Continua with Microstructure in Finite Elasticity for Masonry Modelling

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1 Introduction

Let us consider a masonry wall. We can think of it as a system of rigid blocks arranged in a lattice whose nodes are the centers of the blocks.

The position of each block is given by the position of its center and by its *attitude* (which could be described by a vector "attached" to each block, or two vectors, depending on the dimension of the space).

If the blocks are very small, compared with the extention of the wall, it could be conveniet to model the wall as a continuum. What continuum model to choose? A continuum model should retain information about both the position of the center and the attitude of the block. So it is natural to choose a continuum with structure. It's not a new idea (such a model has been used for granular material as well) but we want to address some questions regarding the constitutive relations.

2 Discrete Model

Let us look at the descrete model. We describe the external mechanical actions on each rigid block by a force applied at its center and a couple. The interactions are described in this way: the mechanical action exerted by the block $\mathcal{B}$ on the block $\mathcal{A}$ is defined by a force applied at a given point and by a couple. This point has no particular meaning, it's just a mean for describing the interaction.

The action exerted by the block $\mathcal{B}$ on the block $\mathcal{A}$ can be assumed to be determined only by the relative positions of the two blocks (we can state a *Principle of Determinism* and a *Principle of Local Action*. We could also consider the histories of the relative positions, instead of their present values.

We can obtain reduced forms of these constitutive relations through the *Principle of Materisl Frame Indifference*. A change of frame is defined by this relation: ...

A different observer see any vector rotated by Q, so any rotation is composed with Q.

By substituting the new values we get an equation that should be satisfied by any orthogonal tensor Q. If we choose $Q = R^\mathsf{T}$ we get a new form of the constitutive relation. Viceversa, any relation of this form satisfies the *Principle of Material Frame Indifference*.

A useful version of this reduced form can be obtained by defining two new objects, which we can see as kind of *strain measures*. Then we can extend this form to histories of strain measures. The point is: whatever the function $\hat{t}$ we choose, the *Principle of Material Frame Indifference* is satisfied. Note that there is no assumptions on the regularity of $\hat{t}$.

3 Continuum Model

Let us look now at the continuum model. A placement assigns to each substantial point a point in a Euclidean space and a rotation. Once a reference placement has been chosen, a transplacement is described by two functions x and R, which transform the reference place and rotation into the present ones.

The velocity is defined by a vector field v and a tensor field V. The mechanical power has this form.

Here b is the density of body forces, t the density of contact forces, B the density of body couples and C the density of contact couples.

The balance equations can be given as an axiom or derived as conditions for the power to be frame indifferent. It is then possible to define a stress tensor T and a couple stress tensor $\mathbb{C}$, and hence obtain the local form of the balance equations.

By substituting the balance equations into the expressions of the power we get an expression in term of stress and couple stress tensors.

We define also the two strain measures U and $\mathbb{U}$. Note that the polar decomposition of U gives the *pure stretch* composed with the pull-back of the macro-rotation through the micro-rotation R. The tensor $\mathbb{U}$ compares the covariant derivatives of a vector field in the reference place with the covariant derivative of the same field transported by R to the present shape.

A general constitutive relation for an elastic material is characterized by a response function depending on the deformation gradients F and $\mathbb{F}$.

By the *Principle of Material frame Indifference* it is possible to get a reduced form for an elastic material. This form can be extended to the more general case of a simple material by considering histories F^t and $\mathbb{U}^t$.

4 Identification of the Constitutive Relations for the Continuum Model

We relate the two models in this way: assume that the wall, in the reference shape has a modular structure; then assume that the motion of a module is related to a neighborhood of a substantial point of the continuous model through this relations, and assume that the average power of the module equals the power density of the continuum at the corresponding point.

The relations between the velocities allow us to compare the expressions of the two power and identify the stress and couple stress tensors.

This identification produces two constitutive relations for T and $\mathbb{C}$. The question is:

do these relations satisfy the Principle of Material Frame Indifference and, if they do, what kind of material do they define?

If we assume that the placement of the models are related in this way: .... then

- by using the definitions of the strain measures for the continuum,
- by substituting the reduced forms of the constitutive relations for the interactions between blocks,

we get the corresponding reduced forms for the continuum model, either for an elastic material or for a simple material in general, according to the interaction response functions used.

This is important because we are free to choose, at least in principle, any response function for the interactions between blocks as long as it has the form given by the reduced constitutive relation derived above.