$(13-14)_3$ Monday [2014-03-10]

16:00 → 18:00

A1.3

Local deformation

$$c_1(h) = \phi(p_0) + F(\bar{p}_0)\left(\bar{c}_1(h) - \bar{c}_1(0)\right) + o(h)$$

$$o(h) := c_1(h) - \left(\underbrace{\phi(p_0)}_{c_1(0)} + F(\bar{p}_0)\left(\bar{c}_1(h) - \bar{c}_1(0)\right)\right)$$

$$\lim_{h \to 0} o(h) = 0$$

$$\lim_{h \to 0} \frac{c_1(h) - c_1(0)}{h} - F(\bar{p}_0)\,\bar{c}_1'(0) = 0$$

$$\lim_{h \to 0} \frac{o(h)}{h} = c_1'(0) - F(\bar{p}_0)\,\bar{c}_1'(0) = 0$$

$o(h) \to 0$ faster than h

That is why we study AFFINE DEFORMATIONS

Polar decomposition of the deformation gradient

$$F = RU \qquad R^T R = I \quad \det R = 1$$

$$U \in \mathrm{Psym}$$

$$C := F^T F = (RU)^T (RU) = U^2$$

$$U = \sqrt{F^T F}$$

$$[C] = [A] \operatorname{diag}(\eta_1, \eta_2, \eta_3) [A]^T$$

$$[u_1, u_2, u_3]^T [u_1] = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$[u_1, u_2, u_3] \operatorname{diag}(\eta_1, \eta_2, \eta_3) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = [u_1] \eta_1$$

$$C = \eta_1 u_1 \otimes u_1 + \eta_2 u_2 \otimes u_2 + \eta_3 u_3 \otimes u_3$$

$$C u_1 = \eta_1 u_2 \qquad \dots$$

$(15-16)_3$ Tuesday [2014-03-11]
A1.3 09:00 – 11:00

$vol(u_1, u_2, u_3) \qquad vol : \mathcal{V} \times \mathcal{V} \times \mathcal{V} \to \mathbb{R}$

$$\begin{cases} vol(u_1 + v, u_2, u_3) = vol(u_1, u_2, u_3) + vol(v, u_2, u_3) \\ vol(\alpha u_1, u_2, u_3) = \alpha\, vol(u_1, u_2, u_3) \\ vol(u_2, u_1, u_3) = -\, vol(u_1, u_2, u_3) \end{cases}$$

$$\Rightarrow \quad vol(u_1, u_1, u_3) = -vol(u_1, u_1, u_3) = 0$$

$$vol(\underbrace{u_1 - u_1}_{0}, u_2, u_3) = vol(u_1, u_2, u_3) - vol(u_1, u_2, u_3) = 0$$

$$vol(\alpha_2 u_2 + \alpha_3 u_3, u_2, u_3) = 0$$

$vol(u_1, u_2, u_3) = 0$ for u_1, u_2, u_3 linearly independent

$\Rightarrow \quad vol = 0$

$$A_{g_1} := (n_1, u_2, u_3)$$

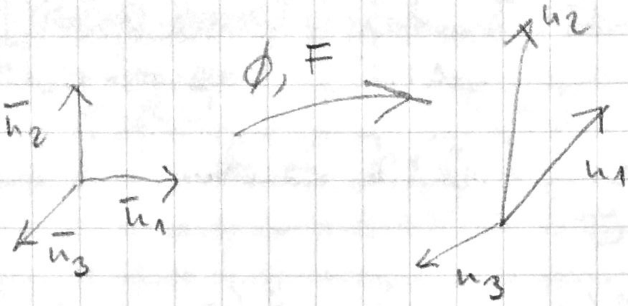
$$V = vol(u_1, u_2, u_3) = vol(\underbrace{w_1 + (u_1 \cdot n_1) n_1}, u_2, u_3)$$

$$w_1 := u_1 - (u_1 \cdot n_1) n_1$$

n_1 unit normal vector $\quad \Rightarrow \quad w_1 \cdot n_1 = u_1 \cdot n_1 - (u_1 \cdot n_1)(n_1 \cdot n_1) = 0$

$$w_1 = \alpha_2 u_2 + \alpha_3 u_3$$

$$\Rightarrow \quad V = h_1 A_{g_1} \qquad h_1 := (u_1 \cdot n_1)$$



How volumes and areas are changed by F ?

$$V = \mathrm{vol}(u_1, u_2, u_3) = \mathrm{vol}(F\bar{u}_1, F\bar{u}_2, F\bar{u}_3)$$

$$\frac{V}{\bar{V}} = \frac{\mathrm{vol}(F\bar{u}_1, F\bar{u}_2, F\bar{u}_3)}{\mathrm{vol}(\bar{u}_1, \bar{u}_2, \bar{u}_3)}$$

Choose any basis $\{e_1, e_2, e_3\}$

For example $e_1 = \bar{u}_1$, $e_2 = \bar{u}_2$, $e_3 = \bar{u}_3$

In the very simple case of planar deformation

$$Fe_1 = f_{11}\, e_1 + f_{21}\, e_2 + 0\, e_3$$

$$Fe_2 = f_{12}\, e_1 + f_{22}\, e_2 + 0\, e_3$$

$$Fe_3 = e_3$$

we get

$$\frac{\mathrm{vol}(Fe_1, Fe_2, Fe_3)}{\mathrm{vol}(e_1, e_2, e_3)} = f_{11} f_{22} - f_{21} f_{12}$$

The ratio does not depend on the basis (1)

It does not depend on the volume function

That is why we define

$$\det F = \frac{\mathrm{vol}(F\bar{u}_1, F\bar{u}_2, F\bar{u}_3)}{\mathrm{vol}(\bar{u}_1, \bar{u}_2, \bar{u}_3)}$$

(1) just because the definition of vol does not rely on any basis

Wednesday [2014-03-12] 11:00–13:00 A1.3 $(17-18)_3$

$$\frac{V}{\bar{V}} = \det F$$

Let us consider

$$V = h_1 \Delta_{\mathcal{F}_1} \qquad h_1 := u_1 \cdot n_1$$

$$\bar{V} = \bar{h}_1 \bar{\Delta}_{\bar{\mathcal{F}}_1} \qquad \bar{h}_1 := \bar{u}_1 \cdot \bar{n}_1$$

$$\frac{\Delta_{\mathcal{F}_1}}{\bar{\Delta}_{\bar{\mathcal{F}}_1}} = \frac{V}{h_1} \frac{\bar{h}_1}{\bar{V}} = \frac{\bar{h}_1}{h_1} \det F$$

$$h_1 := u_1 \cdot n_1 \qquad u_2 \cdot n_1 = 0 \qquad u_3 \cdot n_1 = 0$$

$$h_1 = F\bar{u}_1 \cdot n_1 \qquad F\bar{u}_2 \cdot n_1 = 0 \qquad F\bar{u}_3 \cdot n_1 = 0$$

$F\bar{u}_2$ and $F\bar{u}_3$ are the edges of $\mathcal{F}_1$

$$h_1 = \bar{u}_1 \cdot F^T n_1 \qquad \bar{u}_2 \cdot F^T n_1 = 0 \qquad \bar{u}_3 \cdot F^T n_1 = 0$$

$\Rightarrow$ $F^T n_1$ is orthogonal to

$\bar{u}_2$ and $\bar{u}_3$ which are the edges of $\bar{\mathcal{F}}_1$

Hence F^T pulls back n_1 to a vector normal to $\bar{\mathcal{F}}_1$

$$F^T n_1 = k_1^{-1} \bar{n}_1$$

$$n_1 = k_1^{-1} F^{-T} \bar{n}_1 \quad \Rightarrow \quad h_1 = \bar{u}_1 \cdot F^T n_1 = \bar{u}_1 \cdot \bar{n}_1 \, k_1^{-1}$$

$$h_1 = \bar{h}_1 \, k_1^{-1}$$

$$k_1 = \frac{\bar{h}_1}{h_1} \quad \Leftarrow$$

$$F^{-T} \bar{n}_1 = k_1 n_1 = \frac{A_{g_1}}{\bar{A}_{\bar{g}_1}} \frac{1}{\det F} n_1$$

$$\underbrace{(\det F) F^{-T}}_{\mathrm{cof}\, F} \bar{n}_1 = \frac{A_{g_1}}{A_{\bar{g}_1}} n_1$$

Matrix of cof F

vol $(e_1, e_2, e_3) = 1$ orthonormal basis

$$(\mathrm{cof}\, F) e_1 = \frac{A_{g_1}}{\bar{A}_{\bar{g}_1 \to 1}} n_1 = A_{g_1} n_1$$

$$(\mathrm{cof}\, F) e_1 \cdot e_i = A_{g_1} n_1 \cdot e_i = (n_1 \cdot e_i) \operatorname{vol}(n_1, F e_2, F e_3)$$

$$e_i = w_{1i} + (e_i \cdot n_1) n_1 \quad \Rightarrow \quad w_{1i} \cdot n_1 = 0$$

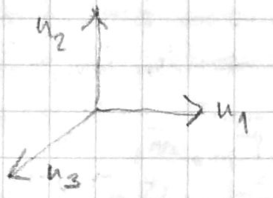
$$\begin{aligned}(\mathrm{cof}\, F) e_1 \cdot e_i &= \operatorname{vol}((n_1 \cdot e_i) n_1, F e_2, F e_3) \\ &= \operatorname{vol}(e_i - w_{1i}, F e_2, F e_3) \\ &= \operatorname{vol}(e_i, F e_2, F e_3)\end{aligned}$$

$$\underbrace{(\mathrm{cof}\,F)\,e_2}_{=\frac{A_{\mathfrak{Z}_2}}{\bar{A}_{\bar{\mathfrak{Z}}_2}} n_2}\cdot e_i = A_{\mathfrak{Z}_2}\, n_2\cdot e_i \qquad \bar{A}_{\bar{\mathfrak{Z}}_2} = 1$$

$$(\mathrm{cof}\,F)\,e_2\cdot e_i = A_{\mathfrak{Z}_2}(n_2\cdot e_i) = (n_2\cdot e_i)\,\mathrm{vol}(Fe_1, n_2, Fe_3)$$

$$A_{\mathfrak{Z}_2} = \mathrm{vol}(e_1, n_2, e_3) \qquad\qquad e_i = w_{2i} + (e_i\cdot n_2)\,n_2$$

$$\Rightarrow w_{2i}\cdot n_2 = 0$$



$$(\mathrm{cof}\,F)\,e_2\cdot e_i = \mathrm{vol}(Fe_1, e_i - w_2, Fe_3)$$

$$= \mathrm{vol}(Fe_1, e_i, Fe_3)$$

recall that $\mathrm{vol}(u_1, u_2, u_3)$ is equal to the det of the matrix made up with the vectors components

$$= \det\begin{pmatrix} f_{11} & \cdot & f_{13} \\ f_{21} & \cdot & f_{23} \\ f_{31} & \cdot & f_{33} \end{pmatrix}$$

$$= Fe_1\cdot e_i\times Fe_3 = -e_i\cdot Fe_1\times Fe_3 \qquad [\ldots]$$

$$\operatorname{tr} A = \frac{\operatorname{vol}(Ae_1, e_2, e_3)}{\operatorname{vol}(e_1, e_2, e_3)} + \ldots$$

$$\frac{d}{dt} \det F(t) = \frac{\operatorname{vol}(\dot{F}\bar{u}_1, F\bar{u}_2, F\bar{u}_3)}{\operatorname{vol}(\bar{u}_1, \bar{u}_2, \bar{u}_3)} + \ldots$$

$$= \frac{\operatorname{vol}(\dot{F}F^{-1}u_1, u_2, u_3)}{\operatorname{vol}(\bar{u}_1, \bar{u}_2, \bar{u}_3)} + \ldots$$

$$= \frac{\operatorname{vol}(\dot{F}F^{-1}u_1, u_2, u_3)}{\operatorname{vol}(\bar{u}_1, \bar{u}_2, \bar{u}_3)} \frac{\operatorname{vol}(u_1, u_2, u_3)}{\operatorname{vol}(u_1, u_2, u_3)} + \ldots$$

$$= \frac{\operatorname{vol}(\dot{F}F^{-1}u_1, u_2, u_3)}{\operatorname{vol}(u_1, u_2, u_3)} \det F + \ldots$$

$$= \det F \operatorname{tr}(\dot{F}F^{-1})$$

$$\det AB = \frac{\operatorname{vol}(ABe_1, ABe_2, Ae_3)}{\operatorname{vol}(e_1, e_2, e_3)}$$

$$\det B = \frac{\operatorname{vol}(Be_1, Be_2, Be_3)}{\operatorname{vol}(e_1, e_2, e_3)}$$

$$\det A = \frac{\operatorname{vol}(ABe_1, ABe_2, ABe_3)}{\operatorname{vol}(Be_1, Be_2, Be_3)}$$

$$\Rightarrow \det A \det B = \det AB$$

$$F^T n_1 = k_1^{-1}\,\bar{n}_1 = \frac{h_1}{\bar{h}_1}\,\bar{n}_1 \qquad h_1 = u_1\cdot n_1\,;\ \bar{h}_1 = \bar{u}_1\cdot\bar{n}_1$$

There are 2 couples of dual bases

$$\begin{cases} F^T n_i = a_{1i}\,u_1 + a_{2i}\,u_2 + a_{3i}\,u_3 \\ n_i = b_{1i}\,\bar{u}_1 + b_{2i}\,\bar{u}_2 + b_{3i}\,\bar{u}_3 \end{cases} \quad \binom{*}{0} \qquad \begin{matrix}\{u_i\},\{n_j\}\\ \{\bar{u}_i\},\{\bar{n}_j\}\end{matrix}$$

$$\begin{cases} F^T n_1\cdot n_2 = a_{21}\,u_2\cdot n_2 = a_{21}\,h_2\ ; & F^T n_1\cdot n_1 = a_{11}\,h_1 \quad \binom{*}{1} \\ n_2\cdot\bar{n}_1 = b_{12}\,\bar{u}_1\cdot\bar{n}_1 = b_{12}\,\bar{h}_1\ ; & n_1\cdot\bar{n}_1 = b_{11}\,\bar{h}_1 \quad \binom{*}{2} \end{cases}$$

$$\Uparrow$$

$$\left.\begin{matrix} \boxed{\begin{matrix} F^T n_1\cdot\bar{u}_2 = n_1\cdot u_2 = 0\ ;\ F^T n_1\cdot\bar{u}_3 = 0 \\ F^T n_2\cdot\bar{u}_1 = 0\ ;\ F^T n_2\cdot\bar{u}_3 = 0 \\ F^T n_3\cdot\bar{u}_1 = 0\ ;\ F^T n_3\cdot\bar{u}_2 = 0 \end{matrix}} & \begin{matrix} \Rightarrow F^T n_1 = k_1^{-1}\,\bar{n}_1 \\ \Rightarrow F^T n_2 = k_2^{-1}\,\bar{n}_2 \\ \Rightarrow F^T n_3 = k_3^{-1}\,\bar{n}_3 \end{matrix} \end{matrix}\right\} \binom{*}{3}$$

$$\left\{\begin{matrix} F^T n_1\cdot n_1 \underset{\binom{*}{3}}{=} k_1^{-1}\,\bar{n}_1\cdot n_1 \underset{\binom{*}{1}}{=} a_{11}\,h_1 \underset{\binom{*}{2}}{\Rightarrow} \dfrac{h_1}{\bar{h}_1}\,b_{11}\,\bar{h}_1 = a_{11}\,h_1 \\ F^T n_1\cdot n_2 = k_1^{-1}\,\bar{n}_1\cdot n_2 = a_{21}\,h_2 \Rightarrow \dfrac{h_1}{\bar{h}_1}\,b_{12}\,\bar{h}_1 = a_{21}\,h_2 \end{matrix}\right. \quad (*)$$

$$\det F^T = \frac{\mathrm{vol}(F^T n_1, F^T n_2, F^T n_3)}{\mathrm{vol}(n_1, n_2, n_3)} = \frac{\langle \det a_{ij}\rangle}{\langle \det b_{ij}\rangle}\,\frac{\mathrm{vol}(u_1,u_2,u_3)}{\mathrm{vol}(\bar{u}_1,\bar{u}_2,\bar{u}_3)} = \det F$$

$$\begin{pmatrix} h_1 & & \\ & h_2 & \\ & & h_3 \end{pmatrix}\begin{pmatrix} b_{11} & \cdot & \cdot \\ b_{21} & \cdot & \cdot \\ b_{31} & \cdot & \cdot \end{pmatrix}\begin{pmatrix} h_1 & & \\ & h_2 & \\ & & h_3 \end{pmatrix}^{-1} = \begin{pmatrix} a_{11} & \cdot & \cdot \\ a_{21} & \cdot & \cdot \\ a_{31} & \cdot & \cdot \end{pmatrix}^T \qquad h_i\,b_{ij} = a_{ji}\,h_j \quad (*)$$

It is much easier to rely on the determinant's independence of any basis, and choose an orthonormal basis and exploit the property of the matrices of being the transpose of each other thus resulting in the same expression for the determinant.

