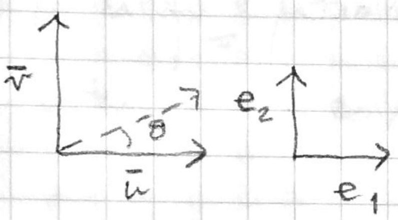
$(7\text{-}8)_2$ Monday [2014-03-03]

16-18 A1.3

Matrix representation of the rotation tensor and coordinate representation of a rigid deformation

$$\phi(\bar{p}_1) = \phi(\bar{p}_0) + R\,(\bar{p}_1 - \bar{p}_0)$$

↑ vector



basis vectors

$$R\bar{u} = \cos\vartheta\,\bar{u} + \sin\vartheta\,\bar{v}$$

$$R\bar{v} = -\sin\vartheta\,\bar{u} + \cos\vartheta\,\bar{v}$$

$$R\bar{u}\cdot R\bar{u} = \cos^2\vartheta\,\bar{u}\cdot\bar{u} + \sin^2\vartheta\,\bar{v}\cdot\bar{v} + 2\cos\vartheta\sin\vartheta\,\bar{u}\cdot\bar{v}$$

if $\bar{u}\cdot\bar{v} = 0$
$\bar{u}\cdot\bar{u} = \bar{v}\cdot\bar{v}$ $\Rightarrow$ $R\bar{u}\cdot R\bar{u} = \bar{u}\cdot\bar{u} = \bar{v}\cdot\bar{v}$

$\Downarrow$ $R\bar{v}\cdot R\bar{v} = \bar{v}\cdot\bar{v} = \bar{u}\cdot\bar{u}$

$$R\bar{u}\cdot R\bar{v} = -\sin\vartheta\cos\vartheta\,\bar{u}\cdot\bar{u} + \cos^2\vartheta\,\bar{u}\cdot\bar{v} - \sin^2\vartheta\,\bar{v}\cdot\bar{u}$$
$$+ \sin\vartheta\cos\vartheta\,\bar{v}\cdot\bar{v} = 0$$

It is convenient to choose an orthonormal basis

Orthonormal basis

$R e_1 = \cos\vartheta\, e_1 + \sin\vartheta\, e_2$

$R e_2 = -\sin\vartheta\, e_1 + \cos\vartheta\, e_2$

matrix of R $\qquad \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}$

we arrange components in columns

$R e_1 \cdot e_1 = \cos\vartheta$

$R e_1 \cdot e_2 = \sin\vartheta$

...

$R^T e_1 \cdot e_1 = e_1 \cdot R e_1 = R e_1 \cdot e_1 = \cos\vartheta$

$R^T e_1 \cdot e_2 = R e_2 \cdot e_1 = -\sin\vartheta$

matrix of R^T

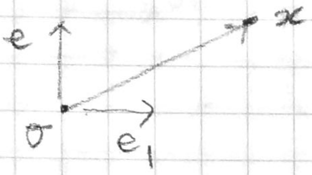
In general

$A e_1 = a_{11} e_1 + a_{21} e_2$

$A e_2 = a_{12} e_1 + a_{22} e_2$

$$[A] = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Coordinates (cartesian)



$$(x - o) = x_1 e_1 + x_2 e_2$$

$$p_A = p_0 + R(\bar{p}_A - \bar{p}_0)$$

$$\begin{pmatrix} x_{1A} \\ x_{2A} \end{pmatrix} = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} + \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix} \begin{pmatrix} \bar{x}_{1A} - \bar{x}_{10} \\ \bar{x}_{2A} - \bar{x}_{20} \end{pmatrix}$$

extending rotations to 3D

$$R a = \lambda a \qquad \|Ra\| = \|a\|$$

$$\Rightarrow |\lambda| = 1 \qquad \lambda = \alpha \pm \beta i \qquad \lambda = \pm 1$$

[2014-03-04] Tuesday $(9-10)_2$

9:00 – 11:00 (9:00 - 10:45, no breaks)

$$R a = \lambda a$$

$$R a \cdot R a = a \cdot a \Rightarrow \lambda^2 = 1$$

extending $\mathcal{V}$ to the complex field $\mathbb{C}$

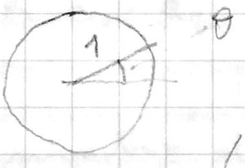
$$u \cdot v = \overline{v \cdot u}$$

$$(\alpha u) \cdot v = \alpha (u \cdot v)$$

$$\Rightarrow \quad u \cdot (\beta v) = \overline{\beta v \cdot u} = \bar{\beta}\, \overline{v \cdot u} = \bar{\beta}\, u \cdot v$$

$$R a \cdot R a = \lambda \bar{\lambda} = |\lambda|^2$$

$$\lambda_{1,2} = \alpha \pm i \beta \qquad \alpha^2 + \beta^2 = 1 \qquad \lambda_3 = \pm 1$$

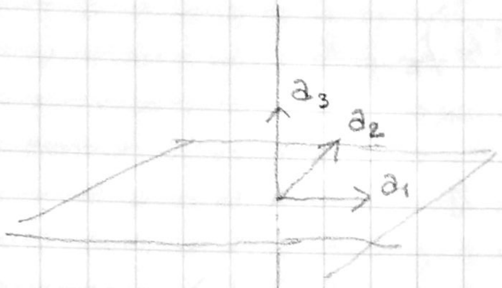


$$\alpha = \cos\theta$$
$$\beta = \sin\theta$$

$$\begin{pmatrix} \cos\theta + i \sin\theta & 0 & 0 \\ 0 & \cos\theta - i \sin\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

orthogonal to each other



$$R a_3 = a_3$$

$$R a_1 \cdot a_3 = 0$$

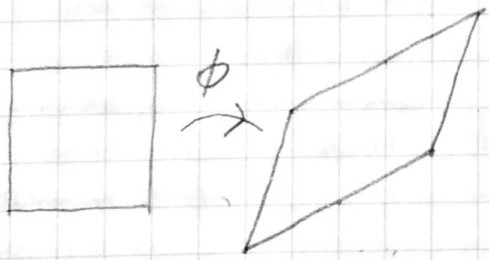
$$R a_2 \cdot a_2 = 0$$

We generalize rigid deformations to <u>affine deformations</u>

$$\phi(\bar{p}_A) = \phi(\bar{p}_0) + F(\bar{p}_A - \bar{p}_0)$$

where F is a non singular tensor $\left(F(\bar{u}) = 0 \Leftrightarrow u = 0\right)$

- straight lines are deformed into straight lines
- parallel segments are deformed into parallel segments
- a square is deformed into a parallelogram
- a cube is deformed into a parallelepiped
- a circle is deformed into an ellipse
- a sphere is deformed into an ellipsoid

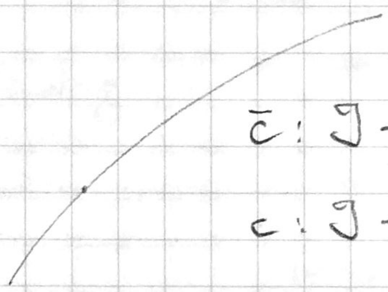


An <u>affine</u> deformation is also called a <u>homogeneous</u> deformation

[2014-03-05] Wednesday $(11-12)_2$

11:00 - 13:00 A1.3

Tangent vectors



$\bar{c} : \mathcal{I} \to \mathcal{E}$ curve on $\bar{\mathcal{R}} = \operatorname{im} \bar{p}$

$c : \mathcal{I} \to \mathcal{E}$ curve on $\mathcal{R} = \operatorname{im} p$

Parameterization of the <u>reference shape</u>

$$\kappa(s_1, s_2, s_3) \in \bar{\mathcal{R}}$$

in general it is a piecewise parameterization

so κ^{-1} does exist locally in general

$$\kappa : \mathcal{D} \to \bar{\mathcal{R}} \subset \mathcal{E}$$

parameterization domain

(s_1, s_2, s_3) are also called "material coordinates"

$$\phi : \bar{\mathcal{R}} \to \mathcal{R} \subset \mathcal{E}$$

$$\uparrow \bar{p}$$

$$\mathcal{B}$$

$$\begin{array}{c} \phi : \bar{\mathcal{R}} \to \mathcal{R} \\ \uparrow \kappa \\ \mathcal{D} \end{array}$$

$$\phi_\kappa := \phi \circ \kappa : \mathcal{D} \to \mathcal{R}$$

$$\phi(\kappa(s_1, s_2, s_3)) \in \mathcal{R}$$

parameterization of the current shape (convected coordinates)

By using a coordinate system

$$\phi_\kappa(s_1, s_2, s_3) = o + \phi_{\kappa 1}(s_1, s_2, s_3)\, e_1$$

$$= \phi(\kappa(s_1, s_2, s_3)) = \qquad + \phi_{\kappa 2}(s_1, s_2, s_3)\, e_2$$

$$+ \phi_{\kappa 3}(s_1, s_2, s_3)\, e_3$$

$$\bar{c}_1(h) = \underbrace{\bar{p}_0}_{x\, \in \bar{\mathcal{R}}} + h\, e_1 = \kappa(s_1, s_2) + h\, e_1$$

Let us define κ, for example, by using coordinates in $\mathcal{E}$

$$\kappa(s_1, s_2, s_3) = o + s_1 e_1 + s_2 e_2 + s_3 e_3$$

Thus

$$\bar{c}_1(h) = o + s_1 e_1 + \cdots + h\, e_1 = \kappa(s_1 + h, s_2, s_3)$$

$$c_1(h) = \phi(\bar{c}_1(h)) = \phi(\kappa(s_1+h, s_2, s_3))$$

$$c_1(h) = \phi_\kappa(s_1+h, s_2, s_3)$$

$$c_1(h) = o + \phi_{\kappa 1}(s_1+h, s_2, s_3)\, e_1$$
$$+ \phi_{\kappa 2}(s_1+h, s_2, s_3)\, e_2$$
$$+ \phi_{\kappa 3}(s_1+h, s_2, s_3)\, e_2$$

$$c_1'(0) = \lim_{h\to 0} \frac{1}{h}\left(c_1(h) - c_1(0)\right) = \ldots$$

$$= \partial_1 \phi_{\kappa 1}\, e_1 + \partial_1 \phi_{\kappa 2}\, e_2 + \partial_1 \phi_{\kappa 3}\, e_3$$

[...]

$$c_2'(0) = \ldots$$

$$c_3'(0) = \ldots$$

$$\bar{c}(h) = \kappa(s_1 + h\alpha_1, s_2 + h\alpha_2, s_3 + h\alpha_3)$$

$$c(h) = o + \phi_{\kappa 1}(s_1 + h\alpha_1, s_2 + h\alpha_2, s_3 + h\alpha_3)\, e_1$$
$$+ \ldots$$

$\Rightarrow$ there exists a linear transformation (tensor)

$F : \mathcal{V} \to \mathcal{V}$, called the <u>deformation gradient</u>, such that

$$F\bar{c}_1' = c_1' \quad , \quad F\bar{c}_2' = c_2' \quad , \quad F(\alpha_1 \bar{c}_1' + \alpha_2 \bar{c}_2') = \alpha_1 c_1' + \alpha_2 c_2'$$