \tilde{\lambda}_c t + \frac{1}{2}\tilde{\tilde{\lambda}}_c t^2 + o(t^2),
\end{aligned} \tag{19}$$

where the subscript c denotes values at $t = 0$.

By using (18) and (19), the problem (17) can be recast in a sequence of linear systems of differential equations with appropriate boundary conditions. The first system results in an *eigenvalue problem* whose solution gives λ_c and $\tilde{v}_c$ while $\tilde{\lambda}_c$ is obtained by imposing a solvability condition to the second order system whose solution leads to $\tilde{\tilde{v}}_c$ and so on. As the aim of this section is just to introduce the symbols used in the next sections, the interested reader is referred to Budiansky (1974).

4. SIMPLY SUPPORTED BEAM BENT BY TERMINAL COUPLES

In this section we will perform a bifurcation analysis of an initially straight beam of length ℓ, simply supported at both ends, where the warping is free and the axial rotation is restrained, loaded by two opposite conservative bending couples.

Remembering that dead couples are not conservative, we will recover such a property if each couple is obtained by attaching two (opposite) dead forces to two (different) points of the same end section. Denoting by d the distance between the lines of action of the two forces and by f their intensity, we set

$$\mathbf{T}(\ell) = \lambda \mathbf{D}_3 \wedge \mathbf{d}_1, \qquad \mathbf{T}(0) = -\mathbf{T}(\ell), \qquad \text{with } \lambda := fd. \tag{20}$$

In order to simplify the analysis, we assume the beam to be flexural indeformable in the plane spanned by the applied forces in the reference shape, by adding the following internal constraint

$$\mu_3 = 0. \tag{21}$$

The appropriate boundary conditions for the case in hand, are

$$\begin{aligned} &\mathbf{u} = \mathbf{0}, \quad M_1 = M_3 = \theta_1 = \varpi = 0, \quad \text{at } s = 0, \\ &v_2 = v_3 = Q_1 = M_1 = M_3 = \theta_1 = \varpi = 0, \quad \text{at } s = \ell. \end{aligned} \tag{22}$$

It can be easily seen that the stated boundary value problem, due to the assumed constitutive functions, has at least the following solution

$$\begin{aligned} &\mathbf{u} = \mathbf{0}, \quad \mathbf{R} = \mathbf{I}, \quad \alpha = 0, \\ &\mathbf{s} = \mathbf{0}, \quad \varpi = 0, \quad \mathbf{S} = \lambda \mathbf{D}_3 \wedge \mathbf{D}_1, \end{aligned} \tag{23}$$

which means that the reference configuration is balanced for any value of λ.

In order to investigate the possible bifurcations from the given solution, we linearize the field equations together with the boundary conditions near the above solution and arrive at the following equations

$$\begin{aligned} &b_2 \tilde{v}_3'''' - \lambda \tilde{\theta}_1'' = 0, \\ &k^2 h \tilde{\theta}_1'''' - c \tilde{\theta}_1'' - \lambda \tilde{v}_3'' = 0, \end{aligned} \tag{24}$$

with the boundary conditions

$$\begin{aligned} &\tilde{v}_3 = \tilde{v}_3'' = 0 \quad \text{at } s = 0, \text{ and } s = \ell, \\ &\tilde{\theta}_1 = \tilde{\theta}_1'' = 0 \quad \text{at } s = 0, \text{ and } s = \ell, \end{aligned} \tag{25}$$

where ˜ has the meaning introduced in section 3.

By solving the eigenvalue problem (24), (25), one obtains the buckling load

$$\lambda_c = \frac{\pi}{\ell} \sqrt{b_2 c} \sqrt{1 + \frac{k^2 h}{c} \left(\frac{\pi}{\ell}\right)^2}, \tag{26}$$

with the associate buckling mode

$$\tilde{v}_3 = -\frac{\ell}{\pi}\sqrt{\frac{c}{b_2}}\sqrt{1 + \frac{k^2 h}{c}\left(\frac{\pi}{\ell}\right)^2}\sin\left(\frac{\pi s}{\ell}\right), \qquad \tilde{\theta}_1 = \sin\left(\frac{\pi s}{\ell}\right). \tag{27}$$

normalized by setting $\tilde{\theta}_1\left(\frac{\ell}{2}\right) = 1$.

Note that, despite the approaches being completely different, the (26) can recover the classical result obtained by Timoshenko (1910) (see also Vlasov, 1961) by assuming: $b_2 = EI, c = GJ$ (the Saint-Venant flexural and torsional rigidities) and $hk^2 = EJ_\omega$ (called by Vlasov the 'sectorial warping rigidity').

The increment of the load parameter corresponding to the solution of the eigenvalue problem, is given by

$$\tilde{\lambda}_c = \frac{\pi}{\ell^2}\left(q_2 + \frac{\pi}{\ell}\frac{^2 b_2 k r}{\lambda_c}\right), \tag{28}$$

showing that the beam considered can have an asymmetric bifurcation, that is, it *can be imperfection sensitive*. It is worth noting, however, that such a behavior depends on the coupling between torsion, bending couple and bimoment and that it can be shown (Tatone, 1992) that such a coupling may occur *only* if the beam cross section has no axis of symmetry.

Note that $\tilde{\lambda}_c \neq 0$ even if we assume, as usually done, a linear dependence between the bimoment and the torsion, due to the constant q_2. The terms containing q_2 in the constitutive functions, are responsible for the coupling between flexure and torsion and play the same role as the terms containing the coefficient p. In fact these terms can be given the same physical meaning as the 'unitary warping' of Wagner (1929).

5. AXIALLY COMPRESSED CANTILEVER BEAM

We consider next a straight beam of length ℓ clamped at one end, where the warping is constrained to be zero, and compressed by an axial force λ applied at the free end. Thus the boundary conditions are

$$\begin{aligned} \mathbf{u} = \mathbf{0}, \quad \mathbf{R} = \mathbf{I}, \quad \alpha = 0, \quad \text{at } s = 0, \\ \mathbf{s} = -\lambda \mathbf{D}_1, \quad \mathbf{S} = \mathbf{0}, \quad \varpi = 0, \quad \text{at } s = \ell. \end{aligned} \tag{29}$$

In the case of double symmetric cross sections, ($q_2 = q_3 = r = 0$ in the (15), (16)), the resulting boundary value problem can easily be seen to have the following solution

$$\begin{aligned} v_1 = -\lambda/a, \quad v_2 = 0, \quad v_3 = 0, \quad \alpha = 0, \quad \mathbf{R} = \mathbf{I}, \\ Q_1 = -\lambda, \quad Q_2 = 0, \quad Q_3 = 0, \quad \mathbf{S} = 0, \quad \varpi = 0. \end{aligned} \tag{30}$$

It can be shown that the previous solution has bifurcating branches that usually correspond to an axial-torsional behavior for the beam or to an axial-flexural behavior. By selecting from each of the two classes the branch corresponding to the smallest critical load, the following results are obtained.

The flexural mode is characterized by

$$\begin{aligned} \lambda_c &= \frac{a}{2}\left(1 - \sqrt{1 - \frac{\pi^2}{\ell^2}\frac{b_3}{a}}\right), \\ \tilde{\lambda}_c &= 0, \\ \frac{\tilde{\tilde{\lambda}}_c}{\lambda_c} &= \frac{\pi^2 a^2}{16\ell^2}\frac{a - 4\lambda_c}{(a-\lambda)^2(a - 2\lambda_c)}. \end{aligned} \tag{31}$$

There is nothing interesting about it but the fact that λ_c, in order to exists, has to be such that $0 \leq \lambda_c \leq a/2$. As a result, the ratio $\tilde{\tilde{\lambda}}_c/\lambda_c$ becomes negative whenever $\lambda_c > a/4$. The bifurcated branch, up to the second order, is given by

$$\begin{aligned} \tilde{u}_1 = 0, \qquad & \tilde{u}_2 = \cos\frac{\pi s}{2\ell} - 1, \\ \tilde{\tilde{u}}_2 = 0, \qquad & \tilde{\tilde{u}}_1 = \frac{\pi}{2\ell}\frac{a(a-2\lambda)}{4(a-\lambda)^2}(\sin\frac{\pi s}{\ell} - \frac{\pi s}{\ell}), \end{aligned} \tag{32}$$

being $u_3 = \theta_1 = 0$. Here the normalization condition $\tilde{u}_2(\ell) = -1$ has been assumed.

The torsional bifurcation mode is characterized by

$$\begin{aligned} \lambda_c &= \frac{a}{p}\left(c + \frac{\pi^2}{4}\frac{hk^2}{\ell^2}\right) = \frac{a(4c\ell^2 + \pi^2 hk^2)}{4\ell^2 p}, \qquad \tilde{\lambda}_c = 0, \\ \frac{\tilde{\tilde{\lambda}}_c}{\lambda_c} &= -\frac{1}{\lambda_c}\frac{3\pi^2 p}{16\ell^2} = -\frac{3\pi^2 p^2}{4a(4c\ell^2 + \pi^2 hk^2)}, \end{aligned} \tag{33}$$

and the associate bifurcated solution up to the second order is

$$\begin{aligned} \tilde{u}_1 = 0, \qquad & \tilde{\theta}_1 = \cos\frac{\pi s}{2\ell} - 1, \\ \tilde{\tilde{u}}_1 = \frac{\pi p}{8a\ell}\left(\sin\frac{\pi s}{\ell} - \frac{\pi s}{\ell}\right), \qquad & \tilde{\tilde{\theta}}_1 = 0, \end{aligned} \tag{34}$$

being $u_2 = u_3 = 0$. The normalization has been chosen so that $\tilde{\theta}_1(\ell) = -1$. Note that such a bifurcation can occur only if $p > c + \pi^2 hk^2/4\ell^2$ and the value of $\tilde{\tilde{\lambda}}_c/\lambda_c$ is negative no matter the values of the constants of the constitutive functions are. This means that, despite what can be guessed by looking only at the results of the linearized analysis, torsional buckling due to an axial force is *imperfection sensitive*.

If the constitutive parameters and the length of the beam are such that the buckling loads in (30) and (31) take the same value, the two buckling modes occur simultaneously and the first order part of the bifurcation branches can be put in the form

$$\tilde{u} = \nu_1 \tilde{u}_I + \nu_2 \tilde{u}_{II}, \tag{35}$$

where $\tilde{u}_I$ and $\tilde{u}_{II}$ stand for the torsional mode and the flexural mode respectively.

The resulting mode interaction problem has been solved by following the procedure outlined by Pignataro and Rizzi (1982). It turns out that $\tilde{\lambda}_c = 0$ while, besides the torsional and flexural buckling modes occurring separately, two more solutions appear. Assuming that the coefficients ν_1, ν_2 are such that ${\nu_1}^2 + {\nu_2}^2 = 1$, the expressions for $\tilde{\tilde{\lambda}}_c$ and ${\nu_1}^2$ can be easily obtained by solving the following two equations

$$\begin{aligned} \frac{\tilde{\tilde{\lambda}}_c}{\lambda_c} = &-\frac{3p{\omega_1}^2{\nu_1}^2}{4\lambda_c} - \left(\frac{16\tan(\ell\omega_3){\omega_1}^2(\omega_1^2 - \omega_3^2)^2}{\ell p\,(a-\lambda_c)\,{\omega_3}^3(4\omega_1^2 - \omega_3^2)^2}\right. \\ &\left. + \frac{2\,(a-\lambda_c)\,(\omega_3^2 - \omega_1^2)\,(\omega_3^2 + 8\omega_1^2) - 3{\omega_1}^2{\omega_3}^2 p\,(4{\omega_1}^2 - {\omega_3}^2)}{4p(a-\lambda_c)^2(4\omega_1^2 - \omega_3^2)\omega_3{}^2}\right) a^2\ell^2\,(1 - {\nu_1}^2) \\ \\ &\frac{a^2\ell^2{\omega_1}^2(a - 4\lambda_c)}{8p(a-\lambda_c)^3}(1 - \nu_1^2) + \left(\frac{8\tan(\ell\omega_3){\omega_1}^2(\omega_1^2 - \omega_3^2)^2}{\ell p\,{\omega_3}^3(4\omega_1^2 - \omega_3^2)^2}\right. \\ &\left. + \frac{2\,(a-\lambda_c)(\omega_3^2 - \omega_1^2)\,(\omega_3^2 + 8\omega_1^2) - 3{\omega_1}^2{\omega_3}^2 p(4{\omega_1}^2 - {\omega_3}^2)}{8p(a-\lambda_c)\,(4\omega_1^2 - \omega_3^2)\omega_3{}^2}\right){\nu_1}^2 \\ &- \frac{(a - 2\lambda_c)}{2p\,(a-\lambda_c)}\frac{\tilde{\tilde{\lambda}}_c}{\lambda_c} = 0 \end{aligned} \tag{36}$$

where $\omega_1 = \pi/(2\ell)$ and $\omega_3 = \left|\sqrt{\lambda_c(a-\lambda_c)/(ab_2)}\right|$.

It must be noted that, depending on the values of ω_1 and ω_3, the value of $\tilde{\tilde{\lambda}}_c/\lambda_c$ given by the expression (36) can become negative and hence the beam can be imperfection sensitive.

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